Operator Algebras
Homework 1

Exercise 1

Let $(A_\lambda)_{\lambda\in\Lambda}$ be a collection of algebras and let $\prod_{\lambda\in\Lambda} A_\lambda$ be their Cartesian product. We define an addition, scalar multiplication and multiplication on $\prod_{\lambda\in\Lambda} A_\lambda$ by

$$\begin{aligned}(a_\lambda)+(b_\lambda) &:= (a_\lambda+b_\lambda)\\ \mu(a_\lambda) &:= (\mu a_\lambda)\\ (a_\lambda)(b_\lambda) &:= (a_\lambda b_\lambda),\end{aligned}$$

where $(a_\lambda),(b_\lambda)\in\prod_{\lambda\in\Lambda} A_\lambda$ and $\mu\in\mathbb{C}$. It is clear that this gives $\prod_{\lambda\in\Lambda} A_\lambda$ the structure of an algebra (you don't have to demonstrate this). From now on we will assume that each A_λ is in fact a Banach algebra with norm $\|.\|_{A_\lambda}$. For $(a_\lambda)\in\prod_{\lambda\in\Lambda} A_\lambda$ we write $\|(a_\lambda)\|$ to denote the expression

$$\|(a_\lambda)\| := \sup_{\lambda\in\Lambda}\|a_\lambda\|_{A_\lambda},$$

which obviously takes values in $\mathbb{R}_{\geq 0}\cup\{\infty\}$, and we define

$$\bigoplus_{\lambda\in\Lambda} A_\lambda := \left\{(a_\lambda)\in\prod_{\lambda\in\Lambda} A_\lambda : \|(a_\lambda)\|<\infty\right\}.$$

(a) Show that $\|.\|$ defines a norm on $\bigoplus_{\lambda\in\Lambda} A_\lambda$ and prove that $\bigoplus_{\lambda\in\Lambda} A_\lambda$ becomes a Banach algebra with this norm. *Warning: although you are not asked to show that $\prod_{\lambda\in\Lambda} A_\lambda$ is an algebra, you do have to show that $\bigoplus_{\lambda\in\Lambda} A_\lambda$ is an algebra.*

(b) Show that $\bigoplus_{\lambda\in\Lambda} A_\lambda$ is a unital Banach algebra if each A_λ is a unital Banach algebra. Also show that the converse is true under the assumption that all Banach algebras A_λ are non-zero.

Now let $\bigoplus^{c_0}_{\lambda\in\Lambda} A_\lambda$ be the subset of $\bigoplus_{\lambda\in\Lambda} A_\lambda$ consisting of those (a_λ) that satisfy the following property: for each $\varepsilon>0$ there exists a finite set $F_\varepsilon\subset\Lambda$ such that $\|a_\lambda\|_{A_\lambda}<\varepsilon$ for all $\lambda\in\Lambda\backslash F_\varepsilon$.

(c) Prove that $\bigoplus^{c_0}_{\lambda\in\Lambda} A_\lambda$ is a closed ideal in $\bigoplus_{\lambda\in\Lambda} A_\lambda$.

Exercise 2

Let A be a unital Banach algebra.
(a) If $a\in A$ is invertible, show that $\sigma(a^{-1})=\{\lambda^{-1}:\lambda\in\sigma(a)\}$.
(b) Show that $r(a^n)=(r(a))^n$ for all $a\in A$.
(c) If $B\subset A$ is a maximal abelian subalgebra of A, show that B is closed and contains the unit of A. Also show that $\sigma_B(b)=\sigma_A(b)$ for all $b\in B$.

Exercise 3

Let A be a unital commutative Banach algebra.
(a) Show that if A contains a non-trivial idempotent, i.e. an element $e\in A\backslash\{0,1\}$ with $e^2=e$, then $\Omega(A)$ is disconnected.
(b) Prove that the Gelfand representation is isometric if and only if $\|a^2\|=\|a\|^2$ for all $a\in A$.
(c) Prove that $\sigma(a+b)\subset\sigma(a)+\sigma(b)$ and $\sigma(ab)\subset\sigma(a)\sigma(b)$. Also demonstrate that both of these inclusions might fail if A is non-commutative.