Operator Algebras
Homework 2
Deadline: 21st of October

Exercise 1

Let A be a unital commutative Banach algebra and suppose that A is generated (as a Banach algebra) by n elements $a_1, \ldots, a_n$. Define $\sigma(a_1, \ldots, a_n) \subset \mathbb{C}^n$ by

$$\sigma(a_1, \ldots, a_n) = \{(\tau(a_1), \ldots, \tau(a_n)) : \tau \in \Omega(A)\}.$$

Show that the map $\Omega(A) \to \sigma(a_1, \ldots, a_n)$ given by $\tau \mapsto (\tau(a_1), \ldots, \tau(a_n))$ is a homeomorphism.

Exercise 2

Let A be a unital C^*-algebra.
(a) If $a, b \in A$ are positive, show that $\sigma(ab) \subset \mathbb{R}_{\geq 0}$.
(b) If $a \in \mathrm{Inv}(A)$, show that there exists a unique unitary element $u \in A$ such that $a = u|a|$. Here $|a|$ is defined as $|a| = \sqrt{a^*a}$ using the functional calculus[1].
(c) Suppose that $a \in \mathrm{Inv}(A)$. Prove that a is unitary if and only if $\|a\| = \|a^{-1}\| = 1$.

Exercise 3

For any two topological spaces X and Y we will write $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ to denote the set of continuous functions from X to Y, and for any two unital C^*-algebras A and B we will write $\mathrm{Hom}_{C^*}(A, B)$ to denote the set of unital $*$-homomorphisms from A to B.

If X and Y are non-empty topological spaces and if $\varphi \in \mathrm{Hom}_{\mathrm{Top}}(X, Y)$ then we define

$$\begin{aligned} F_\varphi : C(Y) &\to C(X) \\ f &\mapsto f \circ \varphi. \end{aligned}$$

(a) Show that $F_\varphi \in \mathrm{Hom}_{C^*}(C(Y), C(X))$ in case X and Y are compact Hausdorff spaces. Also prove that in this case the map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Top}}(X, Y) &\to \mathrm{Hom}_{C^*}(C(Y), C(X)) \\ \varphi &\mapsto F_\varphi \end{aligned}$$

is a bijection.

If A and B are unital commutative C^*-algebras and if $\alpha \in \mathrm{Hom}_{C^*}(A, B)$ then we define

$$\begin{aligned} G_\alpha : \Omega(B) &\to \Omega(A) \\ \tau &\mapsto \tau \circ \alpha. \end{aligned}$$

(b) Show that $G_\alpha \in \mathrm{Hom}_{\mathrm{Top}}(\Omega(B), \Omega(A))$. Also prove that if A and B are unital then the map

$$\begin{aligned} \mathrm{Hom}_{C^*}(A, B) &\to \mathrm{Hom}_{\mathrm{Top}}(\Omega(B), \Omega(A)) \\ \alpha &\mapsto G_\alpha \end{aligned}$$

is a bijection.

Now suppose that X and Y are locally compact Hausdorff spaces and let $\varphi \in \mathrm{Hom}_{\mathrm{Top}}(X, Y)$. As discussed during one of the lectures, it is not necessarily true that $F_\varphi(f) \in C_0(X)$ if $f \in C_0(Y)$. Thus $F_\varphi : C(Y) \to C(X)$ does not restrict to a map $C_0(Y) \to C_0(X)$.
(c) Prove that, under the assumption that φ is proper[2], we do have $F_\varphi(f) \in C_0(X)$ whenever $f \in C_0(Y)$.

[1] We will soon prove that $a^*a \geq 0$.
[2] Recall that $\varphi : X \to Y$ is called proper if $\varphi^{-1}(K)$ is compact for all compact $K \subset Y$.