Operator Algebras
Homework 3a
Deadline: 4th of November

Exercise 1

The aim of this exercise is to prove the following remarkable

Theorem If A is a C^*-algebra, $a, b \in A$ satisfy $ab = ba$, and b is normal then $a^*b = ba^*$.

(Note that this is a total triviality under the stronger assumption that b is self-adjoint.)

(a) Let A be a unital C^*-algebra. For $a, b \in A$ we define the map $f_{a,b} : \mathbb{C} \to A$ by

$$f_{a,b}(\lambda) := e^{i\lambda b} a e^{-i\lambda b}.$$

Prove that for any $a, b \in A$ and for any $\lambda \in \mathbb{C}$ the derivative

$$f'_{a,b}(\lambda) := \lim_{h \to 0} \frac{f_{a,b}(\lambda + h) - f_{a,b}(\lambda)}{h}$$

exists and equals $i[b, f_{a,b}(\lambda)]$. Here, and in the rest of this exercise, $[.,.]$ denotes the commutant in A, i.e. $[c, d] := cd - dc$ for $c, d \in A$.
(b) Let W be a closed vector subspace of A that satisfies $uWu^* \subset W$ for all unitaries $u \in A$. Show that for $a \in A$ and $w \in W$ we have $[a, w] \in W$.
(c) Let $P \subset A$ denote the closed linear span of all projections in A. Prove that for $a \in A$ and $p \in P$ we have $[a, p] \in P$.

In what follows, $b \in A$ denotes a fixed normal element and $a \in A$ is a fixed element that commutes with b.
(d) Prove that

$$f_{a,b^*}(\lambda) = e^{2i\mathrm{Re}(\lambda b^*)} a e^{-2i\mathrm{Re}(\lambda b^*)}$$

for all $\lambda \in \mathbb{C}$, where $\mathrm{Re}(c) = \frac{1}{2}(c + c^*)$ denotes the real part of any $c \in A$, of course. Also prove that $\|f_{a,b^*}(\lambda)\| = \|a\|$ for all $\lambda \in \mathbb{C}$.

By a similar argument as in the proof of theorem 1.2.5 on page 9 of Murphy (using Liouville's theorem), we can now conclude that f_{a,b^*} is a constant function.

(e) Prove that a^* commutes with b. Argue in a few sentences that this result remains true if A is non-unital.

Exercise 2 (alternative proof of theorem 3.4.3 on page 95 of Murphy's book)

Next week we will encounter Murphy's proof of Theorem 3.4.3 to the effect that a self-adjoint element a of a C^*-algebra A is positive if and only if $\tau(a) \geq 0$ for every state τ on A. (The 'only if' part is trivial.)

Give a different proof of the 'if' part using the Gelfand isomorphism applied to the abelian C^*-algebra $C^*(a) \subset A$ generated by a, combined with Murphy's Theorem 3.3.8 (which we will prove next week).