Operator Algebras
Homework 3b
Deadline: 11th of November

Exercise 1

Let A be a C^*-algebra and let $\tau : A \to \mathbb{C}$ be a positive linear functional. We define

$$N_\tau := \{a \in A : \tau(a^*a) = 0\}.$$

During the lectures we have seen that N_τ is a closed left ideal.

(a) Show that the map

$$\begin{array}{ccc} A/N_\tau \times A/N_\tau & \to & \mathbb{C} \\ (a + N_\tau, b + N_\tau) & \mapsto & \tau(b^*a) \end{array}$$

is well-defined, i.e. that it does not depend on the chosen representatives a and b.

For $a + N_\tau, b + N_\tau \in A/N_\tau$ we will write

$$\langle a + N_\tau, b + N_\tau \rangle := \tau(b^*a).$$

As we have just shown, $\langle ., . \rangle : A/N_\tau \times A/N_\tau \to \mathbb{C}$ is a well-defined map (which is obviously a sesquilinear form on the vector space A/N_τ). By positivity of τ, we have that $\langle a + N_\tau, a + N_\tau \rangle \in \mathbb{R}_{\geq 0}$ for all $a + N_\tau \in A/N_\tau$, and by definition of N_τ we have that $\langle a + N_\tau, a + N_\tau \rangle = 0$ (if and) only if $a + N_\tau = N_\tau$, i.e. the sesquilinear form is non-degenerate. Hence[1], $\langle ., . \rangle$ defines an inner product on A/N_τ. If $a \in A$, we define a map $\varphi(a) : A/N_\tau \to A/N_\tau$ by

$$\varphi(a)(b + N_\tau) := ab + N_\tau.$$

(b) Prove that $\varphi(a)$ is well-defined, i.e. that it does not depend on the chosen representative b.

As explained on page 94 of Murphy, this map is bounded[2], i.e. $\varphi(a) \in B(A/N_\tau)$. Hence it can be uniquely extended to a bounded operator $\varphi_\tau(a) \in B(H_\tau)$, where H_τ denotes the Hilbert space obtained by completing the inner product space A/N_τ.

(c) Show that

$$\begin{array}{rcl} \varphi_\tau : A & \to & B(H_\tau) \\ a & \mapsto & \varphi_\tau(a) \end{array}$$

is a $*$-homomorphism.

(d) Prove that $\ker(\varphi_\tau) \subset \ker(\tau)$. Also prove that if $I \subset A$ is a closed ideal, then $I \subset \ker(\tau)$ implies that $I \subset \ker(\varphi_\tau)$.

The positive linear functional τ is called *faithful* if for all $a \in A^+$ we have the implication

$$\tau(a) = 0 \quad \Rightarrow \quad a = 0.$$

(e) Show that if τ is faithful, then the GNS representation (H_τ, φ_τ) is faithful[3].

[1] According to the definition of an inner product in some textbooks on functional analysis (including the ones written by Rudin and Conway), we also need to show that the sesquilinear form $\langle ., . \rangle$ is hermitian, i.e. that it satisfies $\langle w, v \rangle = \overline{\langle v, w \rangle}$. However, during the lectures we have shown that a sesquilinear form $\phi : V \times V \to \mathbb{C}$ on a complex vector space V is hermitian if and only if $\phi(v, v) \in \mathbb{R}$ for all $v \in V$. Thus the fact that $\langle a + N_\tau, a + N_\tau \rangle \in \mathbb{R}_{\geq 0}$ for all $a + N_\tau \in A/N_\tau$ already implies that $\langle ., . \rangle$ is hermitian.

[2] Because A/N_τ is an inner product space, it is also a normed vector space. When we speak of a bounded operator on A/N_τ, we mean boundedness with respect to this norm.

[3] Recall that a representation (H, π) of a C^*-algebra A is called *faithful* if $\pi : A \to B(H)$ is injective.

Exercise 2

Let A be a unital C^*-algebra and let α be an automorphism[4] of A such that $\alpha^2 = \mathrm{id}_A$. Recall the definition of the C^*-algebra $M_n(A)$ on pages 94-95 of Murphy. We define a subset $B \subset M_2(A)$ by

$$B := \left\{ \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} : a, b \in A \right\}.$$

(a) Show that B is a C^*-subalgebra of $M_2(A)$.

We define a map $\varphi : A \to B$ by

$$\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix},$$

which is clearly an injective $*$-homomorphism.

We now define the element $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B$. Then $u^* = u = u^{-1}$, so u is a self-adjoint unitary element of B. Note that for all $a, b \in A$,

$$\begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} = \varphi(a) + \varphi(b)u,$$

so $B = \varphi(A) + \varphi(A)u$. In fact, each $\widetilde{b} \in B$ can be written as $\widetilde{b} = \varphi(a) + \varphi(b)u$ for uniquely determined $a, b \in A$.

(b) Show that $\varphi(\alpha(a)) = u\varphi(a)u^*$ for all $a \in A$.

Remark: The identity in part (b) can simplify your computations in part (c) significantly. In fact, when you use this identity in a clever way, you don't have to draw a single matrix in part (c).

(c) Now let C be a unital C^*-algebra with a self-adjoint unitary element $v \in C$. Prove that if $\psi : A \to C$ is a unital $*$-homomorphism satisfying

$$\psi(\alpha(a)) = v\psi(a)v^*$$

for all $a \in A$, then there exists a unique $*$-homomorphism $\psi' : B \to C$ such that $\psi' \circ \varphi = \psi$ and $\psi'(u) = v$.

[4] I.e. a $*$-isomorphism from A onto itself.