

**Operator Algebras**  
**Homework 5**  
**Deadline: 9th of December**

## Exercise 1

In this exercise you might want to recall the following standard result from functional analysis:

**Theorem (Principle of uniform boundedness).** *Let  $X$  be a Banach space, let  $Y$  be a normed vector space and let  $S \subset B(X, Y)$  be such that for each (fixed)  $x \in X$  we have  $\sup\{\|ax\| : a \in S\} < \infty$ . Then  $\sup\{\|a\| : a \in S\} < \infty$ .*

Let  $H$  be a Hilbert space.

- (a) Prove that if  $(a_n)_{n \in \mathbb{N}}$  is a sequence of elements in  $B(H)$  that converges strongly to some  $a \in B(H)$ , then this sequence is norm-bounded, i.e.  $\sup\{\|a_n\| : n \in \mathbb{N}\} < \infty$ .
- (b) Let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be sequences of elements in  $B(H)$  converging strongly to  $a \in B(H)$  and  $b \in B(H)$ , respectively. Prove that the sequence  $(a_n b_n)_{n \in \mathbb{N}}$  converges strongly to  $ab$ .
- (c) Let  $v \in K(H)$  and let  $(u_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $B(H)$  converging strongly to  $u \in B(H)$ . Prove that the sequence  $(u_n v)_{n \in \mathbb{N}}$  converges in norm to  $uv$ .

## Exercise 2

Let  $H$  be a Hilbert space and let  $M \subset B(H)$  be a Von Neumann algebra containing the unit operator  $1 = id_H$ . We fix two orthogonal<sup>2</sup> projections  $p, q \in M$  and define  $r_n = (pqp)^n$  for  $n \geq 0$  (note that  $r_0 = 1$ ).

- (a) Prove that for all  $n \geq 0$  we have  $r_n \geq 0$  and  $r_n \geq r_{n+1}$ . Also prove that  $(r_n)$  converges strongly to an orthogonal projection  $r \in M$ .
- (b) Prove that  $r$  is in fact the orthogonal projection onto  $pH \cap qH$ . It is convenient to write this orthogonal projection as  $p \wedge q$ .
- (c) Prove that  $p \wedge q = \lim_{n \rightarrow \infty} (pqp)^n$ , where the limit denotes the limit in the strong operator topology (remark: there is no factor of  $p$  missing inside the bracket).

Let  $p^\perp$  and  $q^\perp$  denote the orthogonal projections onto the subspaces  $(pH)^\perp$  and  $(qH)^\perp$ , respectively.

- (d) Prove that  $(p^\perp \wedge q^\perp)^\perp$  is the orthogonal projection onto  $\overline{pH + qH}$ . It is convenient to write this orthogonal projection as  $p \vee q$ . In this notation, we have shown that  $p \vee q = (p^\perp \wedge q^\perp)^\perp$ .
- (e) Show that if  $p$  and  $q$  commute, then  $p \wedge q = pq$  and  $p \vee q = p + q - pq$  (this should be trivial now).

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<sup>1</sup>Here  $K(H) \subset B(H)$  denotes the space of compact operators on  $H$ .

<sup>2</sup>A bounded linear operator on a Hilbert space is an orthogonal projection if and only if it is a projection in the sense of  $C^*$ -algebras (i.e. a self-adjoint idempotent).

### Exercise 3

Let  $A$  be a  $C^*$ -algebra. Recall that two projections  $p, q \in A$  are called (*Murray-Von Neumann*) *equivalent* if there exists an element  $u \in A$  such that  $p = u^*u$  and  $q = uu^*$ . If  $p$  and  $q$  are equivalent projections, we write  $p \sim q$ .

(a) Let  $p, q \in A$  be equivalent projections and let  $r \in A$  be a projection that is orthogonal to both  $p$  and  $q$ , i.e.  $rp = rq = 0$ . Prove that the projections  $r + p$  and  $r + q$  are equivalent.

If  $p \in A$  is a projection, then  $p$  is called *finite* if for any projection  $q$  we have the implication

$$q \sim p \text{ and } q \leq p \quad \Rightarrow \quad q = p.$$

A projection that is not finite is called *infinite*.

(b) Let  $p \in A$  be a finite projection. Prove that if  $q \in A$  is a projection with  $q \leq p$ , then  $q$  is finite.

If  $p \in A$  is a projection, we recall from example 3.2.1 on page 84 of Murphy's book that  $pAp$  is a hereditary  $C^*$ -subalgebra of  $A$ .

(c) If  $M$  is a Von Neumann algebra and  $p \in M$  is a projection, we call  $p$  *abelian* if  $pAp$  is abelian. Prove that if  $p \in M$  is an abelian projection, then it is finite.