Operator algebras Homework 5
Deadline: 30th of April

Exercise 1

Let A be a C*-algebra and let $\varphi : A \to B(H)$ be a representation of A. Let $x \in H$ be a unit vector and define the state τ by

$$\tau(a) = \langle \varphi(a)x, x \rangle.$$

Show that there is an isometry $V : H_\tau \to H$ such that

$$\varphi(a)V = V\varphi_\tau(a)$$

for all $a \in A$.

Exercise 2

(a) Let A be a C*-algebra and let $\varphi : A \to B(H)$ be an irreducible representation of A. Use the theorems in section 5.1 of Murphy to deduce that for each $x \in H$ the state τ_x defined by

$$\tau_x(a) = \langle \varphi(a)x, x \rangle$$

is a pure state.

(b) Let H be a separable infinite-dimensional Hilbert space. Show that not all pure states of $B(H)$ are of the form

$$B(H) \ni A \mapsto \langle Ax, x \rangle$$

with $x \in H$ a unit vector.

Exercise 3

Let A be a C*-algebra and let $\varphi_1 : A \to B(H_1)$ and $\varphi_2 : A \to B(H_2)$ be two irreducible representations of A. Furthermore, let $T : H_1 \to H_2$ be a bounded operator such that

$$T\varphi_1(a) = \varphi_2(a)T$$

for all $a \in A$. Prove that if φ_1 and φ_2 are not unitarily equivalent, then we must have $T = 0$.

Exercise 4

Let H be a Hilbert space and let ρ be a trace-class operator on H. We define a linear functional $\omega : B(H) \to \mathbb{C}$ by

$$\omega(a) = \mathrm{Tr}(\rho a).$$

Because for each $a \in B(H)$ we have $|\mathrm{Tr}(\rho a)| \leq \|a\|\mathrm{Tr}(|\rho|)$, it follows that ω is bounded (see for example theorem 2.4.16 of Murphy or lemma 3.4.10 of Pedersen's book 'Analysis Now' for a proof of this inequality).

(a) Prove that if ρ is self-adjoint, then

$$\|\omega\| = \sum_{\lambda \in \sigma_p(\rho)} m_\lambda |\lambda|,$$

where $\|\omega\| = \sup\{|\omega(a)| : a \in B(H), \|a\| = 1\}$ is the operator norm, $\sigma_p(\rho)$ denotes the set of eigenvalues of ρ and m_λ is the dimension of the eigenspace corresponding to λ.

Let $v_1, v_2 \in H$ be two unit vectors and let $\tau_1, \tau_2 : B(H) \to \mathbb{C}$ be the "vector states" given by $\tau_j(a) = \langle av_j, v_j\rangle$. Define the linear functional $\omega : B(H) \to \mathbb{C}$ by $\omega = \tau_1 - \tau_2$.

(b) Prove that $\|\omega\|^2 = 4(1 - |\langle v_1, v_2\rangle|^2)$. Hint: use part (a).

Remark: Note that the formula in part (b) implies that $|\langle v_1, v_2\rangle|^2 = 1 - \frac{1}{4}\|\tau_1 - \tau_2\|^2$. *When we replace* $B(H)$ *by an abstract* C^**-algebra* A *and* τ_j *by arbitrary pure states on* A*, then the right-hand side of this equation still makes sense. When the right-hand side vanishes, we will say that* τ_1 *and* τ_2 *are orthogonal to each other (generalizing the notion of orthogonality that is present in the case of* $B(H)$*). This turns out to be very useful, since it allows us to partition (in a way that we will not explain here) the set of pure states into so-called sectors, each of which turns out to be precisely the set* $\{a \mapsto \langle \varphi(a)v, v\rangle : v \in H, \|v\| = 1\}$ *of vector states corresponding to some irreducible representation* $\varphi : A \to B(H)$*. Furthermore, the set of vector states corresponding to two unitarily equivalent irreducible representations define the same sector, whereas the vector states corresponding to two unitarily inequivalent representations define different sectors (which are mutually orthogonal in the sense defined above). Perhaps we will prove all this in the next homework set.*