Operator algebras Homework 6
Deadline: 14th of May

Exercise 1

Let A be a C*-algebra and $Z(A)$ its center. Since $Z(A)$ is abelian, we can find a locally compact Hausdorff space X and an isometric isomorphism $\lambda : Z(A) \to C_0(X)$. Let $\mathrm{Irr}(A)$ denote the class of irreducible representations of A.

(a) For $(H, \varphi) \in \mathrm{Irr}(A)$, show that $\varphi(Z(A)) \subset \mathbb{C}1_H$ and (in case A is unital and φ non-zero) that $\varphi(1) = 1_H$.

(b) Prove that there exists a unique map $\alpha : \mathrm{Irr}(A) \to X$ such that for all $(H, \varphi) \in \mathrm{Irr}(A)$ and for all $a \in Z(A)$ we have $\varphi(a) = \lambda(a)(\alpha(\varphi))1_H$.

Exercise 2

Let H be a Hilbert space and let $M \subset B(H)$ be a Von Neumann algebra containing the unit operator $1 = id_H$. Take two orthogonal[1] projections $p, q \in M$ and define $r_n = (pqp)^n$ for $n \geq 0$ (note that $r_0 = 1$).

(a) Prove that for all $n \geq 0$ we have $r_n \geq 0$ and $r_n \geq r_{n+1}$. Also prove that (r_n) converges strongly to an orthogonal projection $r \in M$.

(b) Prove that r is in fact the orthogonal projection onto $pH \cap qH$. It is convenient to write this orthogonal projection as $p \wedge q$.

(c) Prove that $p \wedge q = \lim_{n\to\infty}(pq)^n$, where the limit denotes the limit in the strong operator topology (remark: there is no factor of p missing inside the bracket).

Let $p^\perp$ and $q^\perp$ denote the orthogonal projections onto the subspaces $(pH)^\perp$ and $(qH)^\perp$, respectively.

(d) Prove that $(p^\perp \wedge q^\perp)^\perp$ is the orthogonal projection onto $\overline{pH + qH}$. It is convenient to write this orthogonal projection as $p \vee q$. In this notation, we have shown that $p \vee q = (p^\perp \wedge q^\perp)^\perp$.

(e) Show that if p and q commute, then $p \wedge q = pq$ and $p \vee q = p + q - pq$ (this should be trivial now).

[1] A bounded linear operator on a Hilbert space is an orthogonal projection if and only if it is a projection in the sense of C*-algebras (i.e. a self-adjoint idempotent).