Operator algebras Homework 7
Deadline: 28th of May

Exercise 1

Let A be a C*-algebra and let $\varphi : A \to B(H)$ be a representation and write $\overline{\varphi(A)}^s$ to denote the closure of $\varphi(A)$ in $B(H)$ with respect to the strong operator topology.

Use the Kaplansky density theorem, together with the rest of your knowledge of operator algebras, to prove that $\varphi(A_1)$ is strongly dense in $\left(\overline{\varphi(A)}^s\right)_1$, where in both expressions (and in the rest of this homework set) the subindex 1 denotes the closed unit ball (i.e. the set of elements with norm ≤ 1).

Exercise 2

Let A be a unital C*-algebra and write $PS(A)$ for the set of pure states on A. For $\tau_1, \tau_2 \in PS(A)$ we define $\tau_1 \cdot \tau_2 \in [0,1]$ by

$$\tau_1 \cdot \tau_2 = 1 - \frac{1}{4}\|\tau_1 - \tau_2\|^2,$$

where $\|.\|$ denotes the operator norm on the dual A^* of A, i.e. $\|\phi\| = \sup\{|\phi(a)| : a \in A_1\}$ for $\phi \in A^*$. Note that $\tau_1 \cdot \tau_2 = 1$ if and only if $\tau_1 = \tau_2$.

We will say that two states τ_1 and τ_2 are *orthogonal* if and only if $\tau_1 \cdot \tau_2 = 0$. Two subsets $S_1, S_2 \subset PS(A)$ are called *mutually orthogonal* if $\tau_1 \cdot \tau_2 = 0$ whenever $\tau_1 \in S_1$ and $\tau_2 \in S_2$. If $S \subset PS(A)$, we say that S is *indecomposable* if S cannot be written as a disjoint union of two non-empty mutually orthogonal subsets of $PS(A)$.

(a) We define a relation on $PS(A)$ by: $\tau_1 \sim \tau_2$ if and only if there exists an indecomposable subset of $PS(A)$ that contains both τ_1 and τ_2. Prove that $\sim$ defines an equivalence relation on $PS(A)$.

Let $S \subset PS(A)$ be indecomposable. Then we will call S a *sector* if for any indecomposable set $S' \subset PS(A)$ the inclusion $S \subset S'$ implies that $S = S'$ (so a sector is a maximal indecomposable subset of $PS(A)$).

(b) Prove that $PS(A)$ is the disjoint union of mutually orthogonal sectors.

Let $\varphi : A \to B(H)$ be an irreducible representation, let $v_1, v_2 \in H$ be two unit vectors and let $\tau_1, \tau_2 : A \to \mathbb{C}$ be the vector states $\tau_j(a) = \langle \varphi(a)v_j, v_j \rangle$. Recall that we have shown in exercise 2a of the 5th homework set that the vector states in an irreducible representation are pure.

(c) Prove that $\tau_1 \cdot \tau_2 = |\langle v_1, v_2 \rangle|^2$. Hint: use the result from exercise 4b of the 5th homework set together with the result from exercise 1 above.

Let $PS_\varphi(A) = \{a \mapsto \langle \varphi(a)v, v \rangle : v \in H, \|v\| = 1\}$ be the set of vector states associated with the irreducible representation φ.

(d) Prove that $PS_\varphi(A)$ is indecomposable.

Let $\varphi_1 : A \to B(H_1)$ and $\varphi_2 : A \to B(H_2)$ be two irreducible representations that are not unitarily equivalent, and let $\pi = \varphi_1 \oplus \varphi_2 : A \to B(H_1 \oplus H_2)$ be their direct sum. The orthogonal projections from $H_1 \oplus H_2$ onto H_1 and H_2 are denoted by E_1 and E_2, respectively.

(e) Prove that the unitary operator $U = E_1 - E_2 \in B(H_1 \oplus H_2)$ lies in the closure of $\pi(A)$ with respect to the strong operator topology. Hint: first compute the commutant $\pi(A)'$ (warning: here it matters that φ_1 and φ_2 are not unitarily equivalent) and then consider the double commutant $\pi(A)''$.

Let $h_1 \in H_1$ and $h_2 \in H_2$ be unit vectors and let $\omega_1, \omega_2 : A \to \mathbb{C}$ be the vector states $\omega_j(a) = \langle \varphi_j(a) h_j, h_j \rangle$.

(f) Prove that $\|\omega_1 - \omega_2\| = 2$. Hint: consider the operator U above and use the result of exercise 1.

(g) Use the results above to prove that:
(1) Each sector is equal to $PS_\varphi(A)$ for some irreducible representation φ of A.
(2) Two unitarily equivalent irreducible representations define the same sector.
(3) Two unitarily inequivalent irreducible representations define mutually orthogonal sectors.