ON SOLITON AUTOMORPHISMS IN MASSIVE AND CONFORMAL THEORIES

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Received 6 April 1998

For massive and conformal quantum field theories in 1+1 dimensions with a global gauge group we consider soliton automorphisms, viz. automorphisms of the quasilocal algebra which act like two different global symmetry transformations on the left and right spacelike complements of a bounded region. We give a unified treatment by providing a necessary and sufficient condition for the existence and Poincaré covariance of soliton automorphisms which is applicable to a large class of theories. In particular, our construction applies to the QFT models with the local Fock property — in which case the latter property is the only input from constructive QFT we need — and to holomorphic conformal field theories. In conformal QFT soliton representations appear as twisted sectors, and in a subsequent paper our results will be used to give a rigorous analysis of the superselection structure of orbifolds of holomorphic theories.

1. Introduction

Solitons in massive quantum field theories in 1+1 dimensions continue to attract the interest of quantum field theorists. Primarily this is due to the fact that they constitute topological excitations of a QFT which are non-trivial yet amenable to thorough understanding. They occupy a prominent position in the analysis of exactly soluble classical and quantum models, and there are strong indications [38] that soliton sectors are the only interesting sectors of massive QFTs in $1+1$ dimensions. The first rigorous approach to the study of solitonic sectors in the framework of general QFT was given by Roberts [45], and a very general analysis of soliton sectors and their composition structure has been provided by Fredenhagen [22, 23].

As is well known, massive QFTs with a spontaneously broken symmetry group give rise to inequivalent vacua and thereby to soliton representations. (Of course, spontaneous symmetry breakdown, which occurs only for discrete groups [12], is not the only possible origin for the existence of inequivalent vacua.) Rigorous constructions of soliton sectors and soliton automorphisms for several models have been

*Supported by EU TMR Network "Noncommutative Geometry".

Reviews in Mathematical Physics, Vol. 11, No. 3 (1999) 337–359

given by Fröhlich [24, 25] relying on methods from algebraic and from constructive quantum field theory.

One aim of this work is to exhibit and exploit the similarities of soliton representations of massive and conformal quantum field theories in $1+1$ dimensions with a global gauge group. Since spontaneous breakdown of inner symmetries is impossible in conformal theories due to the uniqueness of the vacuum [44] and since left and right spacelike infinity coincide, the role of solitons in CQFT is necessarily different. Picking a Minkowski space within the conformal covering space [35] and restricting the theory to this Minkowski space, the soliton condition

$$\begin{aligned} \pi \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) &= \pi_0 \circ \alpha_g \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) \,, \\ \pi \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) &= \pi_0 \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) \,, \end{aligned} \tag{1.1}$$

where $W_{LL}^{\mathcal{O}}, W_{RR}^{\mathcal{O}}$ are the left and right spacelike complements of the double cone $\mathcal{O}$, makes sense. It can be shown [40] that for $g \neq e$ such a representation cannot be unitarily equivalent to π_0 since it is not locally normal at infinity. Since positive energy representations of CQFTs are normal on every double cone [10], thus also at infinity, soliton representations do not constitute proper superselection sectors of the theory $\mathcal{F}$. In restriction to the fixpoint theory $\mathcal{A} = \mathcal{F}^G$, however, the discontinuity at infinity disappears, $\pi \restriction \mathcal{A}$ being localized in $\mathcal{O}$:

$$\pi \restriction \mathcal{A}(\mathcal{O}') = \pi_0 \restriction \mathcal{A}(\mathcal{O}') \,. \tag{1.2}$$

For this reason the solitons, better known as twisted representations, of the field theory $\mathcal{F}$ are relevant for the superselection structure of the fixpoint theory $\mathcal{A}$, cf. [13]. The results of this work will be used in a subsequent paper [40] for giving a rigorous analysis of such 'orbifold models' (of holomorphic models) and clarifying the role of the Dijkgraaf–Witten 3-cocycle ω and of the twisted quantum double $D^\omega(G)$. Since the (chiral) Ising model [36, 4] obviously is not covered by the analysis in [13] although it is an $\mathbb{Z}_2$ orbifold model, there must be an implicit assumption in the latter analysis. Our reconsideration of orbifold models was partially motivated by the desire to clarify which properties the triple $(\mathcal{F}, G, \alpha)$ must possess in order to lead to the results of [13]. As it turns out this is just the existence of 'twisted sectors' in the guise of soliton automorphisms, not just of soliton endomorphisms [4].

We briefly recall the framework of local quantum physics [30, 33]. We consider a QFT to be given in terms of a net of algebras, i.e. a map $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$, where $\mathcal{K}$ is the set of all double cones in Minkowski space and $\mathcal{F}(\mathcal{O})$ is a C^*-algebra. This map being inclusion preserving $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2)$ we can define the quasilocal algebra as the inductive limit: $\mathcal{F} = \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{F}(\mathcal{O})}^{\|\cdot\|}$. There are commuting automorphic actions on $\mathcal{F}$ of the Poincaré group $\mathcal{P}$ and of a locally compact symmetry group G such that $\alpha_{\Lambda,x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O}+x)\,\forall\,(\Lambda,x) \in \mathcal{P}$ and $\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O})\,\forall\, g \in G$. When considering observables we require locality, i.e. $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$ whenever $\mathcal{O}_1, \mathcal{O}_2$ are spacelike to each other. In the presence of fermions we assume the the usual Bose–Fermi commutation relations w.r.t. a $\mathbb{Z}_2$ grading given by the automorphism $\alpha_- = \alpha_v$, where v is an element of order two in the center of G.

The quasilocal algebra being automatically simple, all representations are faithful. All representations π we consider are required to be Poincaré covariant, i.e. there is a unitary representation of $\mathcal{P}$ with positive energy such that $\pi \circ \alpha_x(A) = U_\pi(x)\pi(A)U_\pi(x)^*$. We assume the existence of a vacuum state ω_0 such that the algebras $\pi_0(\mathcal{F}(\mathcal{O})) \subset \mathcal{B}(\mathcal{H}_0)$ in the GNS representation π_0 on the Hilbert space $\mathcal{H}$ are weakly closed. Thus $\mathcal{F}(\mathcal{O})$ is a W^*-algebra and α_g is locally normal. The group of unbroken symmetries (w.r.t. ω_0) is defined as

$$G_0 = \{g \in G \,|\, \omega_0 \circ \alpha_g = \omega_0\}\,. \tag{1.3}$$

If we assume some form of split property (see below) then G_0 is automatically compact when topologized with the strong topology in the representation π_0 [17], and [18, Theorem. 3.6] implies $\mathcal{F} \cap \mathcal{F}^{G_0\prime} = \mathbb{C}\mathbf{1}$. Vacuum representations of local nets are usually required to satisfy Haag duality

$$\pi_0(\mathcal{A}(\mathcal{O})) = \pi_0(\mathcal{A}(\mathcal{O}'))' \quad \forall\, \mathcal{O} \in \mathcal{K}\,, \tag{1.4}$$

which for fermionic nets is replaced by twisted duality $\pi_0(\mathcal{F}(\mathcal{O}))^t = \pi_0(\mathcal{F}(\mathcal{O}'))'$ where $X^t = ZXZ^*$ with $Z = \frac{1+iV}{1+i}$, V being the unitary implementer for $\alpha_- = \alpha_v$ ($v \in G_0$ is automatic [43]). For many purposes it is sufficient to replace (twisted) Haag duality by wedge duality $\mathcal{R}_0(W)^t = \mathcal{R}_0(W')' \;\forall\, W \in \mathcal{W}$, where $\mathcal{R}_0(W) = \pi_0(\mathcal{F}(W))''$ and $\mathcal{W}$ is the set of all wedges, i.e. translates of $W_R = \{x \in \mathbb{R}^2 | x^1 \geq |x^0|\}$ and the spacelike complement $W_L = W_R'$.

We will be interested in *soliton automorphisms* of $\mathcal{F}$, viz. automorphisms $\rho^{\mathcal{O}}_{g,h}$ which coincide with α_g on the left spacelike complement of a double cone $\mathcal{O}$ (i.e. $W^{\mathcal{O}}_{LL}$) and with α_h on the right complement ($W^{\mathcal{O}}_{RR}$). For $g, h \in G_0$ and assuming the existence of a vacuum representation π_0 satisfying Haag duality and the split property for wedges (SPW) such automorphisms can easily be constructed using disorder operators, cf. [37] and the next section. Besides the defining properties of soliton automorphisms the $\rho^{\mathcal{O}}_{g,h}$ obtained in this way have the following remarkable properties:

(a) The map $G \times G \ni (g,h) \mapsto \rho^{\mathcal{O}}_{g,h} \in \operatorname{Aut}\mathcal{F}$ is a group homomorphism.
(b) $\rho^{\mathcal{O}}_{g,g} = \alpha_g \;\forall\, g$.
(c) $\alpha_k \circ \rho^{\mathcal{O}}_{g,h} = \rho^{\mathcal{O}}_{kgk^{-1},khk^{-1}} \circ \alpha_k \;\forall\, g,h,k$. (This property follows from the first two.) In particular, if G is abelian then the soliton automorphisms commute with the global symmetry.

It is easy to see that families of two-sided soliton automorphisms can be equivalently characterized using only left-handed soliton automorphisms. Write $\rho^{\mathcal{O}}_g := \rho^{\mathcal{O}}_{g,e}$. Then properties (a–c) imply:

(A) $\alpha_k \circ \rho^{\mathcal{O}}_g = \rho^{\mathcal{O}}_{kgk^{-1}} \circ \alpha_k \;\forall\, g,h$.
(B) The map $G \ni g \mapsto \rho^{\mathcal{O}}_g \in \operatorname{Aut}\mathcal{F}$ is a group homomorphism.

Conversely, defining $\rho^{\mathcal{O}}_{g,h} \equiv \alpha_h \circ \rho^{\mathcal{O}}_{h^{-1}g} = \rho^{\mathcal{O}}_{gh^{-1}} \circ \alpha_h,\;\; g,h \in G$ one verifies that properties (a–c) follow from (A) and (B).

Postulating the existence of soliton automorphisms with the above properties Rehren [42] has recently derived the modular theoretic assumptions of [41], where a general proof of the cyclic form factor equation is announced. Since these results are quite interesting it seems important to understand better when soliton automorphisms exist. Our aim will thus be to find conditions for the existence of soliton automorphisms $\rho_g^{\mathcal{O}}$ without appealing to the SPW or to $g, h \in G_0$, preferably in such a way that the above properties (A), (B) are valid. The paper is organized as follows. Before turning to the general analysis, we give two results on solitons in massive theories (characterized by the SPW). In particular, in Subsec. 2.2 we reconsider the dual net obtained as in [37] from a massive theory with unbroken abelian symmetry and show that it possesses soliton automorphisms with all desired properties, which allows to reconstruct the original net. Section 3 is the core of the paper and contains the proof of our criterion for the existence of soliton automorphisms and the proof of their Poincaré covariance. While the existence part is not entirely new, our proof of Poincaré covariance is and relies on the uniqueness of soliton sectors up to unitary equivalence, proved in [38]. In Sec. 4 we show that the results of Sec. 3 apply to all QFT models which possess the local Fock property, e.g. the $P(\phi)_2$ and Y_2 theories.

2. Solitons in Massive Theories

2.1. *Soliton automorphisms from soliton sectors*

Given two vacuum representations π_0^L, π_0^R, a representation π is said to be a soliton representation of type (π_0^L, π_0^R) if it is translation covariant and

$$\pi \upharpoonright \mathcal{F}(W_{L/R}) \cong {\pi_0}^{L/R} \upharpoonright \mathcal{F}(W_{L/R}) \,, \tag{2.1}$$

where W_L, W_R are arbitrary left and right handed wedges, respectively. Clearly, a (π_0^L, π_0^R)-soliton representation is locally normal w.r.t. π_0^L and π_0^R. In [47] it was shown that for every pair of mutually locally normal vacuum representations π_0^L, π_0^R there is a soliton representation of type (π_0^L, π_0^R) if the vacua satisfy Haag duality and the split property for wedges (SPW). Recall that a graded local net $\mathcal{F}$ with twisted duality satisfies the SPW if for every double cone $\mathcal{O}$ there is the following isomorphism of von Neumann algebras:

$$\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_{RR}^{\mathcal{O}})^t \simeq \mathcal{R}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{R}(W_{RR}^{\mathcal{O}})^t \,. \tag{2.2}$$

(This isomorphism is automatically spatial, i.e. unitarily implemented.) For free massive scalar and Dirac fields the SPW is satisfied, and it will be assumed in this subsection and the next.

Considering now the case of a broken symmetry let π_g be the GNS representation corresponding to $\omega_g = \omega_0 \circ \alpha_g$. Due to $\pi_g \cong \pi_0 \circ \alpha_g$ all π_g's satisfy Haag duality and the SPW if π_0 does. Assuming the existence of an irreducible soliton representation of type (π_g, π_0) we show that this implies the existence of soliton automorphisms of the (abstract) C^*-algebra $\mathcal{F}$, restricting ourselves to the case of a local net.

Proposition 2.1. *Let $\mathcal{F}$ be a local net and let π be an irreducible soliton representation of type (π_g, π_0), where one of the (thus both) vacuum representations satisfies Haag duality and the SPW. Then for each double cone $\mathcal{O}$ there is an automorphism $\rho_g^{\mathcal{O}}$ of $\mathcal{F}$ such that $\rho_g^{\mathcal{O}} \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) = \alpha_g$ and $\rho_g^{\mathcal{O}} \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) = id$.*

Proof. By [38, Theorem 4.3] π automatically satisfies Haag duality, and the SPW carries over to π by [38, Theorem 5.1]. Thus also wedge duality holds [38, Proposition 2.5]. Let $\mathcal{O} \in \mathcal{K}$. By the soliton criterion there is a representation ρ_1 on $\mathcal{H}_0$ equivalent to π such that $\rho_1 \restriction \mathcal{F}(W_R^{\mathcal{O}}) = \pi_0$, where $W_R^{\mathcal{O}} = (W_{LL}^{\mathcal{O}})'$. Then we have

$$\mathcal{R}(W_R^{\mathcal{O}}) = \rho_1(\mathcal{F}(W_R^{\mathcal{O}}))'' = \pi_0(\mathcal{F}(W_R^{\mathcal{O}}))'' = \mathcal{R}_0(W_R^{\mathcal{O}})\,, \tag{2.3}$$

and wedge duality implies $\mathcal{R}(W_{LL}^{\mathcal{O}}) = \mathcal{R}_0(W_{LL}^{\mathcal{O}})$. Now, by the soliton criterion ρ_1 is equivalent to $\pi_0 \circ \alpha_g \cong \pi_g$ on every left wedge. Thus there is a unitary U on $\mathcal{H}_0$ such that

$$Ad\, U \circ \rho_1 \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) = \pi_0 \circ \alpha_g\,. \tag{2.4}$$

This formula shows that $Ad\, U$ maps the ultraweakly dense subalgebra $\mathcal{F}(W_{LL}^{\mathcal{O}})$ of $\mathcal{R}(W_{LL}^{\mathcal{O}})$ onto another such algebra, and by ultraweak continuity U acts as an automorphism on $\mathcal{R}(W_{LL}^{\mathcal{O}})$. The SPW for π gives rise to a spatial isomorphism between $\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_{RR}^{\mathcal{O}})$ and $\mathcal{R}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{R}(W_{RR}^{\mathcal{O}})$ implemented by a unitary $Y^{\mathcal{O}}$. As in [37] we define $\tilde{U} = Y^{\mathcal{O}*}(U \otimes \mathbf{1})Y^{\mathcal{O}}$ which by wedge duality is contained in $\mathcal{R}(W_L^{\mathcal{O}})$ and has the same adjoint action on $\mathcal{R}(W_{LL}^{\mathcal{O}})$ as U. Define

$$\rho_2 = Ad\, \tilde{U} \circ \rho_1\,. \tag{2.5}$$

By the localization of $\tilde{U}$ we have

$$\rho_2 \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) = \rho_1 \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) = \pi_0\,, \tag{2.6}$$

whereas by construction we have

$$\rho_2 \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) = \pi_0 \circ \alpha_g\,. \tag{2.7}$$

Since π satisfies Haag duality we easily see that $\pi(\mathcal{F}(\hat{\mathcal{O}})) = \pi_0(\mathcal{F}(\hat{\mathcal{O}}))$ whenever $\hat{\mathcal{O}} \supset \mathcal{O}$. (This was already observed in [25, p. 403].) Now the soliton automorphism is obtained by $\rho_g^{\mathcal{O}} = \pi_0^{-1} \circ \rho_2$. $\square$

Remark. Since the proof involves the SPW it is evident that the interpolation region $\mathcal{O}$ between π_0 and $\pi_0 \circ \alpha_g$ cannot be eliminated.

In [24] a soliton *representation* for the $\mathbb{Z}_2$ symmetric $(\phi^4)_2$ theory in the broken phase was constructed by a doubling trick, and the procedure given in [47] is an abstract version of the former. In both references, however, the irreducibility of the constructed soliton representation is left open, such that the above theorem cannot be used. Therefore an alternative approach to the construction of soliton automorphisms will be developed in the next section. But before doing so we will give an instructive direct proof of the existence of soliton automorphisms satisfying conditions (A) and (B) for an interesting special class of models considered first in [37].

2.2. *Soliton automorphisms for dual theories*

We start by recalling some results of [37]. Again we assume $\mathcal{F}$ to satisfy twisted duality and the SPW. Let G be a group of unbroken, i.e. unitarily implemented symmetries. By the split property G must be strongly compact and second countable [17], and the Hilbert space $\mathcal{H}$ is separable. For $g, h \in G$ we define disorder operators by

$$\begin{aligned} U_L^{\mathcal{O}}(g) &= Y^{\mathcal{O}*}(U(g) \otimes \mathbf{1})Y^{\mathcal{O}}\,, \\ U_R^{\mathcal{O}}(h) &= Y^{\mathcal{O}*}(\mathbf{1} \otimes U(h))Y^{\mathcal{O}}\,, \end{aligned} \tag{2.8}$$

where $Y^{\mathcal{O}}$ implements the spatial isomorphism (2.2). One easily verifies $Ad\,U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) = \alpha_g$ and $Ad\,U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) = id$ and similarly for $Ad\,U_R^{\mathcal{O}}(g)$. Since, as already observed in [25], $Ad\,U_L^{\mathcal{O}}(g)$ maps $\pi_0(\mathcal{F}(\hat{\mathcal{O}})), \hat{\mathcal{O}} \supset \mathcal{O}$ into itself we can obtain soliton automorphisms by

$$\rho_{g,h}^{\mathcal{O}} = \pi_0^{-1} \circ Ad\,U_L^{\mathcal{O}}(g)U_R^{\mathcal{O}}(h) \circ \pi_0\,. \tag{2.9}$$

Using the definition (2.8) and $Y^{\mathcal{O}}U(g) = (U(g) \otimes U(g))Y^{\mathcal{O}}$ one can verify that the automorphisms (2.9) satisfy the properties (a–c). This construction clearly relies on the existence of the global implementers $U(g)$ of α_g, which is due to invariance of ω_0. Since these soliton automorphisms are unitarily implemented in the vacuum representation, they may seem uninteresting from the point of view of superselection theory. But, as discussed in [37, 38] they are quite useful for elucidating the structure of the fixpoint net $\mathcal{A} = \mathcal{F}^G$, which violates Haag duality, and its dual net. To this purpose one introduces a nonlocal extension of the net $\mathcal{F}$:

$$\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)''\,. \tag{2.10}$$

Lemma 2.2. *$\hat{\mathcal{F}}(\mathcal{O})$ is isomorphic to the crossed product $\mathcal{F}(\mathcal{O}) \rtimes_{\alpha_L} G$, where $\alpha_L(g) = Ad\,U_L^{\mathcal{O}}(g)$.*

Proof. Recall the result [46, Appendix] according to which the action of G on each $\mathcal{F}(\mathcal{O})$ has full spectrum: $\Gamma(\alpha \restriction \mathcal{F}(\mathcal{O})) = \hat{G}$. The same a fortiori holding for the wedge algebra $\mathcal{R}(W_R^{\mathcal{O}})$ and the latter being factorial [20], $\mathcal{R}(W_R^{\mathcal{O}}) \rtimes_\alpha G$ is a factor by [27, Corollary 6]. But then [31, Corollary 2.3] gives us $\mathcal{R}(W_R^{\mathcal{O}}) \vee U(G)'' \simeq \mathcal{R}(W_R^{\mathcal{O}}) \rtimes_\alpha G$. Since $\mathcal{F}(\mathcal{O})$ is unitarily equivalent to $\mathcal{R}(W_R^{\mathcal{O}}) \otimes \mathcal{R}(W_L^{\mathcal{O}})$ and $U_L^{\mathcal{O}}(g)$ to $U(g) \otimes \mathbf{1}$ we are done. □

Remark. In [37] this result was obtained only for finite groups, but see also [11].

Restricting now to the case of abelian groups G, Takesaki duality gives us continuous actions $\hat{\alpha}^{\mathcal{O}}$ of the dual group $\hat{G}$ on all algebras $\hat{\mathcal{F}}(\mathcal{O})$. These actions being compatible they give rise to an action of $\hat{G}$ on the quasilocal algebra $\hat{\mathcal{F}}$ which is spontaneously broken: $\omega_0 \circ \hat{\alpha}_\chi \neq \omega_0 \ \forall\, \chi \neq \hat{e}$. The dual symmetry $\hat{\alpha}$ commuting with the action α of G it acts on the fixpoint net $\hat{\mathcal{A}}(\mathcal{O}) = \hat{\mathcal{F}}(\mathcal{O})^G$, the restriction of which to the vacuum sector $\mathcal{H}_0$ was shown to be just the dual net of the fixpoint net $\mathcal{A} \restriction \mathcal{H}_0$. The net $\hat{\mathcal{A}} \restriction \mathcal{H}_0 = (\mathcal{A} \restriction \mathcal{H}_0)^d$ satisfies Haag duality and the SPW and we are in the scenario introduced above. The discrete abelian group $\hat{G}$ is countable.

Definition 2.3. A local net $\mathcal{B}$ satisfying Haag duality and the SPW with a completely broken countable abelian symmetry group K is called dual if it arises from a net with unbroken compact abelian symmetry by the above construction.

Remark. This notation is consistent since $\mathcal{B}$ is the dual net in the conventional sense of $\mathcal{B} = (\mathcal{F}^G \restriction \mathcal{H}_0)^d$.

In [38] it was shown that the representation of $\hat{\mathcal{A}}$ on the charged sectors $\mathcal{H}_\chi \subset \mathcal{H}$ is the representation of the (unique up to unitary equivalence) soliton interpolating between the vacua ω_0 and $\omega_\chi \equiv \omega_0 \circ \hat{\alpha}_\chi^{-1}$. But we can do better:

Theorem 2.4. *Dual nets admit soliton automorphisms satisfying the properties* (A), (B).

Proof. Let $\mathcal{F}$ be the field net with unbroken symmetry G from which $\mathcal{B}$ arises. By [46, Appendix] the action of G on each $\mathcal{F}(\mathcal{O})$ has full spectrum, i.e. $\forall\, \mathcal{O} \in \mathcal{K}\ \forall\, \chi \in \hat{G}\, \exists\, \psi_\chi \in \mathcal{U}(\mathcal{F}(\mathcal{O})) : \alpha_g(\psi_\chi) = \chi(g)\psi_\chi \ \forall\, g \in G$. Due to the split property $\mathcal{H}$ is separable and G is second countable (called separable by many authors). Since the fixpoint algebra $\mathcal{A}(\mathcal{O})$ is properly infinite due to the Borchers property we can apply [48, Proposition 20.12] due to Connes and Takesaki which tells us that there is a strongly continuous homomorphism $s : K = \hat{G} \to \mathcal{U}(\mathcal{F}(\mathcal{O}))$ such that $\alpha_g(s(\chi)) = \chi(g)s(\chi) \ \forall\, g \in G,\, \chi \in \hat{G}$. (Since $\hat{G}$ is discrete in our case continuity is trivial, but the homomorphism property is not.) Due to the defining properties of disorder operators $\rho_\chi = Ad\, s(\chi)$ implements an automorphism of $\hat{\mathcal{A}}_L$ which is the identity on $W_{LL}^{\mathcal{O}}$ and $\hat{\alpha}_\chi^{-1}$ on $W_{RR}^{\mathcal{O}}$, thus a right soliton automorphism. Property (B) is fulfilled by construction, and (A) is true since ρ_χ commutes with $\hat{\alpha}_\xi$. The soliton automorphisms are clearly transportable with intertwiners in $\mathcal{A} = \mathcal{F}^G = \mathcal{B}^K$. $\square$

Given soliton automorphisms satisfying properties (A), (B) one can construct, along the lines of [14], a "crossed product theory" acted upon by the quantum double $D(K)$. For K abelian this was sketched in [42]. Applying this construction to a dual net in the above sense and restricting to net of K fixpoints one re-obtains the original theory with unbroken symmetry under $G = \hat{K}$. In fact, it seems likely that every Haag dual net $\mathcal{B}(\mathcal{O})$ with completely broken countable abelian symmetry and with soliton automorphisms satisfying (A) and (B) is a dual net in the above sense under some additional conditions. In particular, this should be true if $\mathcal{B}$ satisfies the SPW (and thus by [38, Proposition 4.1] also property 4 of the following section). We refrain from going into details since this would lead us to far away from the main subject of the present investigation.

In this section we have written $\mathcal{B}, K$ in order to avoid confusion with the original net with from which $\mathcal{B}$ arises as the dual net. From now on we return to $\mathcal{F}$ and G.

3. General Approach to Soliton Automorphisms

3.1. *Assumptions and preliminary results*

In our considerations of soliton automorphisms we will allow for graded-local nets since solitons appear in the Yukawa$_2$ model, and since also for the consideration of conformal orbifold theories the fermionic case is quite interesting. Our assumptions on the field net $\mathcal{F}$ in the vacuum representation are the following:

1. (Twisted) Haag duality.
2. Split property (for double cones).
3. The local algebras $\mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ factors.
4. Minimality of twisted relative commutants, i.e.

$$\mathcal{F}(\hat{\mathcal{O}}) \wedge \mathcal{F}(\mathcal{O})^{t\prime} = \mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2) \quad \forall\, \mathcal{O} \subset\subset \hat{\mathcal{O}}\,, \tag{3.1}$$

   where $\mathcal{O}_1$, $\mathcal{O}_2$ are related to $\mathcal{O}$, $\hat{\mathcal{O}}$ as in Fig. 1.
5. The automorphisms $\alpha_- \restriction \mathcal{F}(\mathcal{O})$ are outer for all double cones $\mathcal{O}$.

(In the pure Bose case condition 5 and the twists in condition 1 and 4 disappear.) We give a few motivating remarks for these conditions. As to massive theories, all the above properties follow from (twisted) duality and the SPW, as was shown for 5 (only for unbroken symmetries) and 3 in [37] and (in the local case) for 2 and 4 in [38]. We refrain from giving the easy proofs that the latter properties follow from the SPW also in the fermionic case. Free massive scalar and Dirac fields satisfying (twisted) duality and the SPW, they fulfill our assumptions 1–5. Unfortunately, up to now the SPW has not been proven for any interacting theory, but as we will see in Sec. 4, assumptions 1–5 hold in all models with the local Fock property.

In the case of local conformal fields local factoriality [6, 26] and condition 5 [40, 44] are automatic. The split property is a very weak assumption since it follows [26] from finiteness of the conformal characters. It is well known that condition 1,

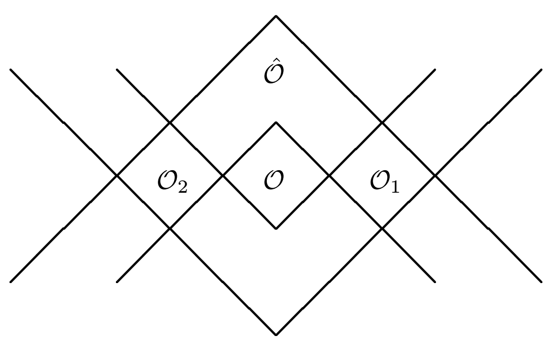

Fig. 1. Relative spacelike complement of double cones.

i.e. duality on Minkowski space (or the line) is equivalent to strong additivity, which will be shown below to follow from the split property and property 4. So we remain with the latter property which is quite restrictive since (in the local case) it implies the absence of DHR sectors [21, 38]. Twisted duality on the conformal spacetime and local factoriality will be taken for granted also in the fermionic case since they can be shown by reconsidering the arguments in [6, 26].

In [38] it was shown that Haag duality and the SPW imply strong additivity. Since instead of the latter we assume only the conditions 1–5 it is reassuring that we still have the following.

Lemma 3.1. *The conditions* 1–5 *imply strong additivity, i.e.* $\mathcal{F}(\mathcal{O}_1)\vee\mathcal{F}(\mathcal{O}_2) = \mathcal{F}(\tilde{\mathcal{O}})$ *whenever the double cones* $\mathcal{O}_1$, $\mathcal{O}_2$ *are spacelike with a common boundary point, and* $\tilde{\mathcal{O}}$ *is the smallest double cone containing* $\mathcal{O}_1$, $\mathcal{O}_2$.

Proof. Consider $\mathcal{O} \subset\subset \hat{\mathcal{O}}$. We will prove

$$(\mathcal{F}(\hat{\mathcal{O}}) \wedge \mathcal{F}(\mathcal{O})^{t\prime}) \vee \mathcal{F}(\mathcal{O}) = \mathcal{F}(\hat{\mathcal{O}})\,, \tag{3.2}$$

which due to condition 4 is equivalent to $\mathcal{F}(\mathcal{O}_1)\vee\mathcal{F}(\mathcal{O})\vee\mathcal{F}(\mathcal{O}_2) = \mathcal{F}(\hat{\mathcal{O}})$. The first two algebras being contained in the algebra of the smallest double cone containing $\mathcal{O}_1$ and $\mathcal{O}$, strong additivity follows. The split property provides us with spatial isomorphisms (implemented by the same operator Y^Λ)

$$\begin{aligned} \mathcal{F}(\mathcal{O}) &\cong \mathcal{F}(\mathcal{O}) \otimes \mathbf{1}\,, \\ \mathcal{F}(\hat{\mathcal{O}})' &\cong \mathbf{1} \otimes \mathcal{F}(\hat{\mathcal{O}})'\,. \end{aligned} \tag{3.3}$$

With $\mathcal{F}(\mathcal{O})^t \cong \mathcal{F}(\mathcal{O})_+ \otimes \mathbf{1} + \mathcal{F}(\mathcal{O})_- V \otimes V$ we compute

$$\mathcal{F}(\hat{\mathcal{O}}) \wedge \mathcal{F}(\mathcal{O})^{t\prime} \cong \mathcal{F}(\mathcal{O})^{t\prime} \otimes \mathcal{F}(\hat{\mathcal{O}})_+ + \mathcal{F}(\mathcal{O})' \otimes \mathcal{F}(\hat{\mathcal{O}})_-\,. \tag{3.4}$$

Condition 3 implies $\mathcal{F}(\mathcal{O})\vee\mathcal{F}(\mathcal{O})' = \mathcal{B}(\mathcal{H})$, and condition 5 entails $\mathcal{F}(\mathcal{O})\vee\mathcal{F}(\mathcal{O})^{t\prime} = \mathcal{B}(\mathcal{H})$ as a consequence of $\mathcal{F}(\mathcal{O}) \wedge {\mathcal{F}(\mathcal{O})_+}' = \mathbb{C}\mathbf{1}$. Thus (3.2) follows. □

Remark. Note that Haag duality has not been used. To the contrary, in the conformal case the above result implies Haag duality on Minkowski space when combined with conformal duality.

Since the group acts locally normally we have also in the broken symmetry case:

Lemma 3.2. *For every* $\Lambda = (\mathcal{O}, \hat{\mathcal{O}})$, $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ *and* $g \in G$ *there is a bosonic unitary implementer* $X_g^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})$ *for* $\alpha_g \restriction \mathcal{F}(\mathcal{O})$.

Proof. Since the vacuum vector is cyclic and separating for $\mathcal{F}(\mathcal{O})$, the algebra is in standard form and there is a unitary representation X_g of G on $\mathcal{H}$ which implements $\alpha \restriction \mathcal{F}(\mathcal{O})$, cf. [9, Sec. 2] (the construction given there coincides with Haagerup's canonical implementation [48, p. 41]). The vacuum state being invariant under α_- [43] there is also a GNS implementer V, with which X_v is easily seen to coincide. Since v is in the center of G, the X_g commute with $V = X_v$, i.e. are bosonic. Using the split property for double cones we can define a representation of G in $\mathcal{F}(\hat{\mathcal{O}})$ [8] by $X_g^\Lambda = Y^{\Lambda *}(X_g \otimes \mathbf{1})Y^\Lambda$ which still implements $\alpha \restriction \mathcal{F}(\mathcal{O})$. Due to $Y^\Lambda V = (V \otimes V)Y^\Lambda$ also X_g^Λ is bosonic. □

In theories with spontaneously broken symmetries the analog of the preceding result for wedges is false. This fact is responsible for additional difficulties in the treatment of soliton automorphisms as compared to sectors which are localizable in double cones or (left *and* right) wedges.

Lemma 3.3. *Assuming G to be compact, broken symmetries α_g, $g \in G - G_0$ act non-normally on the algebras of wedges.*

Proof. Let $\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G$ and $\mathcal{B}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{G_0}$. As shown in [43, Proposition 9] $\mathcal{B}(\mathcal{O}) \subset \mathcal{A}(W)''$ whenever $\mathcal{O} \subset W$ and thus $\mathcal{B}(W)'' = \mathcal{A}(W)''$. (In the case $G_0 = \{e\}$ this implies that the subnet $\mathcal{A}$ still satisfies wedge duality on the Hilbert space $\mathcal{H}$.) Assuming that α_g acts normally on $\mathcal{F}(W)$, it acts triviality on $\mathcal{B}(W)$ since by definition it is trivial on the ultraweakly dense subalgebra $\mathcal{A}(W)$. Then by translation invariance α_g acts trivially on $\mathcal{B}$. Now by [18, Theorem 3.6(b)] every automorphism of $\mathcal{F}$ which acts trivially on $\mathcal{B}$ is a gauge automorphism α_g with $g \in G_0$. □

Remarks. 1. We emphasize that [18, Theorem 3.6] is true also in $1+1$ dimensions with the possible exception of twisted duality for $\mathcal{F}$, which we do not need to prove anyway since it is one of our axioms.

2. The only step where compactness of G is used is the argument in [43, Proposition 9] leading to $\mathcal{B}(\mathcal{O}) \subset \mathcal{A}(W)''$ for $\mathcal{O} \subset W$. The latter is seen without difficulty to work also for locally compact abelian groups acting integrably on the algebras $\mathcal{F}(\mathcal{O})$. This is due to the fact [48, Corollary 21.3] that $\mathcal{F}(\mathcal{O})$ is generated by the operators transforming under the action of G by multiplication with a character. Finally, since $\mathcal{H}$ is separable due to the split property, the integrability property follows (for separable G) by an application of [48, Proposition 20.12] if we assume that for every $\gamma \in \hat{G}$ there is a field operator in $\mathcal{F}(\mathcal{O})$ transforming according to γ. While for non-compact groups the argument in [46, Appendix] does not work, this assumption is physically very reasonable. It is satisfied, e.g., in the sine-Gordon model where $G = \mathbb{Z}$. Lemma 3.3 will not be used in this paper.

The considerations in the sequel are considerably more transparent in the case of purely bosonic, i.e. local nets. For this reason I prefer as in [39] to treat the pure Bose case first in order to avoid confusion by the inessential complications of the general Bose/Fermi situation.

3.2. *Existence of soliton automorphisms*: *Bose case*

The following is a version of a well-known result from [16]. It follows by plugging condition 4 into [39, Proposition 4.2], but incorporates simplifications using strong additivity. We state the proof mainly for reference purposes in the next subsection were we treat the fermionic case.

Lemma 3.4. *Let $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ fulfill conditions* 1–4. *Then every locally normal endomorphism ρ of the quasilocal algebra $\mathcal{F}$ satisfying $\rho \restriction \mathcal{F}(\mathcal{O}') = id$ for some $\mathcal{O} \in \mathcal{K}$ is an inner endomorphism of $\mathcal{F}$, i.e. a direct sum of copies of the identity morphism.*

Proof. Let ρ be localized in $\mathcal{O}$ and choose a double cones K fulfilling $\mathcal{O} \subset\subset K \subset\subset \hat{\mathcal{O}}$. Thanks to the split property there exist type I factors M_1, M_2 such that

$$\mathcal{F}(\mathcal{O}) \subset M_1 \subset \mathcal{F}(K) \subset M_2 \subset \mathcal{F}(\hat{\mathcal{O}})\,. \tag{3.5}$$

By Haag duality ρ maps $\mathcal{F}(\tilde{\mathcal{O}})$ into itself whenever $\mathcal{O} \subset \tilde{\mathcal{O}}$, in particular $\rho(M_1) \subset \mathcal{F}(K)$. Being localized in $\mathcal{O}$, ρ acts trivially on $M_1' \cap \mathcal{F}(\hat{\mathcal{O}}) \subset \mathcal{F}(\mathcal{O})' \cap \mathcal{F}(\hat{\mathcal{O}}) = \mathcal{F}(\hat{\mathcal{O}} \cap \mathcal{O}')$, where we have used condition 4. This implies

$$\rho(M_1) \subset (M_1' \cap \mathcal{F}(\hat{\mathcal{O}}))' \cap M_2 \subset (M_1' \cap M_2)' \cap M_2 = M_1\,, \tag{3.6}$$

the last identity following from M_2 being type I. Thus ρ restricts to an endomorphism of M_1. Now every normal endomorphism of a type I factor is inner [34, Corollary 3.8], i.e. there is a (possibly infinite) family of isometries $V_i \in M_1,\ i \in I$ with $V_i^* V_j = \delta_{i,j}, \sum_{i \in I} V_i V_i^* = \mathbf{1}$ such that $\rho \restriction M_1 = \eta \restriction M_1$, where

$$\eta(A) \equiv \sum_{i \in I} V_i\, A\, V_i^* \tag{3.7}$$

is well-defined on $\mathcal{B}(\mathcal{H})$, the sums over I being understood in the strong sense. Again ρ acts trivially on $\mathcal{F}(\mathcal{O})' \cap M_1 \subset \mathcal{F}(\mathcal{O})' \cap \mathcal{F}(\hat{\mathcal{O}})$, which implies $V_i \in (\mathcal{F}(\mathcal{O})' \cap M_1)' \cap M_1 = \mathcal{F}(\mathcal{O})\ \forall\, i \in I$. Therefore in addition to $\rho = \eta$ on $\mathcal{F}(\mathcal{O})$ we have $\rho = \eta = id$ on $\mathcal{F}(\mathcal{O}')$. But now local normality of ρ and strong additivity, cf. Lemma 3.1, imply $\rho = \eta$ on all local algebras, thus on $\mathcal{F}$. □

Now we can identify a necessary condition for the existence of soliton automorphisms.

Proposition 3.5. *Let $\mathcal{F}$ satisfy the assumptions* 1–4. *Let $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ and let $\mathcal{O}_1, \mathcal{O}_2$ be as in Fig.* 1. *Let $\rho_g^{\mathcal{O}_1}, \rho_g^{\mathcal{O}_2}$ be locally normal soliton automorphisms. Then there is a unitary $U_g^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})$, unique up to a phase, such that $\rho_g^{\mathcal{O}_1} \circ (\rho_g^{\mathcal{O}_2})^{-1} = Ad\, U_g^\Lambda$. Furthermore, $Ad\, U_g^\Lambda \restriction \mathcal{F}(\mathcal{O}) = \alpha_g$ and $Ad\, U_g^\Lambda$ leaves $\mathcal{F}(\mathcal{O}_1)$ and $\mathcal{F}(\mathcal{O}_2)$ stable.*

Proof. It is obvious from the definition of soliton automorphisms that $\gamma = \rho_g^{\mathcal{O}_1} \circ (\rho_g^{\mathcal{O}_2})^{-1}$ acts trivially on $\mathcal{F}(\hat{\mathcal{O}}')$ and as α_g on $\mathcal{F}(\mathcal{O})$. Now by Lemma 3.4 this implies that $\gamma = Ad\, U_g^\Lambda$ where $U_g^\Lambda \in \mathcal{U}(\mathcal{F}(\hat{\mathcal{O}}))$. U_g^Λ is unique up to a phase by irreducibility of $\mathcal{F}$. Now, $\rho_g^{\mathcal{O}_1}$ acts as an inner symmetry on $\mathcal{F}(\mathcal{O}_2)$ and leaves $\mathcal{F}(\mathcal{O}_1)$ stable due to Haag duality. Arguing similarly for $\rho_g^{\mathcal{O}_2}$ we see that $\gamma = Ad\, U_g^\Lambda$ leaves $\mathcal{F}(\mathcal{O}_1)$ and $\mathcal{F}(\mathcal{O}_2)$ stable. □

Remarks. 1. This result is the analog in the broken symmetry case of [37, Lemma 2.3] which stated the uniqueness of disorder operators up to localized unitaries.

2. Given a family of soliton automorphisms we see that transportability is automatic. This fact will play a crucial role in our proof of Poincaré covariance.

By Lemma 3.2 local implementers exist also in the case of broken symmetry. Whereas a local implementer X_g^Λ acts as an automorphism on $\mathcal{F}(\hat{\mathcal{O}}) \cap \mathcal{F}(\mathcal{O})' = \mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)$, the above property of leaving $\mathcal{F}(\mathcal{O}_1)$ and $\mathcal{F}(\mathcal{O}_2)$ *separately* stable is stronger. We will now show the converse, viz. the existence of local implementers with this additional property implies the existence of soliton automorphisms.

Proposition 3.6. *Let $\mathcal{F}$ satisfy assumptions* 1–4. *If for some $g \in G$ and all $\Lambda = (\mathcal{O}, \hat{\mathcal{O}})$, $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ there is $U_g^\Lambda \in \mathcal{U}(\mathcal{F}(\hat{\mathcal{O}}))$ such that $Ad\, U_g^\Lambda \restriction \mathcal{F}(\mathcal{O}) = \alpha_g$ and such that $Ad\, U_g^\Lambda$ restricts to automorphisms of $\mathcal{F}(\mathcal{O}_i), i = 1, 2$ then there are locally normal left soliton automorphisms $\rho_g^{\mathcal{O}}$ $\forall\, \mathcal{O} \in \mathcal{K}$.*

Proof. Choose two double cones $\mathcal{O}_1$ and $\mathcal{O}_2 \ll \mathcal{O}_1$ (i.e. $\mathcal{O}_2 \subset\subset W_{LL}^{\mathcal{O}_1}$). Let $\hat{\mathcal{O}}_2$ be the smallest double cone containing $\mathcal{O}_1$ and $\mathcal{O}_2$, and let $\mathcal{O} = \hat{\mathcal{O}}_2 \cap \mathcal{O}_1' \cap \mathcal{O}_2'$. Then there is a unitary z_2 in $\mathcal{F}(\hat{\mathcal{O}}_2)$ implementing α_g on $\mathcal{F}(\mathcal{O})$ and implementing automorphisms of $\mathcal{F}(\mathcal{O}_1), \mathcal{F}(\mathcal{O}_2)$. Our aim will be to construct a soliton automorphism which acts like $Ad\, z_2$ on $\mathcal{F}(\mathcal{O}_1)$ and like α_g on the left complement of $\mathcal{O}_1$. Considering thus another double cone $\mathcal{O}_3 \ll \mathcal{O}_2$ we denote by $\hat{\mathcal{O}}_3$ the smallest double cone containing $\mathcal{O}_1$ and $\mathcal{O}_3$ (thus also $\mathcal{O}_2$) and by $\tilde{\mathcal{O}}_3$ the smallest double cone containing $\mathcal{O}_2$ and $\mathcal{O}_3$. Again there is $\tilde{z}_3$ implementing α_g on the double cone between $\mathcal{O}_1$ and $\mathcal{O}_3$ and acting on $\mathcal{F}(\mathcal{O}_1), \mathcal{F}(\mathcal{O}_3)$ by automorphisms. Now, $X = \tilde{z}_3^* z_2 \in \mathcal{F}(\hat{\mathcal{O}}_3)$ commutes with $\mathcal{F}(\mathcal{O})$, which by condition 4 implies $X \in \mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\tilde{\mathcal{O}}_3)$. This algebra being isomorphic to the tensor product $\mathcal{F}(\mathcal{O}_1) \otimes \mathcal{F}(\tilde{\mathcal{O}}_3)$ and the adjoint action of X implementing an automorphism of $\mathcal{F}(\mathcal{O}_1)$, the lemma below implies $X = X_1 X_3$ where $X_1 \in \mathcal{U}(\mathcal{F}(\mathcal{O}_1))$, $X_3 \in \mathcal{U}(\mathcal{F}(\tilde{\mathcal{O}}_3))$. Defining $z_3 = \tilde{z}_3 X_1$ we have $Ad\, z_3 \restriction \mathcal{F}(\mathcal{O}_1) = Ad\, z_2 \restriction \mathcal{F}(\mathcal{O}_1)$ and z_3 still implements α_g on the double cone between $\mathcal{O}_3$ and $\mathcal{O}_1$. Let now $\mathcal{O}_n$ be a sequence of double cones tending to left spacelike infinity. More precisely, for every $\mathcal{O} \in \mathcal{K}$ there is a $N \in \mathbb{N}$ such that $\mathcal{O}_n < \mathcal{O}$ $\forall\, n > N$. In the above way we can construct operators z_n, $n \in \mathbb{N}$ implementing α_g on the double cone between $\mathcal{O}_1$ and $\mathcal{O}_n$ and such that $Ad\, z_n \restriction \mathcal{F}(\mathcal{O}_1) = Ad\, z_2 \restriction \mathcal{F}(\mathcal{O}_1)$ $\forall\, n$. Defining

$$\rho_g^{\mathcal{O}_1}(A) = \|\cdot\| - \lim_{i \to \infty} z_i A z_i^* \,, \tag{3.8}$$

it is clear that $\rho_g^{\mathcal{O}_1}$ is a locally normal soliton automorphism. □

Lemma 3.7. *Let M_1, M_2 be factors and let $U \in \mathcal{U}(M_1 \otimes M_2)$ be such that $U(M_1 \otimes \mathbf{1})U^* = M_1 \otimes \mathbf{1}$. Then $U = U_1 \otimes U_2$ where $U_i \in \mathcal{U}(M_i)$.*

Proof. Due to factoriality we have $\mathbf{1} \otimes M_2 = (M_1 \otimes M_2) \cap (M_1 \otimes \mathbf{1})'$. Thus $\alpha = Ad\, U$ stabilizes also $\mathbf{1} \otimes M_2$ and factorizes: $\alpha = \alpha_1 \otimes \alpha_2$. α being inner, the same holds for α_1, α_2 by [48, Proposition 17.6]. Thus there are unitaries $U_i \in M_i$ such that $\alpha = Ad\, U_1 \otimes U_2$. Since we are dealing with factors the inner implementer is unique up to a phase and U_1, U_2 can be chosen such that $U = U_1 \otimes U_2$. □

Remark. Obviously the proof of the above proposition is in the spirit of Roberts' local cohomology theory.

Up to now have established a necessary and sufficient criterion for the existence of arbitrarily localizable soliton automorphisms. We conclude this subsection by giving sufficient criteria in terms of the localized implementers U_g^Λ for the soliton automorphisms to satisfy conditions (A), (B).

Proposition 3.8. *If for every* $\Lambda = (\mathcal{O}, \hat{\mathcal{O}})$ *there are localized implementers* U_g^Λ *which besides the properties required in Proposition* 3.6 *satisfy* $\alpha_k(U_g^\Lambda) = U_{kgk^{-1}}^\Lambda$ *then there are soliton automorphisms satisfying property* (A). *If there are* U_g^Λ*'s such that* $U_g^\Lambda U_h^\Lambda = U_{gh}^\Lambda$ *then there are soliton automorphisms satisfying property* (B). *If there are* U_g^Λ*'s satisfying both conditions then there are soliton automorphisms fulfilling* (A) *and* (B).

Proof. By the definition of soliton automorphisms it is obvious that $\rho_g^{\mathcal{O}} \rho_h^{\mathcal{O}} = \rho_{gh}^{\mathcal{O}}$ and $\alpha_k \circ \rho_g^{\mathcal{O}} = \rho_{kgk^{-1}}^{\mathcal{O}} \circ \alpha_k$ are satisfied in restriction to $\mathcal{F}(\mathcal{O}')$. By strong additivity and local normality of the soliton automorphisms it suffices to prove the above relations for $\mathcal{F}(\mathcal{O})$. But since there are soliton automorphisms such that $\rho_g^{\mathcal{O}_1} \restriction \mathcal{F}(\mathcal{O}_1) = Ad\, U_g^\Lambda$ for some Λ, the claimed implications are obvious consequences of the assumptions. $\square$

Remark. The conditions given above are stronger than necessary, since the adjoint action of U_g^Λ on $\mathcal{O}_2$ does not leave traces in the $\rho_g^{\mathcal{O}_1}$ which is constructed in the theorem. For our purposes the above result is sufficient. Anyhow, a detailed investigation of when properties (A), (B) can be satisfied, part of which will be found in [40], would have to be cohomological.

3.3. *Existence of soliton automorphisms*: *General case*

In the local case condition 4 immediately gives us the relative commutant of two double cone algebras. In the general case we instead have the following:

Lemma 3.9. *With the notation of Fig.* 1 *we have*

$$\mathcal{F}(\hat{\mathcal{O}}) \wedge \mathcal{F}(\mathcal{O})' = (\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2))_+ + (\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2))_- \, U_v^\Lambda \,, \tag{3.9}$$

where U_v^Λ *is an arbitrary Bose unitary in* $\mathcal{F}(\hat{\mathcal{O}})$ *implementing* α_- *on* $\mathcal{F}(\mathcal{O})$.

Proof. The Bose part of the left-hand side in (3.9) is given by $\mathcal{F}(\hat{\mathcal{O}})_+ \wedge \mathcal{F}(\mathcal{O})' = \mathcal{F}(\hat{\mathcal{O}})_+ \wedge \mathcal{F}(\mathcal{O})^{t\prime}$, where we have used $(\mathcal{F}(\hat{\mathcal{O}})_+)^t = \mathcal{F}(\hat{\mathcal{O}})_+$. Using condition 4 we see that for the Bose parts (3.9) is correct. As to the Fermi part we know by Lemma 3.2 that a U_v^Λ as needed exists, and it is easy to see that $(\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2))_- U_v^\Lambda \subset \mathcal{F}(\hat{\mathcal{O}})_- \wedge \mathcal{F}(\mathcal{O})'$. Conversely, if $X \in \mathcal{F}(\hat{\mathcal{O}})_- \wedge \mathcal{F}(\mathcal{O})'$ then $XY \in \mathcal{F}(\hat{\mathcal{O}})_+ \wedge \mathcal{F}(\mathcal{O})'$ for a unitary $Y \in (\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2))_- U_v^\Lambda$. Since $\mathcal{F}(\hat{\mathcal{O}}) \wedge \mathcal{F}(\mathcal{O})'$ is an algebra it contains also X. $\square$

In the sequel an automorphism will be called *even* if it commutes with α_-, i.e. respects the $\mathbb{Z}_2$ grading. We begin by reconsidering Proposition 3.6, assuming the assumptions made there plus condition 5. All geometrical notions are as in the proof of Proposition 3.6 and let z_2 be as defined there, but in addition we need to assume that $Ad\, z_2$ acts on $\mathcal{F}(\mathcal{O}_1), \mathcal{F}(\mathcal{O}_2)$ by even automorphisms. By the split property $\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)^t$ is isomorphic to the tensor product, thus a factor. That also $\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)$ is factorial is proved by exactly the same argument as in [37, Corollary 3.13]. (Here condition 5 is used.) By Lemma 3.2 there is a Bose implementer V_1 of $\alpha_- \restriction \mathcal{F}(\mathcal{O}_1)$ which commutes with $\mathcal{F}(\mathcal{O}_2)$. Using the split property it is easy to see that $Ad\, V_1$ acts as an automorphism on $(\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2))_+$ with fixpoint algebra $\mathcal{F}(\mathcal{O}_1)_+ \vee \mathcal{F}(\mathcal{O}_2)_+$. Since $Ad\, z_i \restriction \mathcal{F}(\mathcal{O}_1)$ is an even automorphism, $V_1 X V_1^* X^*$ commutes with $\mathcal{F}(\mathcal{O}_1)$ and the same holds trivially for $\mathcal{F}(\mathcal{O}_2)$. Thus by factoriality of $\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)$ we have $V_1 X V_1^* = \pm X$. In case the minus sign occurs we replace z_2 by $z_2 Y_1 Y_2$ with $Y_i \in \mathcal{F}(\mathcal{O}_i), i = 1, 2$ Fermi unitaries. Since $Y_1 Y_2$ is bosonic the required implementation properties of z_2 are not affected. We may thus assume that z_2 commutes with V_1, and the same holds for the $\tilde{z}_n$'s. Reconsidering $X = \tilde{z}_3^* z_2 \in \mathcal{F}(\hat{\mathcal{O}}_3)_+ \cap \mathcal{F}(\mathcal{O})'$, the Bose part of Lemma 3.9 implies $X \in (\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\tilde{\mathcal{O}}_3))_+$. By the above we can assume that X commutes with V_1 which yields $X \in \mathcal{F}(\mathcal{O}_1)_+ \vee \mathcal{F}(\tilde{\mathcal{O}}_3)_+$. The latter algebra being isomorphic to a tensor product the rest of the proof works as in the Bose case and (3.8) defines an even soliton automorphism $\rho_g^{\mathcal{O}_1}$. Since all the z_n's commute with V_1, we have $\rho_g^{\mathcal{O}_1}(V_1) = V_1$. Returning to our standard notation for localized implementers this means that $\rho_g^{\mathcal{O}_1}$ is bosonic in the following sense.

Definition 3.10. Let ρ be an endomorphism of $\mathcal{F}$ which is localized in $\mathcal{O}$ in the usual or solitonic sense. Let $U_v^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})_+$ be a localized bosonic implementer for $\alpha_- \restriction \mathcal{F}(\mathcal{O})$. Then ρ is called bosonic if $\rho(U_v^\Lambda) = U_v^\Lambda$.

Remarks. 1. This definition is independent of the choice of U_v^Λ since for another choice $\tilde{U}_v^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})_+$ we have $\tilde{U}_v^\Lambda U_v^{\Lambda *} \in \mathcal{F}(\hat{\mathcal{O}})_+ \wedge \mathcal{F}(\mathcal{O})' = \mathcal{F}(\hat{\mathcal{O}} \cap \mathcal{O}')_+$, on which ρ acts trivially.

2. If ρ is implemented (on a bigger Hilbert space) by a multiplet of field operators Ψ_i then ρ is bosonic iff the Ψ_i commute with U_v^Λ, i.e. are Bose fields.

Let conversely the soliton automorphisms $\rho_g^{\mathcal{O}_1}, \rho_g^{\mathcal{O}_2}$ considered in Proposition 3.5 in addition be even and bosonic. Then $\gamma = \rho_g^{\mathcal{O}_1} \circ (\rho_g^{\mathcal{O}_2})^{-1}$ clearly is a bosonic even endomorphism localized in the double cone $\hat{\mathcal{O}}$ of Fig. 1. As shown in [39] evenness implies that γ still maps $\mathcal{F}(\hat{\mathcal{O}})$ into itself and in Lemma 3.4 we have $\gamma(M_1) \subset \mathcal{F}(K)$. Another change in the latter Lemma is due to the fact that in the case with fermions the relative commutant $\mathcal{F}(\mathcal{O})' \cap \mathcal{F}(\hat{\mathcal{O}})$ appearing in the proof is not given by $\mathcal{F}(\hat{\mathcal{O}} \cap \mathcal{O}')$, but instead as in Lemma 3.9. (Note that the $\mathcal{O}$ in Lemma 3.4 corresponds to the above $\hat{\mathcal{O}}$, wheras the $\hat{\mathcal{O}}$ appearing there has nothing to do with that above!) Yet γ still acts trivially on the relative commutant, since it is bosonic and thus acts trivially on the U_v^Λ in Lemma 3.9. Thus the conclusion of Lemma 3.4 holds and we have a Bose unitary $U_g^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})$ such that $\gamma = Ad\, U_g^\Lambda$. Clearly U_g^Λ implements $\alpha_g \restriction \mathcal{F}(\mathcal{O})$ and even automorphisms of $\mathcal{F}(\mathcal{O}_1)$, $\mathcal{F}(\mathcal{O}_2)$.

We summarize the results of the preceding discussion in the following:

Theorem 3.11. *Let $\mathcal{F}$ satisfy conditions* 1–5. *Then the following are equivalent*

(i) *There are [bosonic even] soliton automorphisms $\rho_g^{\mathcal{O}}$ $\forall\, g \in G$, $\mathcal{O} \in \mathcal{K}$.*
(ii) *For every $g \in G$, $\Lambda = (\mathcal{O}, \hat{\mathcal{O}})$, $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ there is a [bosonic] unitary implementer $U_g^\Lambda \in \mathcal{F}(\hat{\mathcal{O}})$ for $\alpha_g \restriction \mathcal{F}(\mathcal{O})$ whose adjoint action implements [even] automorphisms of $\mathcal{F}(\mathcal{O}_i), i = 1, 2$.*

(*In the pure Bose case omit the words within square brackets.*) *Proposition* 3.8 *remains true in the Bose/Fermi case.*

3.4. *Poincaré covariance*

Up to now we have seen that under certain conditions there are even soliton automorphism which are transportable with Bose intertwiners. In [29, Theorem 5.2] it was proved that every transportable sector which is localizable in wedges and has finite statistics is Poincaré covariant provided certain conditions on the net are satisfied. Unfortunately this result cannot applied since soliton automorphisms typically are non-normal on the algebra of the wedge in which they are localized, cf. Lemma 3.3. We will thus adopt another approach, always assuming the conditions 1–5 on the net $\mathcal{F}$.

Lemma 3.12. *Let $\rho_g^{\mathcal{O}}$ be a bosonic even soliton automorphism. Then for every $(\Lambda, x) \in \mathcal{P}$ and a double cone $\hat{\mathcal{O}}$ containing $\mathcal{O}$ and $\Lambda\mathcal{O} + x$ the automorphism $\beta_{(\Lambda,x)}$ defined by*

$$\alpha_{(\Lambda,x)} \circ \rho_g^{\mathcal{O}} \circ \alpha_{(\Lambda,x)}^{-1}(F) = \beta_{(\Lambda,x)} \circ \rho_g^{\mathcal{O}}(F) \quad \forall\, F \in \mathcal{F} \tag{3.10}$$

is implemented by a unitary $Z(\Lambda, x) \in \mathcal{F}(\hat{\mathcal{O}})$ which is determined up to an arbitrary phase. The map

$$\mathcal{P} \ni (\Lambda, x) \mapsto \alpha_{(\Lambda,x)}^{\rho} \equiv \beta_{(\Lambda,x)}^{-1} \circ \alpha_{(\Lambda,x)} \tag{3.11}$$

is a group homomorphism.

Proof. $\alpha_{(\Lambda,x)} \circ \rho_g^{\mathcal{O}} \circ \alpha_{(\Lambda,x)}^{-1}$ is a bosonic even soliton automorphism for g localized in $\Lambda\mathcal{O} + x$. Thus by Lemma 3.4 and the discussion in the preceding subsection there is a unitary $Z(\Lambda, x)$ in $\mathcal{F}(\hat{\mathcal{O}})$ with $\hat{\mathcal{O}}$ as above, such that (3.10) holds with $\beta_{(\Lambda,x)} = Ad\, Z(\Lambda, x)$. By irreducibility of the quasilocal algebra $Z(\Lambda, x)$ is unique up to a phase. Now $\beta_{(\Lambda,x)}^{-1} \circ \alpha_{(\Lambda,x)} = \rho_g^{\mathcal{O}} \circ \alpha_{(\Lambda,x)} \circ (\rho_g^{\mathcal{O}})^{-1}$, and the group property of α^ρ is obvious. □

Proposition 3.13. *The automorphisms $\alpha_{(\Lambda,x)}^{\rho}$ of the preceding lemma extend uniquely to $\mathcal{B}(\mathcal{H})$. The action $\mathcal{P} \ni (\Lambda, x) \mapsto \alpha_{(\Lambda,x)}^{\rho} \in Aut\mathcal{B}(\mathcal{H})$ is continuous.*

Remark. The natural topology on the automorphism group of a von Neumann algebra A is the u-topology, viz. the topology defined by norm convergence on the predual, since it turns $\mathrm{Aut}\, A$ into a topological group. For a type I factor $\mathcal{B}(\mathcal{H})$ this topology coincides with the p-topology, which is the restriction to $\mathrm{Aut}\,\mathcal{B}(\mathcal{H})$ of the pointwise weak topology, cf. [48, pp. 41–43].

Proof. The extendibility statement is obvious since $\alpha^{\rho}_{(\Lambda,x)}$ is unitarily implemented, and uniqueness follows since $\mathcal{F}$ is irreducible. It clearly suffices to prove continuity for a neighbourhood $\mathcal{V} \subset \mathcal{P}$ of the unit element $e = (\mathbf{1}, 0)$. Let $\mathcal{O}$, $\hat{\mathcal{O}}$ be double cones such that $\Lambda\mathcal{O} + x \subset \hat{\mathcal{O}}\ \forall\, (\Lambda, x) \in \mathcal{V}$. Then by the lemma we have

$$\beta_{(\Lambda,x)} = \alpha_{(\Lambda,x)} \circ \rho_g^{\mathcal{O}} \circ \alpha^{-1}_{(\Lambda,x)} \circ (\rho_g^{\mathcal{O}})^{-1} \tag{3.12}$$

with $\beta_{(\Lambda,x)} = Ad\, Z(\Lambda, x)$ and $Z(\Lambda, x) \in \mathcal{F}(\hat{\mathcal{O}})$. Thus the $\beta_{(\Lambda,x)}, (\Lambda, x) \in \mathcal{V}$ are inner automorphisms of every algebra which contains $\mathcal{F}(\hat{\mathcal{O}})$. Furthermore, for every localized F the map $(\Lambda, x) \mapsto \beta_{(\Lambda,x)}(F)$ is strongly continuous since $\rho_g^{\mathcal{O}}$ is locally normal and α is a continuous action. Let now $\tilde{\mathcal{O}} \supset\supset \hat{\mathcal{O}}$ be another double cone and let M be a type I factor such that $\mathcal{F}(\hat{\mathcal{O}}) \subset M \subset \mathcal{F}(\tilde{\mathcal{O}})$, the existence of which is guaranteed by the split property. By the above $\beta_{(\Lambda,x)}, (\Lambda, x) \in \mathcal{V}$ acts continuously on $\mathcal{F}(\tilde{\mathcal{O}})$, thus on M and trivially on $M' \subset \mathcal{F}(\hat{\mathcal{O}})'$. But $\mathcal{B}(\mathcal{H}) = M \vee M' \simeq M \otimes M'$ implies that $\beta_{(\Lambda,x)}, (\Lambda, x) \in \mathcal{V}$ acts continuously on $\mathcal{B}(\mathcal{H})$. (We have used that, w.r.t. the u-topologies, $\alpha_\iota \to \alpha$ implies $\alpha_\iota \otimes \mathrm{id} \to \alpha \otimes \mathrm{id}$ and that M, M', $\mathcal{B}(\mathcal{H})$ are type I, such that the above remark on the topologies applies.) Since $Aut\,\mathcal{B}(\mathcal{H})$ is a topological group also the action $(\Lambda, x) \mapsto \alpha^{\rho}_{(\Lambda,x)} = \beta^{-1}_{(\Lambda,x)} \circ \alpha_{(\Lambda,x)}$ of $\mathcal{P}$ on $\mathcal{B}(\mathcal{H})$ is continuous for $(\Lambda, x) \in \mathcal{V}$, thus for all of $\mathcal{P}$. □

Theorem 3.14. *The soliton automorphism $\rho_g^{\mathcal{O}}$ is Poincaré covariant, i.e. in the sector $\pi_0 \circ \rho_g^{\mathcal{O}}$ there is a strongly continuous representation $U^\rho(\Lambda, x)$ of the Poincaré group with positive energy such that $Ad\, U^\rho(\Lambda, x) \circ \pi_0 \circ \rho_g^{\mathcal{O}} = \pi_0 \circ \rho_g^{\mathcal{O}} \circ \alpha_{(\Lambda,x)}$.*

Proof. By the preceding results we are in a position to apply a result of Kallman and Moore [48, Theorem 15.16] according to which every continuous one parameter group of inner automorphisms of a von Neumann algebra M with separable predual is implemented by a strongly continuous unitary representation in M. (Since $\mathcal{B}(\mathcal{H})$ is a factor the proof by Hansen, reproduced in [48, p. 218], is sufficient for our purposes.) Thus we have

$$\alpha^\rho_\Lambda = Ad\, e^{i\Lambda K^\rho}\ , \quad \alpha^\rho_{+,a} = Ad\, e^{iaP^\rho_+}\ , \quad \alpha^\rho_{-,b} = Ad\, e^{ibP^\rho_-}\ , \tag{3.13}$$

where $\alpha^\rho_\pm$ are the lightlike translations and $K^\rho, P^\rho_+, P^\rho_-$ are self-adjoint operators on $\mathcal{H}$. From now on we omit the superscript ρ. The unitary implementer of an automorphism being unique up to a phase the commutation relation $\alpha_\Lambda \circ \alpha_{+,a} = \alpha_{+,e^\Lambda a} \circ \alpha_\Lambda$ in $\mathcal{P}$ together with (3.13) implies $e^{i\Lambda K} e^{iaP_+} = c(a, \Lambda)\, e^{ie^\Lambda a P_+} e^{i\Lambda K}$, where c is a continuous phase-valued function satisfying $c(a_1 + a_2, \Lambda) = c(a_1, \Lambda) c(a_2, \Lambda)$

and $c(a,\Lambda_1+\Lambda_2)=c(a,\Lambda_1)c(e^{\Lambda_1}a,\Lambda_2)$. Together with continuity the first equation implies $c(a,\Lambda)=e^{iad(\Lambda)}$, where d satisfies $d(\Lambda_1+\Lambda_2)=d(\Lambda_1)+e^{\Lambda_1}d(\Lambda_2)$. The left-hand side of this equation being symmetric in Λ_1,Λ_2 we have $d(\Lambda_1)+e^{\Lambda_1}d(\Lambda_2)=d(\Lambda_2)+e^{\Lambda_2}d(\Lambda_1)$, which for $\Lambda_2\neq 0$ gives $d(\Lambda_1)=(e^{\Lambda_1}-1)d(\Lambda_2)/(e^{\Lambda_2}-1)=A(e^{\Lambda_1}-1)$. Proceding similarly for the other commutation relations we thus have

$$e^{i\Lambda K}e^{iaP_+}=e^{iAa(e^{\Lambda}-1)}\,e^{ie^{\Lambda}aP_+}e^{i\Lambda K}\,, \tag{3.14}$$

$$e^{i\Lambda K}e^{ibP_-}=e^{iBb(e^{-\Lambda}-1)}\,e^{ie^{-\Lambda}bP_-}e^{i\Lambda K}\,, \tag{3.15}$$

$$e^{iaP_+}e^{ibP_-}=e^{iCab}\,e^{ibP_-}e^{iaP_+}\,, \tag{3.16}$$

where $A,B,C\in\mathbb{R}$ are independent. By differention we obtain the commutation relations

$$i[K,P_+]=P_++A\mathbf{1}\,,\quad i[K,P_-]=-(P_-+B\mathbf{1})\,,\quad i[P_+,P_-]=C\mathbf{1}\,, \tag{3.17}$$

which we need not make precise. Now, the generators K,P_+,P_- are determined by (3.13) only up to addition of a multiple of $\mathbf{1}$. Using this freedom we can replace P_+ by $P_++A\mathbf{1}$, which removes the factor $e^{iAa(e^{\Lambda}-1)}$ from (3.14). This fixes P_+ uniquely and achieves that the Lorentz group acts on P_+ as dilatations: $e^{i\Lambda K}P_+e^{-i\Lambda K}=e^{\Lambda}P_+$. Thus the spectrum of P_+ is one of the sets $\{0\},[0,\infty),(-\infty,0],\mathbb{R}$. In any case $0\in\mathrm{Sp}(P_+)$. P_- is treated similarly. For K there is no prefered normalization, since shifting its origin does not affect the relations (3.14–3.16). (This is just the fact that the Poincaré group in 1+1 dimensions has non-trivial one dimensional representations $(\Lambda,a)\mapsto e^{iC\Lambda},C\in\mathbb{R}$.) (3.16) is not affected by shifting $P_\pm$, thus the constant C cannot be changed. (This reflects the fact that the Lie algebra cohomology $H^2(\mathfrak{P},\mathbb{R})$ is one dimensional.) We have a true representation of the Poincaré group iff C vanishes. Now we observe that $C\neq 0$ implies $\mathrm{Sp}(P_+)=\mathrm{Sp}(P_-)=\mathbb{R}$, cf. e.g. [2]. (For $\lambda\in\mathrm{Sp}(P_-)$ the differentiated commutation relation $e^{iaP_+}P_-e^{-iaP_+}=P_-+Ca\mathbf{1}$ implies $\lambda+Ca\in\mathrm{Sp}(P_-)\ \forall\, a\in\mathbb{R}$. Since $\mathrm{Sp}(P_-)$ cannot be empty $C\neq 0$ implies $\mathrm{Sp}(P_-)=\mathbb{R}$.) Before we know that $C=0$ it makes, of course, no sense to consider the joint spectrum of P_+ and P_-. We will however prove that $P_+\geq 0,P_-\geq 0$. By the above this then entails $C=0$ and positivity in the usual sense: $\mathrm{Sp}(P^\mu)\subset\overline{V^+}$.

We consider only P_+ since the argument for P_- is the same. The proof is modeled on the one [15] for DHR sectors. There are simplifications since we are dealing with soliton *auto*morphisms, thus $\bar\rho=\rho^{-1}$ is a α_+-covariant soliton automorphism, too. Yet, we spell the proof out since there we have to use lightlike instead of spacelike clustering. The spectra of P_+^ρ, $P_+^{\bar\rho}$ containing 0, positivity of P_+^ρ, $P_+^{\bar\rho}$ follows from positivity in the vacuum sector if we prove

$$\mathrm{Sp}(P_+^\rho)+\mathrm{Sp}(P_+^{\bar\rho})\subset\mathrm{Sp}(P_+^{\pi_0})\,. \tag{3.18}$$

Let $\mathcal{N}_1$, $\mathcal{N}_2$ be arbitrary open sets in $\mathbb{R}$ intersecting $\mathrm{Sp}(P_+^\rho)$, $\mathrm{Sp}(P_+^{\bar\rho})$, respectively. Then there is a vector $\Psi_1\neq 0$ with P_+^ρ-support in $\mathcal{N}_1$ and, by [15, Lemma 5.1], a $B\in\mathcal{A}$ such that $\Psi_2=B\Omega\neq 0$ has $P_+^{\bar\rho}$-support in $\mathcal{N}_2$. Now $\Psi_a=\rho(B)e^{-iaP_+^\rho}\Psi_1$

has $P_+^{\pi_0}$-support in $\mathcal{N}_1 + \mathcal{N}_2$ for all $a \in \mathbb{R}$ and we are done if there is an a such that $\Psi_a \neq 0$. With

$$\|\Psi_a\|^2 = (\Psi_1, e^{iaP_+^\rho}\rho(B^*B)e^{-iaP_+^\rho}\Psi_1) = (\Psi_1, \rho(\alpha_{+,a}^{\pi_0}(B^*B))\Psi_1) \tag{3.19}$$

such an a exists if the second step in the computation

$$\lim \|\Psi_a\|^2 = \lim(\Psi_1, \rho(\alpha_{+,a}^{\pi_0}(B^*B))\Psi_1) \stackrel{?}{=} \|\Psi_1\|^2 \cdot \omega_0(B^*B) = \|\Psi_1\|^2 \cdot \|\Psi_2\|^2 \neq 0 \tag{3.20}$$

is justified for $a \to +\infty$ or $a \to -\infty$. The cluster result [19, Proposition 1.2] for lightlike translations gives us weak convergence:

$$\mathrm{w} - \lim_{|a| \to \infty} \alpha_{+,a}^{\pi_0}(B^*B) = \omega_0(B^*B)\,\mathbf{1}\ \forall\, B \in \mathcal{A}\,. \tag{3.21}$$

(This result uses that $P_+^{\pi_0}$ has half-sided spectrum and the Reeh-Schlieder theorem in the vacuum representation.) Assume that the soliton automorphism ρ is localized in a left wedge. Then it acts normally on the algebras of all right wedges. If B is localized in a bounded region then there is a right wedge W such that $\alpha_{+,a}(B^*B) \in \mathcal{A}(W)\ \forall\, a \geq 0$ and (3.20) holds for $a \to +\infty$. The maps $B \mapsto (\Psi_1, \rho(\alpha_{+,a}^{\pi_0}(B^*B))\Psi_1)$ being uniformly bounded and the stricly local operators being norm dense in $\mathcal{A}$, (3.20) holds for all $B \in \mathcal{A}$ and $a \to +\infty$ and we are done. If ρ is right-localized then let $a \to -\infty$. □

4. The Solitons of Theories with the Local Fock Property

As an application of Theorem 3.11 we will give in this section a new construction of the soliton sectors of the superrenormalizable quantum field theories with the local Fock property, like the $P(\phi)_2$ theory (the polynomial P is assumed even in order to have $\mathbb{Z}_2$ symmetry) or the Yukawa$_2$ model. We briefly indicate the main steps of the hamiltonian approach to the construction of these models. One begins with a finite number of massive free fields and a formal interaction hamiltonian H_I. We are interested in the case where the non-interacting theory has a group G of inner symmetries which leave H_I formally invariant and which survives the renormalization (i.e. there are no anomalies) but which may be spontaneously broken. Furthermore, we assume that G commutes with the Poincaré action on the free and the interacting theory. In the rigorous construction of the models, which we sketch on the example of the $P(\phi)_2$ model, one first obtains the interacting field $\tilde{\phi}$ on the Fock space of the free field ϕ_0 (as a quadratic form) via

$$\tilde{\phi}(x,t) = U^I(t)\,\phi_0(x,0)\,U^I(t)^* \tag{4.1}$$

for x in an interval I, where U^I is a propagator obtained by smoothly cutting off the interaction outside I. (In particular $\tilde{\phi}(x,0) = \phi_0(x,0)$.) One can show that the field $\tilde{\phi}(x,t)$ is independent of the form of the cut off of the interaction provided (x,t) is contained in the double cone with basis $(I, t=0)$. $\tilde{\phi}$ carries an action of the Poincaré group, but the action is not unitarily implemented and there is no invariant vacuum vector. Still there is an invariant vacuum state ω_0, and by GNS construction one obtains the physical representation π of the quasilocal algebra $\mathcal{F}$ and a unitary representation of the Poincaré group. In restriction to arbitrary double cones ω_0 is normal which implies local normality of π. This in turn implies the existence of the physical field $\phi(x,t)$. Quantum field theories which can be constructed in the above way (including renormalizations of the hamiltonian where necessary) are said to possess the *local Fock property* [28]. This property is the only piece of information on these models which we will need since (4.1) in conjunction with the local normality of π allows to carry over many properties from the free fields to the interacting ones.

Lemma 4.1. *The free massive fields permit local implementers U_g^Λ of the unbroken symmetries with the property required in Theorem* 3.11. *The U_g^Λ can be chosen such that $\alpha_k(U_g^\Lambda) = U_{kgk^{-1}}^\Lambda$ and $U_g^\Lambda U_h^\Lambda = U_{gh}^\Lambda$.*

Proof. Unbroken symmetries are unitarily implemented, and the split property for wedges implies the existence of disorder operators [25, 37] $U_L^{\mathcal{O}}(g)$ implementing the α_g on the left complement of $\mathcal{O}$, acting trivially on the right complement and implementing an even automorphism of $\mathcal{F}(\mathcal{O})$. The latter is even since the disorder operators commute with V. Now let $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ and let $\mathcal{O}_1, \mathcal{O}_2$ be as in Fig. 1. Clearly, $U^\Lambda(g) = U_L^{\mathcal{O}_1}(g) U_L^{\mathcal{O}_2}(g)^*$ acts trivially on $\hat{\mathcal{O}}'$, implements α_g on $\mathcal{O}$ and maps the algebras $\mathcal{F}(\mathcal{O}_i), i = 1,2$ into themselves. By duality, $U^\Lambda \in \mathcal{F}(\hat{\mathcal{O}}')' = \mathcal{F}(\hat{\mathcal{O}})$. Furthermore, $\alpha_k(U_g^\Lambda) = U_{kgk^{-1}}^\Lambda$ follows from $U(k) U_L^{\mathcal{O}_i}(g) U(k)^* = U_L^{\mathcal{O}_i}(kgk^{-1})$, $i = 1,2$ [37] and we have

$$\begin{aligned} U_g^\Lambda U_h^\Lambda &= U_L^{\mathcal{O}_1}(g) U_L^{\mathcal{O}_2}(g)^* \, U_L^{\mathcal{O}_1}(h) U_L^{\mathcal{O}_2}(h)^* \\ &= U_L^{\mathcal{O}_1}(g) U_L^{\mathcal{O}_1}(h) U_L^{\mathcal{O}_2}(h^{-1}g^{-1}h) U_L^{\mathcal{O}_2}(h^{-1}) \\ &= U_L^{\mathcal{O}_1}(gh) U_L^{\mathcal{O}_2}(gh)^* = U_{gh}^\Lambda \,, \end{aligned} \tag{4.2}$$

where we have used $U_g^{\mathcal{O}_2} \in \mathcal{F}(W_{LL}^{\mathcal{O}_1})''$. □

Remark. The unitaries U_g^Λ can be shown to satisfy and to be determined by $U_g^\Lambda AB\eta = \alpha_g(A)B\eta \ \forall\, A \in \mathcal{F}(\mathcal{O}), B = \mathcal{F}(\hat{\mathcal{O}})'$. Here η is the unique vector in $\mathcal{P}^\natural(\mathcal{F}(\mathcal{O}) \vee \mathcal{F}(\hat{\mathcal{O}})', \Omega)$ implementing the state $\omega_\eta(AB) = \omega_0(A)\omega(B), A \in \mathcal{F}(\mathcal{O}), B = \mathcal{F}(\hat{\mathcal{O}})'$, where ω in turn is the product state (existing due to the SPW) which restricts to ω_0 on $\mathcal{F}(W_{LL}^{\hat{\mathcal{O}}})$ and on $\mathcal{F}(W_{RR}^{\hat{\mathcal{O}}})$. Recall that the usual localized implementer [8, 17] is obtained by replacing the product state ω by ω_0.

We summarize the properties of the interacting theory obtained as indicated above.

Lemma 4.2. *Theories with local Fock property satisfy the assumptions* 1–5 *of Sec.* 3 *in their vacuum sector.*

Proof. All these properties are fulfilled by the free scalar and Dirac fields of non-zero mass [1, 7, 49] as well as by theories of finitely many such fields, since they follow from twisted duality and the split property for wedges [38] which is known to be fulfilled. (Apart from perhaps condition 5 all these properties have been known before.) The last four properties are of a purely local nature, thus they carry over immediately to the interacting theory by the local Fock property. The nontrivial fact that also duality, which is a global property, carries over to the interacting theory has been proven in [21] for local nets and in [49, Theorem 2.8] for the twisted case. □

Lemma 4.3. *The theories with local Fock property permit local implementers of their inner symmetries with the property required in Theorem* 3.11 *and satisfying in addition* $\alpha_k(U_g^\Lambda) = U^\Lambda_{kgk^{-1}}$, $U_g^\Lambda U_h^\Lambda = U_{gh}^\Lambda$.

Proof. By Poincaré covariance and the fact that G and $\mathcal{P}$ commute it suffices to prove the existence of soliton automorphisms for double cones $\mathcal{O}$ which have as basis an interval I in the line $t = 0$. Let $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ with respective bases $I \subset\subset \hat{I}$. In view of $\mathcal{F}_0(\mathcal{O}) = \mathcal{F}(I) = \tilde{\mathcal{F}}(\mathcal{O})$ a local implementer U_g^Λ be the free field (provided by the Lemma 4.1) also implements α_g on $\tilde{\mathcal{F}}(\mathcal{O})$. On the Fock space the group G is unitarily implemented, and since the disorder operators for the free field behave covariantly, the localized implementers have all desired properties also in the physical representation π. □

Putting things together we obtain our main result.

Theorem 4.4. *Irreducible quantum field theories in* $1+1$ *dimensions which have the local Fock property w.r.t. a finite number of free massive scalar and/or Dirac fields admit Poincaré covariant locally normal soliton automorphisms which can be chosen even and bosonic and satisfying properties* (A) *and* (B).

Proof. In view of Lemmas 4.2 and 4.3 the existence of even bosonic soliton automorphisms follows from Theorem 3.11 and the Poincaré covariance from Theorem 3.14. Properties (A) and (B) are a consequence of Proposition 3.8. □

Remark. After we proved the above theorem we discovered that the essential idea of the existence part is already contained in [25, pp. 402–404]. Still, our results go beyond those of [25] in several respects. Theorem 3.11 provides a convenient sufficient *and necessary* condition for the existence of soliton automorphisms, which also applies to the twisted sectors of holomorphic conformal theories. Furthermore, in our proof of Poincaré covariance we do not appeal to any result of constructive QFT except the local Fock property, which, to be sure, is a rather deep result. Finally, the fact that one finds soliton automorphisms with the covariance (A) and homomorphism (B) properties is new.

5. Summary and Outlook

Theorem 2.4 completes the abstract treatment of the duality between massive theories (satisfying twisted duality and the SPW) with unbroken compact abelian and broken abelian symmetry groups, respectively, which was begun in [37]. The central theme, however, of this work was the observation that soliton automorphisms of massive quantum field theories in $1+1$ dimensions and twisted sectors of holomorphic conformal field theories can be treated on equal footing. The crucial property shared by these apparently unrelated classes of models is the condition 4 in Sec. 3. Section 4 finally complements and extends the earlier rigorous works on solitons [22–25, 47] by establishing sufficiency of the local Fock property for the existence of Poincaré covariant soliton automorphisms. The results of that section make plain that contrary to a widespread belief a profound understanding of the free fields is quite useful, if not necessary, in the study of interacting models, at least of those with the local Fock property.

Our analysis relies on several deep results from the theory of automorphisms of von Neumann algebras. The theorems of Connes/Takesaki and Kallmann/Moore, used in Theorems 2.4 and 3.14, respectively, make statements on the existence of continuous unitary group representations inside von Neumann algebras. Crucial were furthermore the facts that unital normal endomorphisms of type I factors are inner, and that inner tensor-product automorphisms factorize into inner automorphisms.

As emphasized the results of Sec. 3 apply also to conformal theories. This will be the basis for a rigorous analysis of conformal orbifold models in [40]. As mentioned in the Introduction the present analysis was partially motivated by the desire to understand why the chiral Ising model [36, 4] does not fit into the analysis of orbifold models given in [13]. As will be discussed further in [40] this is due to the fact that real fermions on the circle do not admit soliton automorphisms whereas the existence of the latter — called "twisted sectors" — is implicitly assumed in [13]. (In [32] the existence of twisted sectors is proved for a class of WZNW models.) A chiral theory of complex fermions or, more generally, of an even number of real fermions does possess soliton automorphisms which is why for these theories the fusion rules of the $\mathbb{Z}_2$ orbifold theory [5] are given by $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ in accordance with [13].

As to solitons in massive models, it would be very interesting to have a proof, to the largest possible extent model independent, of bounds on the soliton mass of the sort proven in [3]. Furthermore, one should try to extend Theorem 4.4 to more general models which do not possess the local Fock property.

Acknowledgments

I would like to thank K. Fredenhagen, D. Guido, R. Longo, K.-H. Rehren and J. Roberts for useful discussions concerning this work. I am particulary indebted to D. Guido for pointing out an error in the proof of Theorem 3.14 in an earlier version of this paper.

Note Added in Proof

I thank D. Buchholz for drawing my attention to a small problem in the proof of Theorem 3.14 in the case of massless theories. Namely, in 1+1 dimensions Driessler's lightlike cluster theorem can be proved only assuming a mass gap. As also noted by Buchholz, the problem is easily circumvented considering $\Psi_a = \rho(B)e^{-iaP_1^\rho}\Psi_1$ and the limit $a \to +\infty$ (if ρ is localized in a left wedge). The point is that commutatively of P_+^ρ and P_1^ρ is not needed. Then the usual spacelike cluster theorem applies and the argument goes through otherwise unchanged.

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