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Galois extensions of braided tensor categories and braided crossed G-categories

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Abstract

We show that the author's notion of Galois extensions of braided tensor categories [Adv. Math. 150 (2000) 151], see also [A. Bruguières, Math. Ann. 316 (2000) 215], gives rise to braided crossed G-categories, recently introduced for the purposes of 3-manifold topology [V.G. Turaev, Homotopy field theory in dimension 3 and crossed group-categories, arXiv: math.GT/0005291]. The Galois extensions $\mathcal{C} \rtimes \mathcal{S}$ are studied in detail, in particular we determine for which $g \in G$ non-trivial objects of grade g exist in $\mathcal{C} \rtimes \mathcal{S}$.

1. Introduction

According to the influential paper [15], the notion of braided tensor categories (BTC for short) originated in

(I) considerations in higher-dimensional category theory (BTC as 3-categories with one object and one 1-morphism) and
(II) homotopy theory (braided categorical groups classifying connected homotopy types with only π_2, π_3 non-trivial).

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On the other hand, the

(III) representation categories of quasitriangular (quasi-, weak, etc.) Hopf algebras, cf., e.g., [16], and of
(IV) quantum field theories (QFT) in low-dimensional space times [10,11], in particular conformal field theories [10,23], are BTC.

Finally,

(V) the category of tangles is a BTC, which is the origin of various constructions of invariants of links and 3-manifolds [1,16,33].

It goes without saying that all five areas continue to be very active fields of research and the connections continue to be explored.

In this paper we are concerned with a recent generalization of the notion of BTC which is quite interesting in that can be approached from most of the above viewpoints. (V): In the context of his programme of homotopy TQFT, Turaev [33] introduced *braided G-crossed categories* and showed that, subject to some further conditions, they give rise to invariants of 3-dimensional G-manifolds, to wit 3-manifolds together with a principal G-bundle. Let us state the definition in its simplest form.

Definition 1.1. Let G be a (discrete) group. A strict crossed G-category is a strict tensor category $\mathcal{C}$ together with

- a map $\partial : \mathrm{Obj}\ \mathcal{C} \to G$ constant on isomorphism classes,
- a homomorphism $\gamma : G \to \mathrm{Aut}\mathcal{C}$ (strict monoidal automorphisms of $\mathcal{C}$)

such that

(1) $\partial(X \otimes Y) = \partial X \partial Y$;
(2) $\partial(\gamma_g(X)) = g\partial(X)g^{-1}$.

We write ${}^{Y}\cdot = \gamma_{\partial Y}(\cdot)$. A braiding for a crossed G-category $\mathcal{C}$ is a family of isomorphisms $c_{X,Y} : X \otimes Y \to {}^{X}Y \otimes X$ such that

$$c_{X,Z\otimes T} = \mathrm{id}_{XZ} \otimes c_{X,T} \circ c_{X,Z} \otimes \mathrm{id}_T, \qquad c_{X\otimes Y,Z} = c_{X,{}^{Y}Z} \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes c_{Y,Z},$$
$$c_{X',Y'} \circ s \otimes t = {}^{X}t \otimes s \circ c_{X,Y}, \quad \forall s : X \to X',\ t : Y \to Y'.$$

Of the various generalizations permitted by this definition we will need only the admission of inhomogeneous objects, cf. Section 3. As to subject (III): In [20] it was shown that some crossed G-categories can be obtained from quantum groups. With a view towards applications to algebraic topology (II), in [4] a notion of *categorical G-crossed module* was defined. The latter are simply crossed G-categories which are categorical groups, i.e., monoidal groupoids with invertible objects. In turn, categorical G-crossed modules

generalize Whitehead's ubiquitous notion of crossed modules and Conduché's 2-crossed modules.

The main result of the present paper is to show that braided crossed G-categories arise from a categorical construction, the Galois extensions of braided tensor categories [3,24]. This refers to the construction in [24] which associates to a braided tensor category $\mathcal{C}$ and a full symmetric subcategory $\mathcal{S}$ a tensor category $\mathcal{C} \rtimes \mathcal{S}$. The braiding of $\mathcal{C}$ lifts to a braiding of $\mathcal{C} \rtimes \mathcal{S}$ iff $\mathcal{S}$ is contained in the center $Z_2(\mathcal{C})$ of $\mathcal{C}$, the latter being the full subcategory of objects X satisfying $c_{X,Y} \circ c_{Y,X} = \mathrm{id}$ for all $Y \in \mathcal{C}$. (In [3], where a category equivalent to $\mathcal{C} \rtimes \mathcal{S}$ was defined, the objects of $Z_2(\mathcal{C})$ were called transparent.) Dropping the condition $\mathcal{S} \subset Z_2(\mathcal{C})$, we show in Theorem 3.20 that $\mathcal{C} \rtimes \mathcal{S}$ is a braided crossed G-category, where we also clarify for which $g \in G$ there exist $X \in \mathcal{C} \rtimes \mathcal{S}$ with $\partial X = g$, cf. Theorem 3.26. In the final Section 3.4 we show that a subcategory $\mathcal{S} \subset \mathcal{C}$ where $\mathcal{S} \cong \mathrm{Rep}\, G$ with G finite abelian induces a G-grading on $\mathcal{C}$ compatible with the one on $\mathcal{C} \rtimes \mathcal{S}$. Similar results are obtained in [19], in particular part II. However, our approach is quite different, more suitable for the application to quantum field theory [28] sketched below, and in places somewhat more satisfactory, e.g., concerning the braiding on $\mathcal{C} \rtimes \mathcal{S}$.

We close this introduction with a glance at the applications of this paper in quantum field theory and topology. In a companion paper [28] we will show, in the context of algebraic quantum field theory [14], that a chiral conformal field theory A carrying an action of a finite group G gives rise to a braided crossed G-category $G\text{-Loc}\, A$ of 'G-twisted representations.' The full subcategory $\partial^{-1}(e) \subset G\text{-Loc}\, A$ of grade zero objects is just the ordinary braided representation category $\mathrm{Rep}\, A$, which does not the G-action into account. In [28] we prove the equivalences

$$G\text{-Loc}\, A \simeq \mathrm{Rep}\, A^G \rtimes \mathcal{S}, \qquad \mathrm{Rep}\, A^G \simeq (G\text{-Loc}\, A)^G,$$

where A^G is the 'orbifold theory' [7], i.e., the subtheory of A consisting of the fixpoints under the G-action, and $\mathcal{S} \simeq \mathrm{Rep}\, G$ is a full subcategory of $\mathrm{Rep}\, A^G$. The significance of the first equivalence is that the same braided crossed G-category arises (i) as the—intrinsically defined—category of G-twisted representations of A and (ii) by the crossed product construction of [24] whose braided crossed G-structure is the subject of the present work. The second equivalence computes the representation category of the orbifold theory A^G in terms of $G\text{-Loc}\, A$, i.e., categorical information about A. To put this into context we emphasize the well-known fact that the grade zero subcategory $\mathrm{Rep}\, A \subset G\text{-Loc}\, A$ does *not* contain enough information to determine $\mathrm{Rep}\, A^G$.

Finally, by [17] the categories $\mathrm{Rep}\, A$ and $\mathrm{Rep}\, A^G$ are modular, and Corollary 3.27 implies that $G\text{-Loc}\, A \simeq \mathrm{Rep}\, A^G \rtimes \mathcal{S}$ has full G-spectrum, i.e., there exists an object of grade g for every $g \in G$. Combining this with Turaev's work [33,34] on invariants of (G-)manifolds, we thus obtain an equivariant version of the chain

$$\text{Rational chiral CFT} \rightsquigarrow \text{modular category} \rightsquigarrow \text{3-manifold invariant},$$

of constructions, namely

$$\begin{array}{lll} \text{Rational chiral CFT} & & \text{invariant for 3-manifolds equipped} \\ \text{with symmetry G} & \rightsquigarrow \text{modular crossed} & \rightsquigarrow \text{with principal G-bundle} \\ & \text{G-category} & \end{array}.$$

The above applications of the constructions of this paper place braided crossed G-categories squarely into the context of the areas (I) (higher category theory) and (V) (quantum field theory) mentioned above. Most results of this paper and of [28] were announced in [25].

2. Preliminaries

We briefly recall without proof the facts concerning Tannakian and module categories that will be needed later. Some of those are well-known, while others are relatively recent.

We assume as known the notions of abelian, monoidal (or tensor) braided, symmetric, rigid, and ribbon categories, cf., e.g., [1,15,16,22]. All categories considered in this paper will be $\mathbb{F}$-linear semisimple (thus in particular abelian) over an algebraically closed field $\mathbb{F}$ with finite-dimensional Hom-spaces and monoidal with $\mathrm{End}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$. Unless otherwise stated, tensor categories will be strict, as we are allowed to assume by virtue of the coherence theorems. A $\mathbb{C}$-linear tensor category is a $*$-category if there exists a $*$-operation, i.e., an involutive antilinear contravariant and monoidal endofunctor $*$ that acts trivially on the objects. In other words, $s^* \in \mathrm{Hom}(Y, X)$ if $s \in \mathrm{Hom}(X, Y)$, $s^{**} = s$ and, whenever these expressions are defined, $(s \circ t)^* = t^* \circ s^*$ and $(s \otimes t)^* = s^* \otimes t^*$. A $*$-operation is positive if $s^* \circ s = 0$ implies $s = 0$. A category with positive $*$-operation is called $*$-category [8,13,21] or unitary [33], cf. also [37]. (Since we assume finite-dimensional hom-spaces, a $*$-category in fact is a C^*- and W^*-category in the sense of [8,13], cf. e.g. [24, Proposition 2.1].)

The category of finite-dimensional polynomial representations of a reductive proalgebraic group (in characteristic zero) is a rigid abelian symmetric tensor category with $\mathrm{End}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$. The category of finite-dimensional continuous representations of a compact topological group has the same properties and is in addition a $*$-category. There are converses to these statements due to Doplicher and Roberts [8] and to Deligne [5], respectively. For our purposes in this paper it is sufficient to consider symmetric categories with finitely many (isomorphism classes of) simple objects, corresponding to finite groups.

Definition 2.1.

(1) A TC is a semisimple $\mathbb{F}$-linear spherical tensor category [2] with finite-dimensional Hom-spaces and $\mathrm{End}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$, where $\mathbb{F}$ is an algebraically closed field. It is called finite if the set of isomorphism classes of simple objects is finite. The dimension of a finite TC is given by $\dim \mathcal{C} = \sum_i d(X_i)^2$, where i runs through the set of isomorphism classes of simple objects and d is the dimension function defined by the spherical structure.
(2) A BTC is a semisimple $\mathbb{F}$-linear rigid braided ribbon category with finite-dimensional Hom-spaces and $\mathrm{End}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$, and is automatically a TC.
(3) An STC is a symmetric BTC.
(4) An STC over $\mathbb{F}$ is admissible if either
   (i) $\mathbb{F} = \mathbb{C}$, $\mathcal{C}$ is a $*$-category, and all objects have trivial twist $\Theta(X)$, or
   (ii) $\mathbb{F}$ has characteristic zero and $d(X) \in \mathbb{Z}_+$ for all $X \in \mathcal{C}$.

Remark 2.2.

(Ad 1) Since we work over algebraically closed fields throughout, an object X is simple (every non-zero subobject is isomorphic to X) iff it is absolutely simple ($\mathrm{End}\, X = \mathbb{F}\mathrm{id}_X$). We will therefore just speak of simple objects.
By dropping the assumption of sphericity, one arrives at the notion of fusion categories which were studied in [9]. There are remarkably strong results like the automatic positivity of $\dim\mathcal{C}$ when $\mathbb{F} = \mathbb{C}$. (Yamagami has shown [37] that a $*$-structure gives rise to an essentially unique spherical structure, and one might suspect that this generalizes to fusion categories.)

(Ad 2) A rigid ribbon category gives rise to a spherical structure and conversely in a spherical braided category $\mathcal{C}$ there exists a canonical twist Θ rendering $\mathcal{C}$ a ribbon category. See [2,38].

(Ad 3) At first sight, the supplementary conditions (i) and (ii) on the twists and the dimensions, respectively, look quite different. This is due to the different notions of duality in both formalisms, but ultimately both conditions amount to the same thing. Let $X \in \mathcal{C}$. In [8] one chooses $r_X : \mathbf{1} \to \overline{X} \otimes X$, $\bar{r}_X : \mathbf{1} \to X \otimes \overline{X}$ such that

$$\mathrm{id}_{\overline{X}} \otimes \bar{r}_X^* \circ r_X \otimes \mathrm{id}_{\overline{X}} = \mathrm{id}_{\overline{X}}, \qquad \mathrm{id}_X \otimes r_X^* \circ \bar{r}_X \otimes \mathrm{id}_X = \mathrm{id}_X,$$
$$r_X^* \circ r_X = \bar{r}_X^* \circ \bar{r}_X = d(X)\mathrm{id}_{\mathbf{1}}.$$

One then defines the twist $\Theta(X) \in \mathrm{End}\, X$ by

$$\Theta(X) = r_X^* \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes c_{X,X} \circ r_X \otimes \mathrm{id}_X.$$

For simple X, one finds $\Theta(X) = \pm\mathrm{id}_X$, whereas $d(X) \geqslant 0$ is automatic by positivity of the $*$-operation. In fact, one proves $d(X) \in \mathbb{Z}_+$, and the condition $\Theta(X) = \mathrm{id}_X$ is necessary and sufficient for $\mathcal{C} \simeq \mathrm{Rep}\, G$ for some G.
On the other hand, in [5] one has morphisms $d_X : \mathbf{1} \to X \otimes \overline{X}$, $e_X : \overline{X} \otimes X \to \mathbf{1}$, which are part of the given data and satisfy the usual triangular equations. One then defines

$$\delta_X \otimes \mathrm{id}_X = c_{X,\overline{X}} \circ d_X : \mathbf{1} \to \overline{X} \otimes X, \qquad \eta_X = e_X \circ c_{X,\overline{X}} : X \otimes \overline{X} \to \mathbf{1}.$$

With this definition the twist $\Theta(X) = e_X \otimes \mathrm{id}_X \circ \mathrm{id}_{\overline{X}} \otimes c_{X,X} \circ \delta_X$ is automatically trivial, but $d(X) = \eta_X \circ d_X = e_X \circ \delta_X$ is not necessarily positive. In any case, for a $*$-category one has both notions of duals, and the supplementary conditions are equivalent.

Theorem 2.3 [5,8]. *Let $\mathcal{C}$ be a finite admissible STC over $\mathbb{F}$. Then there exists a finite group G, unique up to isomorphism, such that there is an equivalence $\mathcal{C} \simeq \mathrm{Rep}^{\mathrm{fin}}_{\mathbb{F}} G$ compatible with all structures in sight.*

Remark 2.4.

(1) The proof of [5, Theorem 2.3] roughly consists of two steps: (i) One constructs a faithful tensor functor $E : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}^{\mathrm{fin}}$. Defining $G = \mathrm{Nat}^{\otimes} E$, the set of monoidal natural transformations from E to itself, one finds (a) G is a group, cf. e.g. [15, Proposition 7.1], by virtue of rigidity of $\mathcal{C}$, and (b) $\mathcal{C} \simeq \mathrm{Rep}_{\mathbb{F}} G$.
(2) If $\mathcal{C}$ has objects with non-trivial twists or integral but non-positive dimensions, respectively, it still is the representation category of a supergroup, i.e., a pair (G, k) where G is a group and $k \in Z(G)$ is involutive, cf. [8, Section 7], see also [6]. This generalization will not be used in this paper.

Definition 2.5. Let $\mathcal{C}$ be a strict tensor category. A Frobenius algebra in $\mathcal{C}$ is a quintuple $(\Gamma, m, \eta, \Delta, \varepsilon)$ such that $(\Gamma \in \mathcal{C}, m : \Gamma^2 \to \Gamma, \eta : \mathbf{1} \to \Gamma)$ is a monoid, $(\Gamma, \Delta : \Gamma \to \Gamma^2, \varepsilon : \Gamma \to \mathbf{1})$ is a comonoid and the condition

$$\mathrm{id}_\Gamma \otimes m \circ \Delta \otimes \mathrm{id}_\Gamma = \Delta \circ m = \Delta \otimes \mathrm{id}_\Gamma \circ \mathrm{id}_\Gamma \otimes m$$

holds. A Frobenius algebra in an $\mathbb{F}$-linear category is called strongly separable [26] if

$$m \circ \Delta = \alpha\, \mathrm{id}_\Gamma, \qquad \varepsilon \circ \eta = \beta\, \mathrm{id}_{\mathbf{1}}, \quad \alpha, \beta \in \mathbb{F}^*.$$

Remark 2.6. Following earlier terminology used by the author, which in turn was inspired by F. Quinn, strongly separable Frobenius algebras were called 'special' in [12].

Proposition 2.7 [26]. *Let G be a finite group and $\mathbb{F}$ an algebraically closed field whose characteristic does not divide $|G|$. There exists a strongly separable Frobenius algebra $(\Gamma, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C} = \mathrm{Rep}_{\mathbb{F}} G$ such that*

(1) *$\alpha\beta = |G|$. We normalize such that $\beta = 1$.*
(2) *Γ is (isomorphic to) the left regular representation of G.*
(3) *$\Gamma \otimes X \cong d(X)\Gamma\ \forall X$.*
(4) *$\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \Gamma) = 1$.*

If $\mathbb{F} = \mathbb{C}$, $\mathrm{Rep}\, G$ is a $*$-category and one can achieve $\Delta = m^*$, $\varepsilon = \eta^*$.

Remark 2.8.

(1) See also [3] where a similar, but less symmetric, statement appears.
(2) The proposition generalizes to finite-dimensional Hopf algebras H, where the categorical Frobenius algebra in H-Mod is strongly separable iff H is semisimple and cosemisimple, cf. [26].
(3) Some of the structure survives for infinite compact groups and discrete quantum groups, cf. [29].

Remark 2.9. Given the monoid part of the above Frobenius algebra, one can obtain a fiber functor $E : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ as follows:

$$E(X) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \Gamma \otimes X), \qquad E(s)\phi = s \otimes \mathrm{id}_X \circ \phi, \quad s : X \to Y,\ \phi \in E(X).$$

The natural isomorphisms $d_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y)$ are given by

$$d_{X,Y}(\phi \boxtimes \psi) = m \otimes \mathrm{id}_X \otimes \mathrm{id}_Y \circ \mathrm{id}_\Gamma \otimes \phi \otimes \mathrm{id}_Y \circ \psi, \quad \phi \in E(X),\ \psi \in E(Y).$$

(Similarly, one can use the comonoid structure.) For the details, which are an immediate generalization of [5], see [29]. A similar construction is given in [35, Appendix C]. Defining

$$\mathrm{Aut}(\Gamma, m, \eta) \equiv \{g \in \mathrm{End}\,\Gamma \mid g \circ m = m \circ g \otimes g,\ g \circ \eta = \eta\}$$

it is easy to see that

$$g \mapsto (g_X), \quad g_X(\phi) = g \otimes \mathrm{id}_X \circ \phi, \quad X \in \mathcal{C},\ \phi \in E(X)$$

defines a homomorphism $\mathrm{Aut}(\Gamma, m, \eta) \to \mathrm{Nat}^{\otimes} E = G$. Appealing to the Yoneda lemma, one verifies that this is a bijection, implying that $\mathrm{Aut}(\Gamma, m, \eta)$ is a group. This allows to recover G from the monoid structure on the regular representation without reference to the fiber functor arising from the latter. This will turn out very useful in the sequel.

Remark 2.10. In fact, in [29] a proof of Theorem 2.3 will be given, whose first step is to construct from a category $\mathcal{C}$ (not necessarily finite) a monoid (Γ, m, η) (in $\mathrm{Ind}\mathcal{C}$ if $\mathcal{C}$ is infinite) such that $\Gamma \otimes X \cong d(X)\Gamma$ and $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$. One then obtains G simply as the automorphism group of the monoid as above, the monoid of course turning out to be the regular monoid of G. (This goes beyond the proof in [5] that used a monoid not satisfying the latter condition. This monoid is not the regular representation and gives rise to a fiber functor into $\mathrm{Vect}_{\mathbb{F}}$ only after a quotient operation. Thus one cannot define G as the automorphism group of the monoid.)

Even though the only monoids and Frobenius algebras considered in this paper are those arising from regular representations as in Proposition 2.7, it is natural to give the following considerations in larger generality.

Definition/Proposition 2.11 [31]. *Let $\mathcal{C}$ be a strict tensor category and let (Γ, m, η) be a monoid in $\mathcal{C}$. A Γ-module in $\mathcal{C}$ is a pair (X, μ) where $X \in \mathcal{C}$ and $\mu : \Gamma \otimes X \to X$ satisfies*

$$\mu \circ \mathrm{id}_\Gamma \otimes \mu = \mu \circ m \otimes \mathrm{id}_X, \qquad \mu \circ \eta \otimes \mathrm{id}_X = \mathrm{id}_X.$$

The modules form a category Γ-$\mathrm{Mod}_{\mathcal{C}}$ where $\mathrm{Hom}_{\Gamma\text{-Mod}}((X, \mu), (Y, \lambda)) = \{s : X \to Y \mid s \circ \mu = \lambda \circ \mathrm{id}_\Gamma \otimes s\}$. If $\mathcal{C}$ is braided and has coequalizers, $\otimes$ preserves coequalizers, and (Γ, m, η) is commutative then Γ-Mod is a tensor category with $(X, \mu) \otimes (Y, \eta) = \mathrm{coeq}(\alpha, \beta)$, where $\alpha, \beta : \Gamma \otimes X \otimes Y \to X \otimes Y$ are given by

$$\alpha = \mu \otimes \mathrm{id}_Y, \qquad \beta = \mathrm{id}_X \otimes \eta \circ c_{\Gamma,X} \otimes \mathrm{id}_Y.$$

The full subcategory $\Gamma\text{-Mod}_{\mathcal{C}}^{0} \subset \Gamma\text{-Mod}_{\mathcal{C}}$ *consisting of the objects* (X,μ) *satisfying* $\mu \circ c_{X,\Gamma} \circ c_{\Gamma,X} = \mu$ *is monoidal and braided.*

Remark 2.12.

(1) The above definition and facts are due to Pareigis [31] and were rediscovered in [18]. The special case where $\Gamma \in Z_2(\mathcal{C})$, implying $\Gamma\text{-Mod}_{\mathcal{C}}^{0} = \Gamma\text{-Mod}_{\mathcal{C}}$, was considered in [3].
(2) Note that the coequalizers are unique only up to isomorphism, thus some care is required in the definition of the associativity constraint of $\Gamma\text{-Mod}_{\mathcal{C}}$. In [31] this is handled by showing that $\Gamma\text{-Mod}_{\mathcal{C}}$ is equivalent (as a category) to a full subcategory of the category of M-M bimodules in $\mathcal{C}$. For the latter the associativity constraint had been constructed in [30].
(3) We will exclusively consider semisimple categories with duals. In such categories, coequalizers exist and are preserved by $\otimes$.

Recall that the dimension of a finite TC is the sum over the squared dimensions of its simple objects, cf. e.g. [2,26].

Proposition 2.13. *Let* $\mathcal{C}$ *be a finite BTC and let* $(\Gamma, m, \eta, \Delta, \varepsilon)$ *be a strongly separable Frobenius algebra in* $\mathcal{C}$ *satisfying* $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$. *Then* $\Gamma\text{-Mod}_{\mathcal{C}}$ *is a semisimple* $\mathbb{F}$*-linear spherical tensor category with* $\mathrm{End}_{\Gamma}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$, *and*

$$\dim \Gamma\text{-Mod}_{\mathcal{C}} = (\dim \Gamma)^{-1} \dim \mathcal{C}.$$

Proof. The free module functor $F : \mathcal{C} \to \Gamma\text{-Mod}_{\mathcal{C}}$, $X \mapsto (\Gamma \otimes X, m \otimes \mathrm{id}_X)$ is a left adjoint of the forgetful functor $G : \Gamma\text{-Mod}_{\mathcal{C}} \to \mathcal{C}$, $(X,\mu) \mapsto X$, cf. [3,18]. F is monoidal, implying $F(\mathbf{1}) \cong \mathbf{1}$ and $d(F(X)) = d(X)$. The tensor unit of $\Gamma\text{-Mod}_{\mathcal{C}}$ being (Γ, m) we have $\mathrm{End}_{\Gamma}\,\mathbf{1} = \mathrm{Hom}_{\Gamma}(F(\mathbf{1}), (\Gamma, m)) \cong \mathrm{Hom}(\mathbf{1}, \Gamma)$, implying $\mathrm{End}_{\Gamma}\,\mathbf{1} = \mathbb{F}\mathrm{id}_{\mathbf{1}}$. As a rigid ribbon category, $\mathcal{C}$ is spherical and so is $\Gamma\text{-Mod}_{\mathcal{C}}$ [26], allowing us to talk of dimensions of objects irrespective of whether $\Gamma\text{-Mod}_{\mathcal{C}}$ is braided. Semisimplicity is proven as in [3,18]; it is here that the Frobenius structure is used, cf. also [26]. The fact $GF(X) = \Gamma \otimes X$ together with $d(F(X)) = d(X)$ and additivity of F and G implies $d(G(Y)) = d(\Gamma)d(Y)$. Let now $\{X_i \in \mathcal{C}\}$ and $\{Y_j \in \Gamma\text{-Mod}_{\mathcal{C}}\}$ be complete sets of simple objects in $\mathcal{C}$ and $\Gamma\text{-Mod}_{\mathcal{C}}$, respectively. The computation

$$\begin{aligned}
\dim \mathcal{C} &= \sum_i d(X_i)^2 = \sum_i d(X_i)\, d\big(F(X_i)\big) = \sum_i \sum_j d(X_i)\, d(Y_j) \dim \mathrm{Hom}\big(F(X_i), Y_j\big) \\
&= \sum_i \sum_j d(X_i)\, d(Y_j) \dim \mathrm{Hom}\big(X_i, G(Y_j)\big) = \sum_j d(Y_j)\, d\big(G(Y_j)\big) \\
&= d(\Gamma) \sum_j d(Y_j)^2 = d(\Gamma) \dim \Gamma\text{-Mod}_{\mathcal{C}}
\end{aligned}$$

completes the proof. □

Remark 2.14. A similar result is proven in [3] where $\Gamma \in Z_2(\mathcal{C})$, implying Γ-Mod to be braided, is assumed. The present very simple proof shows that such an assumption is not needed.

While the category Γ-$\mathrm{Mod}_{\mathcal{C}}$ considered above is conceptually very natural, there is an alternative description which occasionally is more convenient. The point is that the tensor product of Γ-$\mathrm{Mod}_{\mathcal{C}}$, while entirely analogous to that in R-Mod, is not very convenient to work with.

Definition/Proposition 2.15. *Let $\mathcal{C}$ be a strict BTC and $(\Gamma, m, \eta, \Delta, \varepsilon)$ a strongly separable Frobenius algebra in $\mathcal{C}$. Then the following defines a tensor category $\widetilde{\mathcal{C}}_\Gamma$:*

- $\mathrm{Obj}\,\widetilde{\mathcal{C}}_\Gamma = \mathrm{Obj}\,\mathcal{C}$.
- $X \mathbin{\widetilde{\otimes}} Y = X \otimes Y$.
- $\mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y)$.
- *Let $s \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y)$ and $t \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(Y, Z) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes Y, Z)$. Then $t \mathbin{\tilde{\circ}} s = t \circ \mathrm{id}_\Gamma \otimes s \circ \Delta \otimes \mathrm{id}_X$ in $\mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X, Z) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Z)$.*
- *Let $s \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y)$ and $t \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(Z, T) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes Z, T)$. Then $s \mathbin{\widetilde{\otimes}} t = s \otimes t \circ \mathrm{id}_\Gamma \otimes c_{\Gamma,X} \otimes \mathrm{id}_Z \circ \Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Z$ in $\mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X \otimes Z, Y \otimes T) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X \otimes Z, Y \otimes T)$.*

The canonical completion $\widehat{\mathcal{C}}_\Gamma = \widetilde{\mathcal{C}}_\Gamma^p$ of $\widetilde{\mathcal{C}}_\Gamma$ to a category with splitting idempotents is semisimple. (Recall that $\mathrm{Obj}\,\widehat{\mathcal{C}}_\Gamma = \{(X, p),\ X \in \mathrm{Obj}\,\widetilde{\mathcal{C}}_\Gamma,\ p = p^2 \in \mathrm{End}_{\widetilde{\mathcal{C}}_\Gamma} X\}$ *etc. Instead of $(X, \mathrm{id}_X) \in \widehat{\mathcal{C}}_\Gamma$ we simply write X.) If $\mathcal{C}$ is a $*$-category and $\Delta = m^*$, $\varepsilon = \eta^*$, then $\widetilde{\mathcal{C}}_\Gamma, \widehat{\mathcal{C}}_\Gamma$ are $*$-categories. The functor $\iota : \mathcal{C} \to \widetilde{\mathcal{C}}_\Gamma$ given by $X \mapsto X, s \mapsto \varepsilon \otimes s$ is monoidal and faithful. The composite of ι with the full embedding $\widetilde{\mathcal{C}}_\Gamma \to \widehat{\mathcal{C}}_\Gamma$ is also denoted by ι.*

Proof. That $\widetilde{\mathcal{C}}_\Gamma$ and therefore $\widehat{\mathcal{C}}_\Gamma$ is a $\mathbb{F}$-linear strict tensor category is almost obvious: One only needs to show associativity of $\tilde{\circ}$, $\widetilde{\otimes}$ on the morphisms and the interchange law, which is left to the reader. The discussion of the $*$-operation on $\widetilde{\mathcal{C}}_\Gamma, \widehat{\mathcal{C}}_\Gamma$ and of semisimplicity of $\widehat{\mathcal{C}}_\Gamma$ is the same as in [24,26], to which we refer for details. □

Proposition 2.16. *Let $\mathcal{C}$ and $(\Gamma, m, \eta, \Delta, \varepsilon)$ be as before. Then there exists a monoidal equivalence $K : \widehat{\mathcal{C}}_\Gamma \to \Gamma$-$\mathrm{Mod}_{\mathcal{C}}$ such that $K \circ \iota \cong F$ as tensor functors.*

Proof. We define $K_0 : \widetilde{\mathcal{C}}_\Gamma \to \Gamma$-$\mathrm{Mod}_{\mathcal{C}}$ by $K_0(X) = F(X)$, and for $s \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_\Gamma}(X, Y) = \mathrm{Hom}(\Gamma \otimes X, Y)$ we put $K_0(s) = \mathrm{id}_\Gamma \otimes s \circ \Delta \otimes \mathrm{id}_X \in K_0(s) \in \mathrm{Hom}_\Gamma(F(X), F(Y))$. The map $s \mapsto K_0(s)$ has inverse $t \mapsto \varepsilon \otimes \mathrm{id}_Y \circ t$. Direct computations show $K_0(s) \bullet K_0(t) = K_0(s \bullet t)$ for $\bullet \in \{\circ, \otimes\}$, thus K is a full and faithful tensor functor and satisfies $K_0 \circ \iota = F$. Since Γ-Mod has splitting idempotents, $K_0 : \widetilde{\mathcal{C}}_\Gamma \to \Gamma$-Mod extends to $K : \widehat{\mathcal{C}}_\Gamma \to \Gamma$-Mod, uniquely up to natural isomorphism. Since every object of Γ-Mod is a retract of an object of the form $K_0(X) = F(X)$, K is essentially surjective, thus an equivalence. □

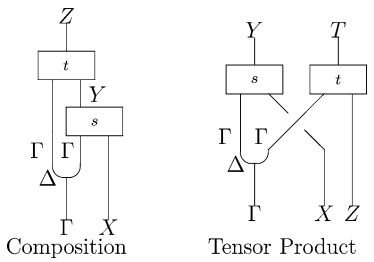


Fig. 1. Composition and tensor product of arrows in $\widetilde{\mathcal{C}}_\Gamma$.

3. Braided crossed G-category from Galois extensions

3.1. Definition of $\mathcal{C} \rtimes \mathcal{S}$ and basic properties

In the rest of the paper we assume $\mathbb{F}$ to satisfy the assumptions of Theorem 2.3 and work exclusively with the category $\widehat{\mathcal{C}}_\Gamma$. Furthermore, $\mathcal{C}$ will be a BTC, not necessarily finite, and $\mathcal{S} \subset \mathcal{C}$ will be a finite admissible full sub-STC.

Lemma 3.1. *Let G be such that $\mathcal{S} \simeq \operatorname{Rep} G$ and let $(\Gamma, \ldots)$ be the corresponding commutative Frobenius algebra in $\mathcal{C}$. We write $p_0 = \eta \circ \varepsilon \in \operatorname{End}\Gamma$ and recall that $G = \operatorname{Aut}(\Gamma, m, \eta)$. For $s \in \operatorname{Hom}(\Gamma \otimes X, Y)$ the following are equivalent*:

(i) $s \circ g \otimes \mathrm{id}_X = s$ *for all* $g \in G$.
(ii) $s \circ p_0 \otimes \mathrm{id}_X = s$.

Proof. (ii) $\Rightarrow$ (i). Obvious consequence of $\varepsilon \circ g = \varepsilon \ \forall g \in G$.

(i) $\Rightarrow$ (ii). If $\widehat{G}$ denotes the set of iso-classes of irreps π_i of G and d_i is the dimension of π_i, we have $\operatorname{End}\Gamma \cong \bigoplus_{i\in\widehat{G}} M_{d_i}(\mathbb{F})$ and $G \ni g = \bigoplus_{i\in\widehat{G}} \pi_i(g)$. Whenever $\widehat{G} \ni i \neq 0$ there exists $g \in G$ such that $\pi_i(g) \neq \mathrm{id}$. If p_i is the unit of $M_{d_i}(\mathbb{F})$, (i) therefore implies $s \circ p_i \otimes \mathrm{id}_X = 0 \ \forall i \neq 0$, and we conclude $s = \sum_i s \circ p_i \otimes \mathrm{id}_X = s \circ p_0 \otimes \mathrm{id}_X$. □

Definition 3.2. Let $\mathcal{C}$ be a strict BTC and $\mathcal{S} \subset \mathcal{C}$ a finite full sub-STC. Let $(\Gamma, \ldots)$ be the Frobenius algebra in $\mathcal{C}$ arising from Theorem 2.3 and Proposition 2.7. Then we write $\mathcal{C} \rtimes_0 \mathcal{S} := \widetilde{\mathcal{C}}_\Gamma$ and $\mathcal{C} \rtimes \mathcal{S} := \widehat{\mathcal{C}}_\Gamma$.

For the sake of legibility, we will continue to write $\widetilde{\mathcal{C}}, \widehat{\mathcal{C}}$ rather than $\mathcal{C} \rtimes_0 \mathcal{S}$, $\mathcal{C} \rtimes \mathcal{S}$ in many places, in particular subscripts.

Proposition 3.3. *$\mathcal{C} \rtimes_0 \mathcal{S}$ and $\mathcal{C} \rtimes \mathcal{S}$ are strict spherical tensor categories and $\mathcal{C} \rtimes \mathcal{S}$ is semisimple. If $\mathcal{C}$ is a $*$-category, then $\mathcal{C} \rtimes_0 \mathcal{S}$ and $\mathcal{C} \rtimes \mathcal{S}$ have a $*$-structure extending that of $\mathcal{C}$. There exists a canonical tensor functor $\iota : \mathcal{C} \to \mathcal{C} \rtimes \mathcal{S}$ which is faithful and injective on the objects, thus an inclusion. The group $G = \operatorname{Aut}(\Gamma, m, \eta)$ acts on $\mathcal{C} \rtimes \mathcal{S}$ via $\gamma_g(s) = s \circ g^{-1} \otimes \mathrm{id}_X$ for $s \in \operatorname{Hom}_{\mathcal{C}\rtimes\mathcal{S}}(X, Y) = \operatorname{Hom}(\Gamma \otimes X, Y)$ and $\gamma_g((X, p)) = (X, \gamma_g(p))$. We have $(\mathcal{C} \rtimes_0 \mathcal{S})^G \cong \mathcal{C}$ and $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$. If $\mathcal{C}$ is finite, then $\dim \mathcal{C} \rtimes \mathcal{S} = \dim \mathcal{C}/|G| = \dim \mathcal{C}/\dim \mathcal{S}$.*

Proof. The first set of statements is obvious. Clearly, $g \mapsto \gamma_g$ is a homomorphism and γ_g is invertible. Now $\gamma_g(s \bullet t) = \gamma_g(s) \bullet \gamma_g(t)$ for $\bullet \in \{\circ, \otimes\}$ follows from $\Delta \circ g = g \otimes g \circ \Delta$. Lemma 3.1 amounts to $(\mathcal{C} \times_0 \mathcal{S})^G = \iota(\mathcal{C}) \cong \mathcal{C}$, and $(\mathcal{C} \rtimes \mathcal{S})^G = \overline{\iota(\mathcal{C})}^p \cong \mathcal{C}^p \simeq \mathcal{C}$. The dimension formula follows from Propositions 2.13 and 2.16. □

Remark 3.4.

(1) Here and in the sequel, $\mathcal{D}^G \subset \mathcal{D}$ denotes the subcategory consisting of the objects and morphisms that are strictly fixed by the action of G. In our strict context this is the right notion, but it presumably needs to be generalized if one works with a less strict notion of G-categories.
(2) For definition of $\mathcal{C} \rtimes \mathcal{S}$ given above for finite $\mathcal{S}$ is equivalent to the one in [24]. Thus Proposition 2.16 proves the equivalence of the approaches to Galois extensions and modularization of braided tensor categories given by the author [24] and A. Bruguières [3]. While both definitions are equally involved, Γ-$\mathrm{Mod}_{\mathcal{C}}$ may be more natural, yet $\widehat{\mathcal{C}_\Gamma}$ has some advantages. On the one hand, the tensor product of $\widehat{\mathcal{C}_\Gamma}$ is canonical, i.e., involving no choices, and strict, making it slightly more convenient to work with. On the other hand, the relationship between the categorical constructions and (algebraic) quantum field theory, cf. the next section, is very easy to establish for $\mathcal{C} \rtimes \mathcal{S}$.
(3) When $\mathcal{S}$ is infinite, the definition of $\mathcal{C} \rtimes \mathcal{S}$ must be changed. While there still is a monoid structure on the regular representation Γ [29], the latter lives in a larger category $\mathrm{Ind}\,\mathcal{S}$ and is no more a Frobenius algebra. Thus the proof of semisimplicity also changes. The somewhat pedestrian definition of $\mathcal{C} \rtimes \mathcal{S}$ in [24] works also for infinite $\mathcal{S}$.
(4) Constructions similar to the one above have been given in [3,12,35,36].

The following is due to Bruguières [3], who proved it for the category of (Γ, m, η)-modules.

Theorem 3.5. *Let $\mathcal{S} \subset \mathcal{C}$ be as before. The tensor functor $\iota : \mathcal{C} \to \mathcal{C} \rtimes \mathcal{S}$ has the following universal property*:

(1) *ι is faithful and for every simple object $Y \in \mathcal{C} \rtimes \mathcal{S}$ there exists $X \in \mathcal{C}$ such that Y is a direct summand of $Y \prec \iota(X)$.*
(2) *For every $X \in \mathcal{S}$ we have $\iota(X) \cong d(X)\mathbf{1}$ in $\mathcal{C} \rtimes \mathcal{S}$.*
(3) *If $\mathcal{D}$ is semisimple and $\iota' : \mathcal{C} \to \mathcal{D}$ satisfies* (1)–(2)*, then there exists a faithful tensor functor $\iota'' : \mathcal{C} \rtimes \mathcal{S} \to \mathcal{D}$, unique up to monoidal natural isomorphism, such that $\iota' = \iota'' \circ \iota$.*

Proof. (1) Obvious by construction.

(2) It is sufficient to show this for $X \in \mathcal{S}$ simple. We have $\mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathbf{1}, \iota(X)) = \mathrm{Hom}_{\mathcal{C}}(\Gamma, X)$, and $\Gamma \cong \bigoplus_i d(X_i)X_i$ implies $\dim \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathbf{1}, \iota(X)) = d(X)$. Thus $\iota(X) \cong d(X)\mathbf{1} \oplus X'$ and $\mathrm{End}\,\iota(X) \cong M_{d(X)} \oplus N$. Now,

$$\dim \mathrm{End}_{\widehat{\mathcal{C}}}\iota(X) = \dim \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, X) = \dim \mathrm{Hom}_{\mathcal{C}}\big(d(X)\Gamma, X\big)$$
$$= d(X)\dim \mathrm{Hom}_{\mathcal{C}}(\Gamma, X) = d(X)^2,$$

thus $N = 0$ and $\iota(X) \cong d(X)\mathbf{1}$.

(3) This follows from the corresponding statement in [3] and Proposition 2.16. (We omit the direct proof for reasons of space.) □

The considerations in the remainder of this section concern the decomposition of $\iota(X) \in \mathcal{C} \rtimes \mathcal{S}$ for simple $X \in \mathcal{C}$, complementing the results in [24, Section 4.1], and will not be used in the rest of the paper.

Definition 3.6. For $X, Y \in \mathcal{C}$ we write $X \sim Y$ iff $\mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y) \neq \{0\}$.

Theorem 3.7. *Restricted to simple objects, the relation $\sim$ is an equivalence relation. Let $X, Y \in \mathcal{C}$ be simple. If $X \nsim Y$, then $\iota(X), \iota(Y)$ are disjoint, to wit $\iota(X), \iota(Y)$ have no isomorphic subobjects. For every equivalence class σ there exist a finite set $\mathcal{I}_\sigma$, mutually non-isomorphic simple objects $Z_i \in \mathcal{C} \rtimes \mathcal{S}, i \in \mathcal{I}_\sigma$, and natural numbers N_X, $X \in \sigma$ such that*

$$\iota(X) \cong N_X \bigoplus_{i \in \mathcal{I}_\sigma} Z_i \quad \forall X \in \sigma.$$

Proof. For all X, Y we have $X \sim X$ (since $\mathbf{1} \prec \Gamma$) and $X \sim Y \Leftrightarrow Y \sim X$ (since $\overline{\Gamma} \cong \Gamma$). Let X, Y, Z be simple and $X \sim Y \sim Z$. $\mathrm{Hom}(\Gamma \otimes X, Y) \neq \{0\}$ implies $Y \prec \Gamma \otimes X$, i.e., Y is a direct summand of $\Gamma \otimes X$. Similarly, $X \prec \Gamma \otimes Y$, $Y \prec \Gamma \otimes Z$, $Z \prec \Gamma \otimes Y$. Thus $X \prec \Gamma \otimes Y \prec \Gamma \otimes \Gamma \otimes Z \cong |G|\Gamma \otimes Z$, where we used $\Gamma^2 \cong |G|\Gamma$. Therefore $\mathrm{Hom}(X, \Gamma \otimes Z) \neq \{0\}$, thus $X \sim Z$, and $\sim$ is transitive. In view of $\mathrm{Hom}(\Gamma \otimes X, Y) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(\iota(X), \iota(Y))$ it is clear that $X \nsim Y$ implies disjointness.

Let $X, Y \in \mathcal{C}$ be simple such that $X \sim Y$ and let $Z_1 \prec \iota(X)$ be simple. Together with $\iota(X) \prec \iota(\Gamma) \otimes \iota(Y)$ this implies $Z_1 \prec \iota(\Gamma) \otimes \iota(Y) \prec |G|\iota(Y)$, where we used $\iota(\Gamma) \cong |G|\mathbf{1}$. Since Z_1 is simple, we have $Z_1 \prec \iota(Y)$. Thus every simple $Z_1 \in \mathcal{C} \rtimes \mathcal{S}$ contained in $\iota(X)$ is also contained in $\iota(Y)$, provided $X \sim Y$. We conclude that $X \sim Y$ implies that $\iota(X)$ and $\iota(Y)$ contain the same simple summands. The rest follows from the fact [24, Proposition 4.2] that, for every simple $X \in \mathcal{C}$, the simple summands of $\iota(X) \in \mathcal{C} \rtimes \mathcal{S}$ appear with the same multiplicity N_X. □

Remark 3.8. If G is abelian, corresponding to all simple objects in $\mathcal{S}$ being invertible, we have $X \sim Y$ iff there exists $Z \in \mathcal{S}$ such that $X \cong Z \otimes Y$. As a consequence, $X \sim Y$ implies $\iota(X) \cong \iota(Y)$ and $N_X = N_Y$. Since in the general case $X \sim Y$ does not imply that X, Y have the same dimensions, the above result, according to which $\iota(X), \iota(Y)$ have the same simple summands, clearly is the best one can hope for.

In the abelian case, the structure of $\mathrm{End}_{\widehat{\mathcal{C}}}\iota(X)$ can be clarified quite explicitly, cf. [24, Section 5.1]. Presently there is no analogous result in the general, non-abelian case.

3.2. $\mathcal{C} \rtimes \mathcal{S}$ as braided crossed G-category

Let c be the braiding of $\mathcal{C}$. For $X, Y \in \mathcal{C}$ it is clear that $\iota(c_{X,Y})$ is an isomorphism $\iota(X) \otimes \iota(Y) \to \iota(Y) \otimes \iota(X)$ satisfying the braid equations. Whether this gives rise to a braiding of $\mathcal{C} \rtimes_0 \mathcal{S}$ (and therefore of $\mathcal{C} \rtimes \mathcal{S}$) depends on whether or not $\iota(c)$ is natural w.r.t. the larger hom-sets of $\mathcal{C} \rtimes_0 \mathcal{S}$. For one variable we in fact have the following lemma.

Lemma 3.9. *Let $X, Y, Z \in \mathcal{C}$ and $s \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y)$. Then*

$$\iota(c_{Y,Z}) \mathbin{\hat{\circ}} s \mathbin{\widehat{\otimes}} \mathrm{id}_Z = \mathrm{id}_Z \mathbin{\widehat{\otimes}} s \mathbin{\hat{\circ}} \iota(c_{X,Z})$$

holds in $\mathcal{C} \rtimes_0 \mathcal{S}$.

Proof. In view of Definition 2.15, the two sides of the desired equation are represented by the following morphisms in $\mathcal{C}$:

$\mathrm{Hom}_{\widehat{\mathcal{C}}}(X \otimes Z,\ Z \otimes Y)$	$\mathrm{Hom}(\Gamma \otimes X \otimes Z,\ Z \otimes Y)$
$\iota(c_{Y,Z}) \mathbin{\hat{\circ}} s \mathbin{\widehat{\otimes}} \mathrm{id}_Z$	$c_{Y,Z} \circ s \otimes \mathrm{id}_Z$
$\mathrm{id}_Z \mathbin{\widehat{\otimes}} s \mathbin{\hat{\circ}} \iota(c_{X,Z})$	$\mathrm{id}_Z \otimes s \circ c_{\Gamma,Z} \otimes \mathrm{id}_X \circ \mathrm{id}_\Gamma \otimes c_{X,Z}$

A trivial computation in $\mathcal{C}$ shows that the expressions on the right-hand side coincide. □

As shown in [24], naturality of c w.r.t. the second variable holds iff $\mathcal{S} \subset Z_2(\mathcal{C})$, which is the case iff $\Gamma \in Z_2(\mathcal{C})$. Here $Z_2(\mathcal{C}) \subset \mathcal{C}$ is the full subcategory of objects X satisfying $c_{X,Y} \circ c_{Y,X} = \mathrm{id}_{YX}$ for all $Y \in \mathcal{C}$, called central in [27] and transparent in [3]. In order to understand the general case $\mathcal{S} \not\subset Z_2(\mathcal{C})$ we need some preliminary considerations.

Lemma 3.10. *Let $X, Y \in \mathcal{C}$, $Z \in \mathcal{C} \cap \mathcal{S}'$, and $s \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes X, Y)$. Then*

$$\iota(c_{Z,Y}) \mathbin{\hat{\circ}} \mathrm{id}_Z \mathbin{\widehat{\otimes}} s = s \mathbin{\widehat{\otimes}} \mathrm{id}_Z \mathbin{\hat{\circ}} \iota(c_{Z,X}).$$

Proof. As above we have:

$\mathrm{Hom}_{\widehat{\mathcal{C}}}(Z \otimes X, Y \otimes Z)$	$\mathrm{Hom}(\Gamma \otimes Z \otimes X, Y \otimes Z)$
$\iota(c_{Z,Y}) \mathbin{\hat{\circ}} \mathrm{id}_Z \mathbin{\widehat{\otimes}} s$	$c_{Z,Y} \circ \mathrm{id}_Z \otimes s \circ c_{\Gamma,Z} \otimes \mathrm{id}_X$
$s \mathbin{\widehat{\otimes}} \mathrm{id}_Z \mathbin{\hat{\circ}} \iota(c_{Z,X})$	$s \otimes \mathrm{id}_Z \circ \mathrm{id}_\Gamma \otimes c_{Z,X}$

Now we find

$$c_{Z,Y} \circ \mathrm{id}_Z \otimes s \circ c_{\Gamma,Z} \otimes \mathrm{id}_X = s \otimes \mathrm{id}_Z \circ \mathrm{id}_\Gamma \otimes c_{Z,X} \circ (c_{Z,\Gamma} \circ c_{\Gamma,Z}) \otimes \mathrm{id}_X.$$

For arbitrary $Z \in \mathcal{C}$ this will not coincide with $s \otimes \mathrm{id}_Z \circ \mathrm{id}_\Gamma \otimes c_{Z,X}$, but for $Z \in \mathcal{C} \cap \mathcal{S}'$ it does since $\Gamma \in \mathcal{S}$, implying $c_{Z,\Gamma} \circ c_{\Gamma,Z} = \mathrm{id}$. □

Let $X \in \mathcal{C}$ and $p \in \mathrm{End}_{\widehat{\mathcal{C}}}(X)$ a minimal idempotent, thus $X_1 = (X, p) \in \mathcal{C} \rtimes \mathcal{S}$ is simple. Let $v : X_1 \to (X, p)$, $v' : (X, p) \to X_1$ satisfy $v' \circ v = \mathrm{id}_{X_1}$, $v \circ v' = p$ and consider

$$\mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} v \mathbin{\hat{\circ}} \iota(c_{X,\Gamma} \circ c_{\Gamma,X}) \mathbin{\hat{\circ}} \mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} v' \in \mathrm{End}_{\widehat{\mathcal{C}}}(\Gamma \otimes X_1). \tag{3.1}$$

In view of $\Gamma \in \mathcal{S} \subset \mathcal{C} \cap \mathcal{S}'$, the preceding lemmas imply that (3.1) equals

$$\mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} p \mathbin{\hat{\circ}} \iota(c_{X,\Gamma} \circ c_{\Gamma,X}) = \iota(c_{X,\Gamma} \circ c_{\Gamma,X}) \mathbin{\hat{\circ}} \mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} p,$$

which in particular implies that (3.1) is invertible, thus is in $\mathrm{Aut}_{\widehat{\mathcal{C}}}(\Gamma \otimes X_1)$. The inverse is given by

$$\partial'' X_1 := \mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} v \mathbin{\hat{\circ}} \iota\big(\tilde{c}(X,\Gamma) \circ \tilde{c}(\Gamma, X)\big) \mathbin{\hat{\circ}} \mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} v' \in \mathrm{Aut}_{\widehat{\mathcal{C}}}(\Gamma \otimes X_1),$$

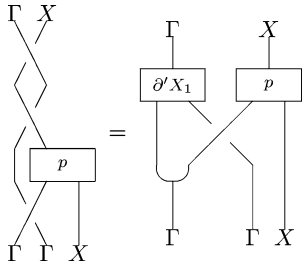
where $\tilde{c}(X, Y) = c_{Y,X}^{-1}$. Since X_1 is simple and $\iota(\Gamma) \cong |G| \mathrm{id}_{\mathbf{1}}$, we have

$$\partial'' X_1 = \partial' X_1 \mathbin{\widehat{\otimes}} \mathrm{id}_{X_1}, \tag{3.2}$$

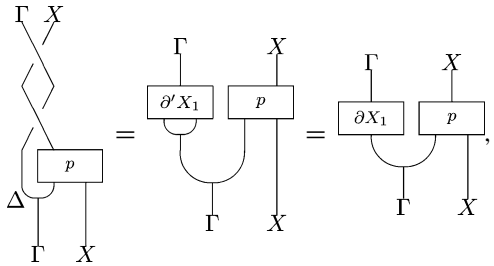
where $\partial' X_1 \in \mathrm{Aut}\, \iota(\Gamma) \cong M_{|G|}(\mathbb{F})$. This equation, which lives in $\widehat{\mathcal{C}}$, corresponds to

$$\iota\big(\tilde{c}(X,\Gamma) \circ \tilde{c}(\Gamma, X)\big) \mathbin{\hat{\circ}} \mathrm{id}_\Gamma \mathbin{\widehat{\otimes}} p = \partial' X_1 \mathbin{\widehat{\otimes}} p$$

in $\widetilde{\mathcal{C}}$ and to



in $\mathcal{C}$. Composing with $\Delta \otimes \mathrm{id}_X$ and using cocommutativity $c_{\Gamma,\Gamma} \circ \Delta = \Delta$, we obtain



(3.3)

where we have defined

$$\partial X_1 := \partial' X_1 \circ \Delta \in \mathrm{End}_{\mathcal{C}}(\Gamma).$$

Before we elucidate the significance of (3.3), we derive an explicit formula for $\partial(X, p)$. In view of (3.2) it is clear that

$$\begin{aligned}\partial' X_1 &= d(X_1)^{-1}(\mathrm{id}_\Gamma \otimes \mathrm{Tr}_{X_1})\partial'' X_1 \\ &= d(X_1)^{-1}(\mathrm{id}_\Gamma \otimes \mathrm{Tr}_X)\big[\iota\big(\tilde{c}(X,\Gamma)\circ\tilde{c}(\Gamma,X)\big)\,\hat{\circ}\,\mathrm{id}_\Gamma \,\widehat{\otimes}\, p\big].\end{aligned}$$

We have $\partial' X_1 \in \mathrm{End}_{\widehat{\mathcal{C}}}\iota(\Gamma)$, and computation shows that $\partial' X_1 \in \mathrm{End}_{\widehat{\mathcal{C}}}\iota(\Gamma)$ is represented by

$$d(X_1)^{-1}\mathrm{id}_\Gamma \otimes \bar{\varepsilon}_X \circ \big(\tilde{c}(X,\Gamma)\circ\tilde{c}(\Gamma,X)\big) \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_\Gamma \otimes p \otimes \mathrm{id}_{\overline{X}} \circ c_{\Gamma,\Gamma} \otimes \varepsilon_X$$

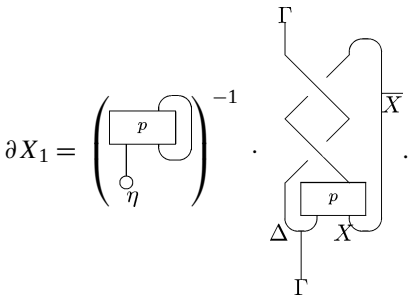
in $\mathrm{Hom}_{\mathcal{C}}(\Gamma\Gamma, \Gamma)$. Furthermore,

$$\begin{aligned}d(X_1) &= \mathrm{Tr}_{X_1}(\mathrm{id}_{X_1}) = \mathrm{Tr}_{\iota(X)}(p) = \mathrm{Tr}_X(p\circ\eta\otimes\mathrm{id}_X) \\ &= \bar{\varepsilon}_X \circ p \otimes \mathrm{id}_{\overline{X}} \circ \eta \otimes \varepsilon_X. \end{aligned} \tag{3.4}$$

For $\partial X_1 = \partial' X_1 \circ \Delta$ we thus obtain

$$\partial X_1 = d(X_1)^{-1}\mathrm{id}_\Gamma \otimes \bar{\varepsilon}_X \circ \big(\tilde{c}(X,\Gamma)\circ\tilde{c}(\Gamma,X)\big) \otimes \mathrm{id}_{\overline{X}} \circ \mathrm{id}_\Gamma \otimes p \otimes \mathrm{id}_{\overline{X}} \circ \Delta \otimes \varepsilon_X,$$

where we have used the cocommutativity $c_{\Gamma,\Gamma}\circ\Delta = \Delta$. In diagrammatic form:

$\partial X_1 =$ 

(3.5)

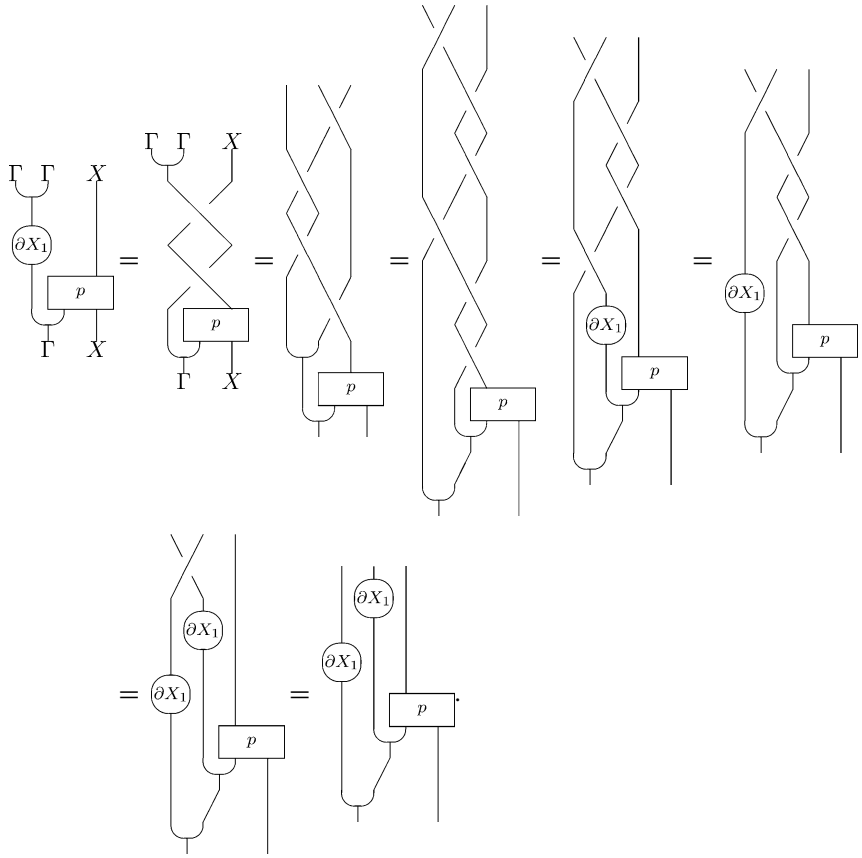
By definition, $\partial(X, p) \in \mathrm{End}_{\mathcal{C}}(\Gamma)$. In fact, we have a much stronger result.

Proposition 3.11. *Let $(X, p) \in \mathcal{C} \rtimes \mathcal{S}$ be simple. Then $\partial(X, p) \in G = \mathrm{Aut}(\Gamma, m, \eta)$.*

Proof. Since $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$, we have $\partial(X, p)\circ\eta = c\eta$ and $\varepsilon\circ\partial(X, p) = c\varepsilon$, where $c = \varepsilon\circ\partial(X, p)\circ\eta$. Thus

$$c = d(X_1)^{-1}\bar{\varepsilon}_X \circ p \otimes \mathrm{id}_{\overline{X}} \circ \eta \otimes \varepsilon_X$$

and comparison with (3.4) shows $c = 1$, thus $\partial(X, p)\circ\eta = \eta$. Next, we compute



Here the first, fourth, and sixth equality are due to (3.3) and the fifth and seventh due to the cocommutativity of Δ. Taking the partial trace over X, we obtain

$$\mathrm{Tr}_{\widehat{\mathcal{C}}}(p)\Delta \circ \partial X_1 = \mathrm{Tr}_{\widehat{\mathcal{C}}}(p)\partial X_1 \otimes \partial X_1 \circ \Delta$$

and thus $\Delta \circ \partial X_1 = \partial X_1 \otimes \partial X_1 \circ \Delta$ since $\mathrm{Tr}_{\widehat{\mathcal{C}}}\, p = d(X_1) \neq 0$. Thus $\partial X_1 \in G = \mathrm{End}(\Gamma, m, \eta)$ is an endomorphism of the monoid (Γ, m, η), and by Remark 2.9, G is a group. □

Definition 3.12. An object of $\mathcal{C} \rtimes \mathcal{S}$ is homogeneous if there exist $g \in G$ and simple objects $X_i \in \mathcal{C} \rtimes \mathcal{S}, i \in \Delta$ such that $X \cong \bigoplus_i X_i$ and $\partial X_i = g$.

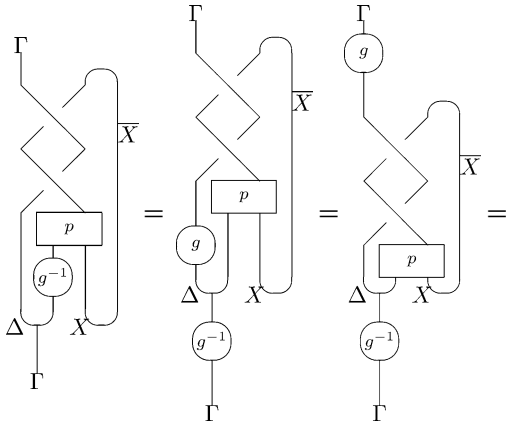
Lemma 3.13. *Let $Z \in \mathcal{C} \rtimes \mathcal{S}$ be homogeneous of grade g. Then g is still given by* (3.5)*. If $(X, p), (Y, q)$ are homogeneous and $(X, p) \cong (Y, q)$, then $\partial(X, p) = \partial(Y, q)$.*

Proof. Let $Z \cong \bigoplus X_1$, where the X_i are simple and $\partial X_i = g$. Reviewing the considerations preceding (3.2) one sees that this equation remains valid with X_1 replaced by Z.

Thus also (3.5) holds for homogeneous Z, and this is all that is used in the proof of Proposition 3.11. That isomorphic homogeneous objects have the same grade is obvious from the definition. □

Proposition 3.14. *Let $X_1 = (X, p) \in \mathcal{C} \rtimes \mathcal{S}$ be homogeneous. Then $\partial\gamma_g(X_1) = g\partial X_1 g^{-1}$ for every $g \in G$.*

Proof. Recall that $\gamma_g((X, p)) = (X, \gamma_g(p)) = (X, p \circ g^{-1} \otimes \mathrm{id}_X)$. Thus

$d(X_1)\,\partial\gamma_g(X_1) =$  $= d(X_1)\, g\partial X_1 g^{-1}.$

Here we have used the equation $\mathrm{id}_\Gamma \otimes g^{-1} \circ \Delta = g \otimes \mathrm{id}_\Gamma \circ \Delta \circ g^{-1}$ which follows from $\Delta \circ g = g \otimes g \circ \Delta$. □

The following definition is a variant of a notion due to Turaev [34].

Definition 3.15. Let G be a (discrete) group. A strict crossed G-category is a strict tensor category $\mathcal{D}$ together with

- a full tensor subcategory $\mathcal{D}_G \subset \mathcal{D}$ of homogeneous objects,
- a map $\partial : \mathrm{Obj}\,\mathcal{D}_G \to G$ constant on isomorphism classes,
- a (strict) homomorphism $\gamma : G \to \mathrm{Aut}\,\mathcal{D}$ (here $\mathrm{Aut}\,\mathcal{D}$ is the group of invertible strict tensor functors $\mathcal{D} \to \mathcal{D}$ respecting the braiding),

such that

(1) $\partial(X \otimes Y) = \partial X \partial Y$ for all $X, Y \in \mathcal{D}_G$;
(2) $\gamma_g(\mathcal{D}_h) \subset \mathcal{D}_{ghg^{-1}}$, where $\mathcal{D}_g \subset \mathcal{D}_G$ is the full subcategory $\partial^{-1}(g)$.

If $\mathcal{D}$ is additive, we require that every object of $\mathcal{D}$ be a direct sum of objects in $\mathcal{D}_G$.

Remark 3.16.

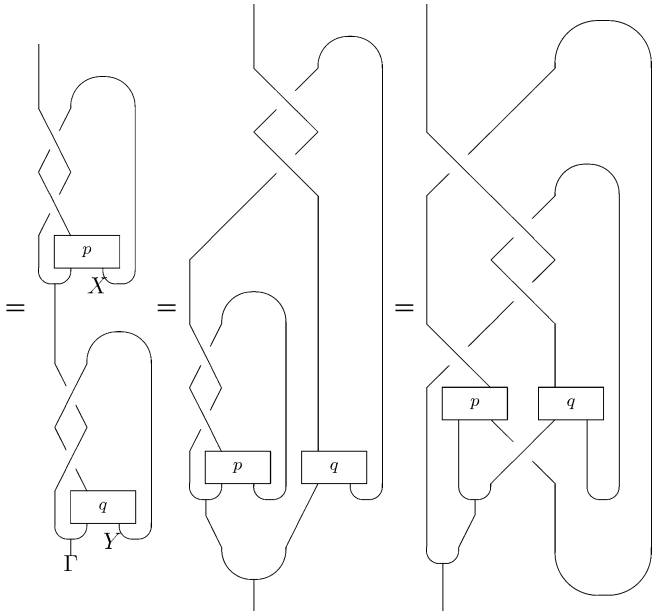
(1) A map $\partial : \mathrm{Obj}\, \mathcal{D}_G \to G$ constant on iso-classes and satisfying $\partial(X \otimes Y) = \partial(X)\partial(Y)$ is the same as a tensor functor $\mathcal{D}_G \to \mathcal{G}$, where $\mathcal{G}$ is the discrete strict monoidal category with $\mathrm{Obj}\, \mathcal{G} = G$.
(2) In [34], $\mathcal{D}_G = \mathcal{D}$ was assumed. Since we are working with additive categories, in particular having all finite direct sums, we must allow inhomogeneous objects. This added generality will be important later on.
(3) Obviously, the definition can be generalized to non-strict tensor categories, cf. [34]. Also the G-action can be generalized by relaxing the γ_g to be self-equivalences satisfying natural isomorphisms $\gamma_g\gamma_h \cong \gamma_{gh}$ with suitable coherence, cf., e.g., [4, p. 238]. For our purposes, in particular the application to conformal field theory [28], the above strict version is sufficient.

In view of Definition 3.15, Propositions 3.11, 3.14 essentially amount to the following statement.

Proposition 3.17. *$\mathcal{C} \rtimes \mathcal{S}$ is a crossed G-category, where $\mathcal{S} \simeq \mathrm{Rep}\, G$.*

Proof. We define $(\mathcal{C} \rtimes \mathcal{S})_G \subset \mathcal{C} \rtimes \mathcal{S}$ to be the full subcategory of homogeneous objects, and we extend ∂ to $(\mathcal{C} \rtimes \mathcal{S})_G$ in the obvious fashion. We have already defined an action γ of G on $\mathcal{C} \rtimes \mathcal{S}$. Now property (2) follows from Proposition 3.14, but property (1) requires proof. Thus let $(X, p), (Y, q) \in \mathcal{C} \rtimes \mathcal{S}$ be homogeneous. In view of Lemma 3.13 we may compute

$$d(X, p)\, d(Y, q)\, \partial(X, p)\partial(Y, q)$$



$$= d\big((X,p)\,\widehat{\otimes}\,(Y,q)\big)\,\partial\big((X,p)\,\widehat{\otimes}\,(Y,q)\big),$$

which is the desired result. □

Definition 3.18. A braiding for a crossed G-category $\mathcal{D}$ is a family of isomorphisms $c_{X,Y} : X \otimes Y \to {}^{X}Y \otimes X$, defined for all $X \in \mathcal{D}_G$, $Y \in \mathcal{D}$, such that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{s\otimes t} & X' \otimes Y' \\ \downarrow{\scriptstyle c_{X,Y}} & & \downarrow{\scriptstyle c_{X',Y'}} \\ {}^{X}Y \otimes X & \xrightarrow[{}^{X}t\otimes s]{} & {}^{X'}Y' \otimes X' \end{array}$$

commutes for all $s : X \to X'$, $t : Y \to Y'$, and

$$c_{X,Z\otimes T} = \mathrm{id}_{{}^{X}Z} \otimes c_{X,T} \circ c_{X,Z} \otimes \mathrm{id}_T, \tag{3.6}$$

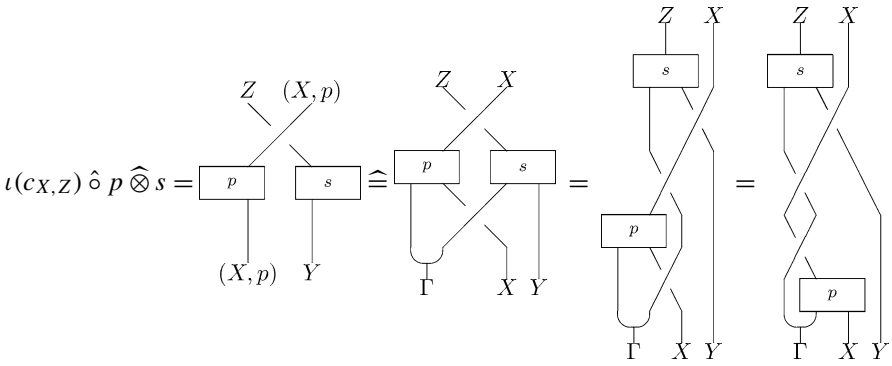
$$c_{X\otimes Y,Z} = c_{X,{}^{Y}Z} \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes c_{Y,Z}, \tag{3.7}$$

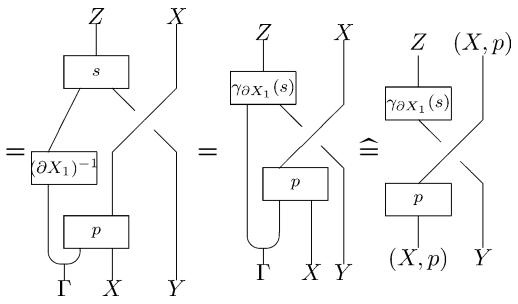
for all $X, Y \in \mathcal{D}_G$, $Z, T \in \mathcal{D}$.

Remark 3.19. Motivated by applications to algebraic topology (rather than 3-manifolds as in [34]), a special class of braided crossed G-categories was introduced independently in [4, Definition 2.1]. The 'categorical G-crossed modules' considered there are braided crossed G-categories that are also categorical groups, i.e., monoidal groupoids whose objects are invertible up to isomorphism w.r.t. $\otimes$.

Theorem 3.20. *$\mathcal{C} \rtimes \mathcal{S} = \widehat{\mathcal{C}}$ is a braided crossed G-category, where $\mathcal{S} \simeq \mathrm{Rep}\, G$.*

Proof. Let $X_1 = (X,p) \in \widehat{\mathcal{C}}_G$, $Y, Z \in \mathcal{C}$, and $s \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(Y,Z) = \mathrm{Hom}_{\mathcal{C}}(\Gamma \otimes Y, Z)$. We calculate





$$= \gamma_{\partial X_1}(s) \mathbin{\widehat{\otimes}} \mathrm{id}_{X_1} \mathbin{\hat{\circ}} \iota(c_{X,Z}) \mathbin{\hat{\circ}} p \mathbin{\widehat{\otimes}} \mathrm{id}_Y.$$

We have used cocommutativity of Δ, Eq. (3.3), Proposition 3.11 according to which $\partial(X, p) \in G$, and the definition of $\gamma_g \in \mathrm{Aut}\,\mathcal{C} \rtimes \mathcal{S}$.

Let now $(X, p) \in (\mathcal{C} \rtimes \mathcal{S})_g$ and $(Y, q) \in \mathcal{C} \rtimes \mathcal{S}$. Then the above computation and Lemma 3.9 imply

$$\iota(c_{X,Y}) \mathbin{\hat{\circ}} p \mathbin{\widehat{\otimes}} q = \gamma_g(q) \mathbin{\widehat{\otimes}} p \mathbin{\hat{\circ}} \iota(c_{X,Y}),$$

thus this expression defines an isomorphism $c_{(X,p),(Y,q)} \in \mathrm{Hom}_{\widehat{\mathcal{C}}}((X, p) \otimes (Y, q), \gamma_g(Y, q) \otimes (X, p))$. By definition, the family $(c_{(X,p),(Y,q)})$ it is natural in the sense of Definition 3.18. The straightforward verification of the braid relations (3.6)–(3.7) is omitted. □

3.3. The G-spectrum of a Galois extension

Definition 3.21. The G-spectrum $\mathrm{Spec}\,\mathcal{D}$ of a G-crossed category $\mathcal{D}$ is set $\{g \in G \mid \mathcal{D}_g \neq \emptyset\}$. The G-spectrum of a crossed G-category is full if it coincides with G and trivial if it is $\{e\}$.

Lemma 3.22. *The G-spectrum of a crossed G-category $\mathcal{D}$ contains the unit, is closed under multiplication and under conjugation with elements of G. It is closed under inverses if $\mathcal{D}$ has duals, in which case* $\mathrm{Spec}\,\mathcal{D}$ *is a normal subgroup of G.*

Proof. The first sentence follows from requirements (1) and (2) in Definition 3.15 and the second from the fact that $\partial \overline{X} = (\partial X)^{-1}$, which follows from $\mathbf{1} \prec X \otimes \overline{X}$. □

Proposition 3.23. *Let $\mathcal{D}$ be a semisimple rigid crossed G-category. Defining* $\dim \mathcal{D}_g$ *to be the sum over the squared dimensions of the simple objects of grade g, we have*

$$\dim \mathcal{D}_g = \dim \mathcal{D}_e \quad \forall g \in \mathrm{Spec}\,\mathcal{D}.$$

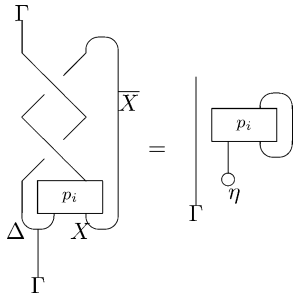
Proof. Let Δ_e, Δ_g be the sets of iso-classes of simple objects in $\mathcal{D}_e, \mathcal{D}_g$, respectively, and let $\{X_i,\ i \in \Delta_e\}$ and $\{Y_j,\ j \in \Delta_g\}$ be representing objects. For $g \in \mathrm{Spec}\,\mathcal{D}$ we may pick a simple object $Z \in \mathcal{D}_g$, and in view of $X_i \otimes Z \in \mathcal{D}_g$ we have

$$d(Z)\sum_{i\in\Delta_e} d(X_i)^2 = \sum_{i\in\Delta_e} d(X_i)\,d(X_i\otimes Z) = \sum_{i\in\Delta_e}\sum_{j\in\Delta_g} d(X_i)\,d(Y_j)\dim\mathrm{Hom}(X_i\otimes Z, Y_j)$$
$$= \sum_{i\in\Delta_e}\sum_{j\in\Delta_g} d(X_i)\,d(Y_j)\dim\mathrm{Hom}(X_i, Y_j\otimes\overline{Z}) = \sum_{j\in\Delta_g} d(Y_j)\,d(Y_j\otimes\overline{Z})$$
$$= d(\overline{Z})\sum_{j\in\Delta_g} d(Y_j)^2.$$

Since $d(Z) = d(\overline{Z}) \neq 0$, the claim follows. □

Proposition 3.24. *Let $\mathcal{C}, \mathcal{S}$ be as in the preceding section. The embedding $(\mathcal{C}\cap\mathcal{S}')\rtimes\mathcal{S}\hookrightarrow\mathcal{C}\rtimes\mathcal{S}$ gives rise to an isomorphism $(\mathcal{C}\rtimes\mathcal{S})_e \cong (\mathcal{C}\cap\mathcal{S}')\rtimes\mathcal{S}$. $\mathcal{C}\rtimes\mathcal{S}$ has trivial G-spectrum iff $\mathcal{S}\subset Z_2(\mathcal{C})$.*

Proof. If $X\in\mathcal{C}\cap\mathcal{S}'$, then $c_{X,\Gamma}\circ c_{\Gamma,X} = \mathrm{id}$, thus every simple summand of $\iota(X)$ has grade e. This implies $(\mathcal{C}\cap\mathcal{S}')\rtimes\mathcal{S}\subset(\mathcal{C}\rtimes\mathcal{S})_e$. As to the converse, every simple object $X_1\in\mathcal{C}\rtimes\mathcal{S}$ is isomorphic to one of the form (X,p), where $X\in\mathcal{C}$ is simple and p is a minimal idempotent. In [24, Proposition 4.2] it was shown that the action γ of G on $\mathcal{C}\rtimes\mathcal{S}$ acts transitively on the minimal central idempotents in $\mathrm{End}_{\widehat{\mathcal{C}}}(\iota(X))$, in particular all simple summands of $\iota(X)$ appear with the same multiplicity N. If $\iota(X)\cong N\bigoplus_i(X,p_i)$ is the decomposition into simples, we conclude from Proposition 3.14 that the set $\{\partial(X,p_i)\}$ is a conjugacy class in G. If $X_1\prec\iota(X)$ has grade e, then this conjugacy class is $\{e\}$, thus $\partial(X,p_i)=e$ for all i. This means



(3.8)

for all minimal central idempotents p_i in $\mathrm{End}_{\widehat{\mathcal{C}}}\iota(X)$. By linearity, (3.8) holds for all central idempotents, in particular for $\mathrm{id}_{\iota(X)} = \varepsilon\otimes\mathrm{id}_X$. Plugging this into (3.8), we obtain $(\mathrm{id}\otimes\mathrm{Tr}_X)(c_{X,\Gamma}\circ c_{\Gamma,X}) = d(X)\mathrm{id}_\Gamma$, and by naturality we conclude

$$S(X,Y) = (\mathrm{Tr}_Y\otimes\mathrm{Tr}_X)(c_{X,Y}\circ c_{Y,X}) = d(X)\,d(Y)$$

for all simple $Y\in\mathcal{S}$. By [27, Proposition 2.5] this is equivalent to $X\in\mathcal{C}\cap\mathcal{S}'$. Now, triviality of the G-spectrum is equivalent to $\mathcal{C}\rtimes\mathcal{S} = (\mathcal{C}\rtimes\mathcal{S})_e = (\mathcal{C}\cap\mathcal{S}')\rtimes\mathcal{S}$, which in turn is equivalent to $\mathcal{C}\cap\mathcal{S}' = \mathcal{C}$ and finally to $\mathcal{S}\subset Z_2(\mathcal{C})$. □

Remark 3.25. We emphasize one observation made in the proof: Whereas every simple object X_1 of $\mathcal{C} \rtimes \mathcal{S}$ defines an element ∂X_1 of G, every simple object $X \in \mathcal{C}$ defines a unique conjugacy class in G.

Let $\mathcal{S}_0 \subset \mathcal{S}$ be a full subcategory, where both categories are finite admissible STCs. Let $(\Gamma, \ldots)$, $(\Gamma_0, \ldots)$ be the corresponding Frobenius algebras in $\mathcal{S}_0, \mathcal{S}$, respectively, with automorphism groups G_0, G. Then $\Gamma \cong \Gamma_0 \oplus Z$ and $\mathrm{Hom}(\Gamma_0, Z) = \{0\}$, thus the projector $q \in \mathrm{End}\,\Gamma$ onto Γ_0 is central. The group

$$N = \{g \in G \mid g \circ q = q\}$$

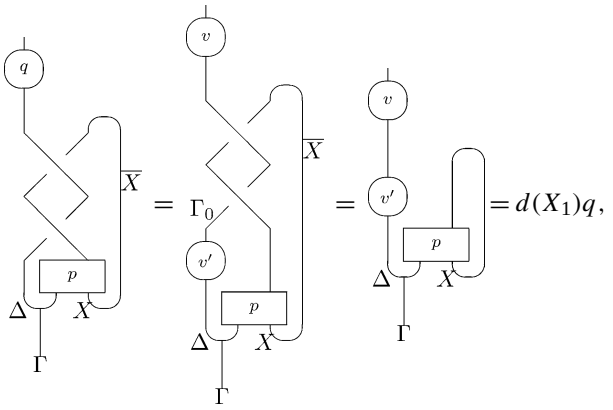
is a normal subgroup of $G = \mathrm{Aut}(\Gamma, m, \eta)$. It coincides with

$$N = \left\{g \in G \mid \pi_X(g) = \mathrm{id}_{E(X)}\ \forall X \in \mathcal{S}_0\right\},$$

where $E : \mathcal{S} \to \mathrm{Vect}_{\mathbb{C}}$ is the fiber functor and π_X is the representation of G on $E(X)$. This is easily deduced from $E(X) = \mathrm{Hom}(\mathbf{1},\ \Gamma \otimes X)$ and the fact that $g \in G$ acts on $E(X)$ by $\pi_X(g) : \phi \mapsto g \otimes \mathrm{id}_X \circ \phi$. This implies $G_0 \cong G/N$.

Theorem 3.26. *Let $\mathcal{S} \subset \mathcal{C}$ with $\mathcal{S} \simeq \mathrm{Rep}\,G$. Then $\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S} = N$, where N is the normal subgroup of G corresponding to the full inclusion $\mathcal{S} \cap Z_2(\mathcal{C}) \subset \mathcal{S}$ as above. $\mathcal{C} \rtimes \mathcal{S}$ has full G-spectrum iff $\mathcal{S} \cap Z_2(\mathcal{C})$ is trivial, i.e., consists only of multiples of* **1**.

Proof. Let $q \in \mathrm{End}_{\widehat{\mathcal{C}}}(\Gamma)$ be the projection onto Γ_0, and let $v : \Gamma_0 \to \Gamma$, $v' : \Gamma \to \Gamma_0$ satisfy $v \circ v' = q$, $v' \circ v = \mathrm{id}_{\Gamma_0}$. Then with $X_1 = (X, p) \in (\mathcal{C} \rtimes \mathcal{S})_G$ we have

$d(X_1)q \circ \partial(X, p) =$  $= d(X_1)q,$

where we used $\Gamma_0 \in \mathcal{S} \cap Z(\mathcal{C})$. We conclude $\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S} \subset N$.

In a braided crossed G-category $\mathcal{D}$ we have isomorphisms $c_{X,Y} : X \otimes Y \to \gamma_g(Y) \otimes X$ whenever $X \in \mathcal{D}_G$. By definition, $g \in \mathrm{Spec}\,\mathcal{D}$, thus in the fixpoint category $\mathcal{D}^{\mathrm{Spec}\,\mathcal{D}}$ the action γ_g disappears and $\mathcal{D}^{\mathrm{Spec}\,\mathcal{D}}$ is braided in the usual sense. We therefore have an intermediate extension

$$\mathcal{C} \subset (\mathcal{C} \rtimes \mathcal{S})^{\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S}} \subset \mathcal{C} \rtimes \mathcal{S}$$

that is braided. On the other hand, in view of Proposition 3.24 it is clear that the maximal intermediate extension of $\mathcal{C}$ that is braided is given by

$$\mathcal{C} \subset \mathcal{C} \rtimes \big(\mathcal{S} \cap Z_2(\mathcal{C})\big) \subset \mathcal{C} \rtimes \mathcal{S}.$$

By the Galois correspondence established in [24, Section 4.2], we have $\mathcal{C} \rtimes (\mathcal{S} \cap Z_2(\mathcal{C})) = (\mathcal{C} \rtimes \mathcal{S})^N$, where N is as defined above. Now the inclusion

$$(\mathcal{C} \rtimes \mathcal{S})^{\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S}} \subset \mathcal{C} \rtimes \big(\mathcal{S} \cap Z_2(\mathcal{C})\big) = (\mathcal{C} \rtimes \mathcal{S})^N$$

implies $N \subset \mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S}$. This completes the proof of $\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S} = N$. The last claim is immediate. □

The following corollary will be very useful in conformal field theory [28].

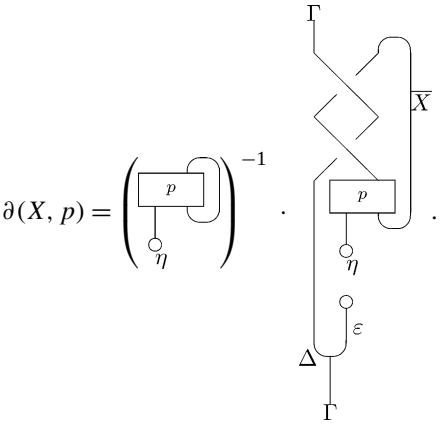
Corollary 3.27. *If $\mathcal{C}$ is modular and* $\mathrm{Rep}\,G \simeq \mathcal{S} \subset \mathcal{C}$*, then $\mathcal{C} \rtimes \mathcal{S}$ has full G-spectrum and $(\mathcal{C} \rtimes \mathcal{S})_e$ is modular.*

Proof. Modularity of $\mathcal{C}$ is equivalent to triviality of $Z_2(\mathcal{C})$, thus the last statement of Theorem 3.26 implies $\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S} = G$. Since $\mathcal{C}$ is modular, [27, Corollary 3.6] implies $Z_2(\mathcal{C} \cap \mathcal{S}') = \mathcal{S}$. Thus $(\mathcal{C} \cap \mathcal{S}') \rtimes \mathcal{S}$ is modular by [24, Theorem 4.4] and coincides with $(\mathcal{C} \rtimes \mathcal{S})_e$ by Proposition 3.24. □

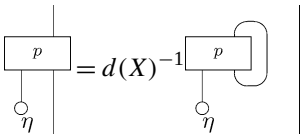
3.4. Abelian case

Let $X \in \mathcal{C}$ be simple and let $X_j \in \mathcal{C} \rtimes \mathcal{S}$, $j \in J$, be simple objects such that $\iota(X) \cong \bigoplus_{j\in J} X_j$. In [24] it was shown that G acts ergodically on the center of the algebra $\mathrm{End}\,\iota(X)$. In view of $\partial\gamma_g(X) = g\partial(X)g^{-1}$ this clearly implies that the set $\{\partial X_j \mid j \in J\}$ is a conjugacy class in G. We thus obtain a map ∂_0 from the simple objects in $\mathcal{C}$ to the conjugacy classes in G. In the case where G is abelian, all simple summands of $\iota(X)$ have the same grade, which induces a G-grading on the category $\mathcal{C}$. In the remainder of this subsection we will give a more explicit description of this grading.

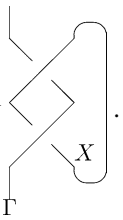
Let thus G be abelian and $K = \widehat{G}$. Then $\Gamma \cong \bigoplus_{k\in K} X_k$, where all $X_k, k \in K$ are invertible, and $\mathrm{End}\,\Gamma \cong \bigoplus_{k\in K} \mathrm{End}\,X_k \cong \bigoplus_{k\in K} \mathbb{F}$. By our normalization $\varepsilon \circ \eta = 1$, $p_e = \eta \circ \varepsilon \in \mathrm{End}\,\Gamma$ is an idempotent, projecting on the summand X_e. Let $X \in \mathcal{C}$ and $(X, p) \in \mathcal{C} \rtimes \mathcal{S}$ be simple. By the above considerations, $\iota(X)$ is homogeneous, thus (3.5) defines an element of $\mathrm{Aut}(\Gamma, m, \eta) \cong G$. In view of $X_k \otimes X_l \cong X_{kl}$ we may insert p_e into (3.5) at the appropriate place, obtaining

$\partial(X,p) =$ 

Now,



and we obtain

$$\partial_0(X) = \partial\big((X,p)\big) = d(X)^{-1}$$ 

(3.9)

We have thus shown the following.

Proposition 3.28. *Consider $\mathcal{S} \subset \mathcal{C}$ where $\mathcal{S}$ is symmetric, even and all its simple objects are one-dimensional, equivalently $\mathcal{S} \simeq \operatorname{Rep} G$ with G abelian. Let (Γ, m, η) be the regular monoid in $\mathcal{S}$. Then* (3.9) *defines an element $\partial_0 X$ of G for every simple $X \in \mathcal{C}$. If we define $\mathcal{C}_G$ to be the full subcategory of homogeneous objects, i.e., of objects all simple summands X_j of which have the same $\partial_0 X_j$, then $\mathcal{C}$ is a G-graded tensor category.* (*To wit, $\mathcal{C}$ is a crossed G-category in the sense of Definition* 1.1 *with trivial G-action.*)

Remark 3.29. This result can be obtained in a more direct way. It suffices to notice that the map $\varphi_X : K \to \mathbb{F}$ defined by $\varphi_X(k)\mathrm{id}_{X_k} = (\mathrm{id}_{X_k} \otimes \mathrm{Tr}_X)(c_{X,X_k} \circ c_{X_k,X})$ is a character of K, thus an element of G. (This goes back at least to [32].) From the above considerations it is then clear that the two definitions yield the same element $\partial_0 X \in G$.

Acknowledgment

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