IMRN International Mathematics Research Notices
2004, No. 51

On the Center of a Compact Group

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1 Introduction

With every compact group G one can associate two canonical compact abelian groups, to wit, the center $Z(G)$ and the abelianization $G_{ab} = G/\overline{[G,G]}$. Since every compact group can be recovered from its (abstract) category of finite-dimensional unitary representations [3], it is natural to ask whether the said abelian groups can be recovered directly from Rep G without appealing to a reconstruction theorem à la Tannaka-Krein-Doplicher-Roberts or Saavedra-Rivano-Deligne-Milne. Since Rep G is a discrete structure, it is clear that one will rather recover the duals $\widehat{G_{ab}}$ and $\widehat{Z(G)}$. In the case of $\widehat{G_{ab}}$ it is well known how to proceed. Writing $\mathcal{C} = \mathrm{Rep}\, G$, let $\mathcal{C}_1 \subset \mathcal{C}$ be the full subcategory of one-dimensional representations. Then the set of isomorphism classes of objects in $\mathcal{C}_1$ is a (discrete) abelian group, and it is easy to see that it is isomorphic to $\widehat{G_{ab}}$. It is natural to ask whether also $\widehat{Z(G)}$ can be recovered directly from Rep G.

Motivated by certain operator algebraic considerations closely related to and inspired by [3], Baumgärtel and Lledó [1, Section 5] defined, for every compact group G, a discrete abelian group $C(G)$ in terms of the representation category Rep G. They identified a surjective homomorphism $C(G) \to \widehat{Z(G)}$ and conjectured the latter to be an isomorphism. They substantiated this conjecture by explicit verification for several finite and compact Lie groups. (According to [1], the case of $SU(N)$ was checked by C. Schweigert.) In this paper we prove that $\widehat{Z(G)} \cong C(G)$ for all compact groups, exploiting a remark made in [4], and we derive two useful corollaries. Despite our general proof, the examples in [1] remain quite instructive.

Received 16 December 2003. Revision received 19 April 2004.
Communicated by Joseph Bernstein.

2 Definitions and preparations

Throughout the paper, G denotes any compact group and $\widehat{G}$ the set of equivalence classes of irreducible representations. We allow ourselves the usual harmless sloppiness of not always distinguishing between an irreducible representation X and its equivalence class $[X] \in \widehat{G}$. (Thus "Let $X \in \widehat{G}$" means "Let $\mathcal{X} \in \widehat{G}$ and let $X \in \operatorname{Rep} G$ be simple such that $[X] = \mathcal{X}$.") While $\widehat{G}$ is a group if and only if G is abelian, there is always a notion of "homomorphism" into an abelian group.

Definition 2.1. Let G be a compact group and A an abelian group. A map $\varphi : \widehat{G} \to A$ is called a t-map (tensor product compatible) if we have $\varphi(Z) = \varphi(X)\varphi(Y)$ whenever $X, Y, Z \in \widehat{G}$ and $Z \prec X \otimes Y$.

Lemma 2.2. If $\varphi : \widehat{G} \to A$ is a t-map, then $\varphi(1) = 1$, where the first 1 denotes the trivial representation of G, and $\varphi(\overline{X}) = \varphi(X)^{-1}$ for every $X \in \widehat{G}$. □

Proof. We have $\varphi(1) = \varphi(1 \otimes 1) = \varphi(1)\varphi(1)$, thus $\varphi(1) = 1$. For any $X \in \widehat{G}$, we have $1 \prec X \otimes \overline{X}$, thus $1 = \varphi(1) = \varphi(X)\varphi(\overline{X})$, which proves the second claim. ■

The following proposition is essentially due to [1].

Proposition 2.3. For every compact group G, there is a universal t-map $p_G : \widehat{G} \to C(G)$. (Thus for every t-map $\varphi : \widehat{G} \to A$, there is a unique homomorphism $\beta : C(G) \to A$ of abelian groups such that

$$\begin{tikzcd} \widehat{G} \arrow[r, "p_G"] \arrow[dr, "\varphi"'] & C(G) \arrow[d, "\beta"] \\ & A \end{tikzcd} \tag{2.1}$$

commutes.) Here the "chain group" $C(G)$ is the free abelian group (written multiplicatively) generated by the set $\widehat{G}$ of isomorphism classes of irreducible representations of G modulo the relations $[Z] = [X] \cdot [Y]$ whenever Z is contained in $X \otimes Y$. The obvious map $p_G : \widehat{G} \to C(G)$ is a t-map. □

Proof. Clearly we must take β to send the generator $[X]$ of $C(G)$ to $\varphi([X])$, proving uniqueness. In view of the definition of a t-map, this is compatible with the relations imposed on $C(G)$, whence existence of β. ■

Remark 2.4. (1) The above elegant definition of $C(G)$ is due to J. Bernstein. In [1], $C(G)$ was defined as $\widehat{G}/\simeq$, where $\sim$ is the equivalence relation given by $X \sim Y$ whenever there exist $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \widehat{G}$ such that both X and Y are contained in $Z_1 \otimes \cdots \otimes Z_n$. Denoting the $\sim$-equivalence class of X by $\langle X \rangle$, $C(G)$ is an abelian group with respect to the operations $\langle X \rangle \langle Y \rangle = \langle Z \rangle$, where Z is any irrep contained in $X \otimes Y$, and $\langle X \rangle^{-1} = \langle \overline{X} \rangle$. The easy verification of the equivalence of the two definitions is left to the reader.

(2) A chain group $C(\mathcal{C})$, in general nonabelian, satisfying an analogous universal property, can be defined for any fusion category $\mathcal{C}$, but we need only the case $\mathcal{C} = \operatorname{Rep} G$ and write $C(G)$ rather than $C(\operatorname{Rep} G)$.

The following, proven in [1], is the most interesting example of a t-map.

Proposition 2.5. The restriction of irreducible representations of G to the center defines a surjective t-map $r_G : \widehat{G} \to \widehat{Z(G)}$. Thus also the homomorphism of abelian groups $\alpha_G : C(G) \to \widehat{Z(G)}$ arising as above is surjective. □

Proof. If $Z \in \widehat{G}$ and $g \in Z(G)$, then $\pi_Z(g)$ commutes with $\pi_Z(G)$, thus by Schur's lemma we have $\pi_Z(g) = \chi_Z(g) 1_Z$, where $\chi_Z(g) \in U(1)$. Clearly, $g \mapsto \chi_Z(g)$ is a homomorphism, thus $\chi_Z \in \widehat{Z(G)}$. This defines a map $r_G : \widehat{G} \to \widehat{Z(G)}$, which is easily seen to be a t-map. Since $Z(G)$ is a closed subgroup of G, [6, Theorem 27.46] says that for every irreducible representation (thus character) χ of $Z(G)$, there is a unitary representation π of G such that $\chi \prec \pi \upharpoonright Z(G)$. We thus have $r_G([\pi]) = \chi$, thus r_G is surjective. Therefore also the map $\alpha_G : C(G) \to \widehat{Z(G)}$ arising from Proposition 2.3 is surjective. ■

For brevity, we denote as fusion categories the semisimple $\mathbb{C}$-linear tensor categories with simple unit and two-sided duals, for example, the C^*-tensor categories with conjugates, direct sums, and subobjects of [3]. (We do not assume finiteness.) All subcategories considered below are full, monoidal, replete (closed under isomorphisms), and closed under direct sums, subobjects, and duals, thus they are themselves fusion categories.

Definition 2.6. Let $\mathcal{C}$ be a fusion category. Then $\mathcal{C}_0$ denotes the full tensor subcategory generated by the simple objects X for which there exists a simple object $Y \in \mathcal{C}$ such that $X \prec Y \otimes \overline{Y}$.

Remark 2.7. The subcategory $\mathcal{C}_0$ of a fusion category seems to have first been considered by Etingof et al. in [4, Section 8.5], where the following fact is remarked in parentheses. The proof might be well known, but we are not aware of a suitable reference.

Proposition 2.8. Let G be a compact group and $\mathcal{C} = \operatorname{Rep} G$. Then the category $\mathcal{C}_0$ coincides with the full subcategory $\mathcal{C}_Z \subset \mathcal{C}$ consisting of those representations that are trivial when restricted to $Z(G)$. Thus $\mathcal{C}_0 \simeq \operatorname{Rep}(G/Z(G))$. □

Proof. If $X, Y \in \widehat{G}$ are simple and $X \prec Y \otimes \overline{Y}$, then the restriction of X to $Z(G)$ is trivial, implying $\mathcal{C}_0 \subset \mathcal{C}_Z$. As to the converse, let $g \in G$ be such that $g \in \ker \pi_X$ for all $X \in C_0$. This holds if and only if $(\pi_Y \otimes \pi_{\overline{Y}})(g) = \mathbf{1}$ for all simple $Y \in \operatorname{Rep} G$. The latter means

$$\pi_Y(g) \otimes \pi_Y(g^{-1})^t = \mathbf{1}, \tag{2.2}$$

which is true if and only if $\pi_Y(g) \in \mathbb{C}\mathbf{1}_Y$. Now, if $g \in G$ is represented by scalars in all irreps $Y \in \widehat{G}$, then $g \in Z(G)$. (This follows from the fact that the irreducible representations separate the elements of G.) In view of the Galois correspondence of full monoidal subcategories $\mathcal{D} \subset \operatorname{Rep} G$ and closed normal subgroups $H \subset G$ given by

$$\begin{aligned} H_{\mathcal{D}} &= \{g \in G \mid \pi_X(g) = \mathrm{id}\ \forall X \in \mathcal{D}\}, \\ \operatorname{Obj} \mathcal{D}_H &= \{X \in \operatorname{Rep} G \mid \pi_X(g) = \mathrm{id}\ \forall g \in H\}, \end{aligned} \tag{2.3}$$

we have $H_{\mathcal{C}_0} \subset Z(G) = H_{\mathcal{C}_Z}$, thus $\mathcal{C}_Z \subset \mathcal{C}_0$ and therefore $\mathcal{C}_0 = \mathcal{C}_Z$. ■

Lemma 2.9. Let G be compact and let $\mathcal{C} = \operatorname{Rep} G$. For a simple object $X \in \mathcal{C}$, $p_G([X]) = 1$ if and only if $X \in \mathcal{C}_0$. □

Proof. If Z and X_i, Y_i, $i = 1, \dots, n$, are simple with $X_i \prec Y_i \otimes \overline{Y}_i$ and $Z \prec X_1 \otimes \cdots \otimes X_n$, then $1, Z \prec Y_1 \otimes \overline{Y}_1 \otimes \cdots \otimes Y_n \otimes \overline{Y}_n$, thus $Z \sim 1$. This implies that $p_G([X]) = \langle X \rangle = 1$ for every simple $X \in \mathcal{C}_0$. Conversely, let $X \in \mathcal{C}$ be simple such that $p_G([X]) = 1$. This is equivalent to $X \sim 1$, thus there are simple $Y_1, \dots, Y_n$ such that $1, X \prec Y_1 \otimes \cdots \otimes Y_n$. Then $X \prec Y_1 \otimes \cdots \otimes Y_n \otimes \overline{Y}_1 \otimes \cdots \otimes \overline{Y}_n$, and therefore $X \in \mathcal{C}_0$. ■

3 Results

Theorem 3.1. The homomorphism $\alpha_G : C(G) \to \widehat{Z(G)}$ is an isomorphism for every compact group G. □

Proof. Since all maps in the diagram

$$\begin{tikzcd} \widehat{G} \arrow[r, "p_G"] \arrow[dr, "r_G"'] & C(G) \arrow[d, "\alpha_G"] \\ & \widehat{Z(G)} \end{tikzcd} \tag{3.1}$$

are surjective, α_G is an isomorphism if and only if $\ker p_G = \ker r_G$. By Lemma 2.9, $[X] \in \ker p_G$ if and only if $X \in \mathcal{C}_0$. On the other hand, $[X] \in \ker r_G$ if and only if $X \in \mathcal{C}_Z$. By Proposition 2.8 we have $\mathcal{C}_0 = \mathcal{C}_Z$, thus we are done. ■

$C(G)$ is defined in terms of the set $\widehat{G}$ and the multiplicities $N_{ij}^k = \dim \mathrm{Hom}(\pi_k, \pi_i \otimes \pi_j)$, $i, j, k \in \widehat{G}$ (the "fusion rules" in physicist parlance). The same information is contained in the representation ring $R(G)$ provided that we take its canonical $\mathbb{Z}$-basis or its order structure [5] into account. We thus have the following corollary.

Corollary 3.2. The center of a compact group G depends only on the (ordered) representation ring $R(G)$, not on the associativity constraint or the symmetry of the tensor category Rep G. (In general, both the associativity constraint and the symmetry are needed to determine G up to isomorphism.) □

Remark 3.3. A considerably stronger result holds for *connected* compact groups. Every isomorphism of the (ordered) representation rings of two such groups is induced by an isomorphism of the groups, (cf. [5]). For nonconnected groups this is wrong: the finite groups D_{8l} and Q_{8l} are nonisomorphic but have isomorphic representation rings (cf. [5]). Yet, as remarked in [1, Section 5.1], the centers are isomorphic (to $\mathbb{Z}/2\mathbb{Z}$), as they must be by Corollary 3.2.

As an obvious consequence of Proposition 2.3 and Theorem 3.1 we have the following corollary.

Corollary 3.4. Let G be a compact group and A an abelian group. Then every t-map $\varphi : \widehat{G} \to A$ factors through $\widehat{Z(G)}$, that is, there is a homomorphism $\beta : \widehat{Z(G)} \to A$ of abelian groups such that

$$\begin{tikzcd} \widehat{G} \arrow[r, "r_G"] \arrow[dr, "\varphi"'] & \widehat{Z(G)} \arrow[d, "\beta"] \\ & A \end{tikzcd} \tag{3.2}$$

commutes. □

Remark 3.5. This result should be considered as dual to the well-known (and much easier) fact that every homomorphism $G \to A$ from a group into an abelian group factors through the quotient map $G \to G_{ab}$.

Remark 3.6. The results of this paper were formulated for compact groups mainly because of the author's taste and background. In view of [2], all results of this paper generalize without change to proreductive algebraic groups over algebraically closed fields of characteristic zero.

Acknowledgments

I am grateful to the DFG-Graduiertenkolleg "Hierarchie und Symmetrie in mathematischen Modellen" for supporting a one-week visit to the RWTH Aachen. In particular I thank Fernando Lledó for the invitation, drawing my attention to [1], many stimulating discussions, comments on the first version of this paper, and the pizza. The author was supported by NWO through the "Pioneer" project no. 616.062.384 of N. P. Landsman.

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