



Representations of Algebraic Quantum Groups and Reconstruction Theorems for Tensor Categories

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Abstract. We give a pedagogical survey of those aspects of the abstract representation theory of quantum groups which are related to the Tannaka–Krein reconstruction problem. We show that every concrete semisimple tensor $*$ -category with conjugates is equivalent to the category of finite-dimensional nondegenerate $*$ -representations of a discrete algebraic quantum group. Working in the self-dual framework of algebraic quantum groups, we then relate this to earlier results of S. L. Woronowicz and S. Yamagami. We establish the relation between braidings and R -matrices in this context. Our approach emphasizes the role of the natural transformations of the embedding functor. Thanks to the semisimplicity of our categories and the emphasis on representations rather than corepresentations, our proof is more direct and conceptual than previous reconstructions. As a special case, we reprove the classical Tannaka–Krein result for compact groups. It is only here that analytic aspects enter, otherwise we proceed in a purely algebraic way. In particular, the existence of a Haar functional is reduced to a well-known general result concerning discrete multiplier Hopf $*$ -algebras.

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1. Introduction

Pontryagin’s duality theory for locally compact Abelian groups and the Tannaka–Krein theory for compact groups are two major results in the theory of harmonic analysis on topological groups [14]. The Pontryagin duality theorem is a statement concerning characters, whereas the Tannaka–Krein theorem is a statement involving irreducible unitary representations. These two notions coincide whenever the group is both Abelian and compact.

Pontryagin’s theorem can be stated more generally for locally compact quantum groups [21], a notion evolving out of Kac algebras [11] and compact matrix pseudogroups [40].

The theory of representations is most naturally developed in the language of tensor categories [24]. The category of finite-dimensional representations of a compact group is a symmetric tensor $*$ -category with conjugates [9], and the Tannaka–Krein theorem tells us how to reconstruct the group from the latter. In 1988, starting from a tensor $*$ -category [12] with conjugates and admitting a generator, but assuming neither a symmetry nor a braiding, S. L. Woronowicz reconstructed [41] a compact matrix pseudogroup [40] having the given category as its category of finite-dimensional unitary corepresentations. No general definition of a compact quantum group existed at the time. Once it did, his proof generalizes to categories without generator. Alternatively, one can use the fact that every $*$ -category and every compact quantum group are inductive limits of categories with generators and of compact matrix pseudogroups, respectively.

For the Tannaka–Krein results mentioned so far, the starting point is a concrete category, i.e. a (nonfull) subcategory of the tensor category of Hilbert spaces. It is conceptually more satisfactory to start from an abstract tensor category together with a faithful tensor functor into the category of Hilbert spaces. In the work of N. Saavedra Rivano [31] the group associated with a concrete symmetric tensor category was identified as the group of natural monoidal automorphisms of the embedding functor as Tannaka had effectively done [33] before the advent of category theory. The idea of considering the natural transformations of the embedding functor was generalized in the work of K.-H. Ulbrich [34], where a given concrete tensor category is identified as the category of comodules over a Hopf algebra, cf. also [45].

While the role of the natural transformations of the embedding functor is obscure in Woronowicz’s approach, they appear at least implicitly in the work of S. Yamagami [43], who considered representations of discrete quantum groups. His approach has two drawbacks. On the one hand, he assumes that the category is equipped with an additional piece of structure, an ‘ ε -structure’. A very similar notion naturally arises in recent axiomatizations of tensor categories with two-sided duals [2], but it is quite superfluous if one works with $*$ -categories. This will become clear in our treatment. (Yamagami has recently proved a result [44, Theorem 3.6] implying that, passing if necessary to an equivalent tensor $*$ -category, there is an essentially unique ε -structure.) On the other hand, the von Neumann algebraic formulation of discrete quantum groups used in [43] is considerably more involved than current definitions, in that it unnecessarily mixes analytic and algebraic structures.

In neither of [40, 43] does one find a complete proof of equivalence between the given category and the representation category of the derived quantum group. Also questions of uniqueness and braidings have not been addressed.

It is well known that the self-duality of the category of finite-dimensional Hopf algebras breaks down in the infinite-dimensional case. Motivated by the desire to find a purely algebraic framework for quantum groups admitting a version of Pontryagin duality, A. van Daele developed the theory of algebraic quantum groups

(AQG) [37]. This is achieved by admitting nonunital algebras and requiring the existence of a positive and left-invariant functional, the Haar functional. All compact and all discrete quantum groups are AQG, and every AQG has a unique analytic extension to a locally compact quantum group [20], having an equivalent von Neumann algebraic version [22]. A discrete multiplier Hopf $*$ -algebra [35] can be shown to have a Haar functional rendering it a discrete AQG.

The purpose of the present paper is to give a coherent and reasonably complete survey of the Tannaka–Krein theory of quantum groups. The only other review we are aware of is [15], which appeared 10 years ago. At the time, no appropriate self-dual category of quantum groups existed. (In the same year, P. Podleś and Woronowicz, in defining a discrete quantum group, took the first step in this direction [28].) The approach to Tannakian categories motivated by algebraic geometry is well reviewed in [6], where, however, only symmetric tensor categories are considered.

Our approach to generalized Tannaka–Krein theory adopts the philosophy on quantum groups in [41, 43], meaning a self-dual category with emphasis on $*$ -categories. In contradistinction to these authors we wish to distinguish categorical from quantum group aspects as well as algebraic from analytic aspects as far as possible. Following [7, 31, 34, 45], we emphasize natural transformations of the embedding functor. Yet, our use of natural transformations is more direct and we work with representations rather than corepresentations. The algebra product is just the composition of natural transformations and the coproduct is defined directly in terms of the tensor structure, the tensor unit giving rise to the counit. This yields a quantum semigroup, whose discreteness is an immediate consequence of the semisimplicity of the category. (The semisimplicity of $*$ -categories avoids appealing to the theorem of Barr and Beck in [7, 31] and to proceed in a pedestrian way, using only the definition of natural transformations.) The coinverse now arises from the conjugation in the category. The result is a discrete multiplier Hopf $*$ -algebra. Thus our reconstruction of the discrete AQG is purely algebraic, the existence of the Haar functional following by quantum group theory.

The selfduality of the category of AQG and the existence of analytic extensions allow us to relate our reconstruction result to those of Woronowicz and Yamagami on one hand and the purely algebraic ones on the other [34, 45]. In particular, making use of the universal unitary corepresentation of an AQG introduced by J. Kustermans [18], we prove that the tensor $*$ -category of $*$ -representations (or modules) of a discrete AQG (A, Δ) is equivalent to the tensor $*$ -category of unitary corepresentations (or comodules) of the dual compact AQG $(\hat{A}, \hat{\Delta})$. We provide these categories with conjugates (for representations of discrete AQG this has not been worked out before). We also show that the tensor $*$ -category of pointwise continuous finite-dimensional $*$ -representations of a sub-Hopf $*$ -algebra of the maximal dual (or Sweedler dual) Hopf $*$ -algebra of a compact quantum group is equivalent to any of the tensor $*$ -categories mentioned above, whenever the Hopf $*$ -algebra separates the regular functions associated with the compact

quantum group. This is a useful result since it applies to the deformed universal enveloping Lie algebras $U_q(\mathfrak{g})$ of M. Jimbo and V. G. Drinfeld. It therefore links these axiomatic quantum group results to the more familiar context of quantum groups given by deformations of semisimple Lie algebras \mathfrak{g} . Indeed, this is how the latter can be shown to produce compact (or discrete) quantum groups and how Woronowicz [41] constructed the compact matrix pseudogroup $SU_q(N)$.

The correspondence between infinite-dimensional representations and corepresentations was already established in [28] and [18], but only for the objects of the respective categories, morphisms and the tensor structure were not considered. In [39] tensor structure and braiding were taken into account in a purely algebraic context and, while studying the amenability of quantum groups, the correspondence between tensor C^* -categories of infinite-dimensional representations and corepresentations was established in [3, 4], for the various analytic versions of AQG and LCQG. The latter results rely on the theory of infinite-dimensional representations and corepresentations and of the construction of the universal corepresentation for LCQG developed in [19]. But none of this work touched on Tannaka–Krein reconstruction, since conjugates do not exist in the infinite-dimensional case. Another type of reconstruction result for tensor C^* -categories involving infinite-dimensional objects was undertaken in [10] from the point of view of multiplicative unitaries and the regular corepresentation.

For a discrete AQG (A, Δ) we establish a bijection between braidings of the category $\text{Rep}_f(A, \Delta)$ and R -matrices in the multiplier algebra $M(A \otimes A)$. To the best of our knowledge this is the first such result rigorously proven in an axiomatic framework for quantum groups. If the category is symmetric and the embedding functor maps the braiding into the canonical braiding of the category of Hilbert spaces, the discrete AQG is cocommutative.

For any discrete AQG there is a compact group G , the intrinsic group, and for cocommutative (A, Δ) we prove an equivalence $\text{Rep}_f(A, \Delta) \simeq \text{Rep}_f G$ of tensor $*$ -categories. This is the only point in our approach where analysis plays a role, in that we use the theorems of Gelfand, Krein–Milman, Stone–Weierstrass, etc. In the theory of locally compact quantum groups it is well known that commutative and cocommutative quantum groups are Kac algebras. Furthermore, commutative (resp. cocommutative) LCQG are of the form $(C(G), \Delta)$ (resp. $(C_r^*(G), \Delta)$) for a locally compact group G . Finally, a commutative compact AQG is the algebra of regular functions on a compact group G , but cocommutative discrete AQG are inconvenient to characterize. This is why, alternatively, we give the more instructive and direct proof of the above equivalence of categories. In passing we give a description of a cocommutative AQG in terms of the intrinsic group G .

It cannot be emphasized enough that all Tannaka–Krein type results discussed so far depend on the tensor $*$ -category being concrete, i.e. coming with a faithful tensor functor into the category of Hilbert spaces. There are applications in pure mathematics and in quantum field theory, where such a functor is not given a priori. For symmetric categories, it was first shown by S. Doplicher and J. E. Roberts [9]

that such a functor always exists. An alternative approach in a more algebraic setting was given by P. Deligne [8]. The questions of existence and uniqueness of an embedding functor will be addressed anew in a sequel to this paper.

The above discussion did not follow the order of presentation. Let us therefore give a brief overview of the organization of this paper. In the next section we provide some preliminaries on tensor $*$ -categories and AQG. Concerning the former we are quite brief, since much of this material is almost universally known. Concerning the latter we focus in particular on the discrete and compact cases and discuss the examples related to groups. In Section 3 we treat the representation and corepresentation theory of discrete and compact AQG, respectively, from a tensor $*$ -category point of view. To this end, we recall the universal corepresentation due to Kustermans, and we discuss conjugates in these categories. The special case of cocommutative discrete AQG is considered in Section 4. Section 5 is the heart of this paper. There we construct a discrete quantum semigroup from a tensor $*$ -category, deriving the coinverse from the conjugation and leading to a discrete AQG. The precise statement of the generalized Tannaka–Krein theorem for quantum groups is then made in Theorem 5.25. In the final Section 6 we establish the bijection between braidings and R -matrices. In the case of a symmetric tensor $*$ -category with symmetric embedding functor we recover the classical Tannaka–Krein theorem for compact groups.

2. Preliminaries

2.1. TENSOR CATEGORIES

For the definitions of categories, functors and natural transformations we refer to [24]. In this subsection we briefly recall some of the less standard notions of category theory which will be needed here, others will be introduced as we proceed. We may occasionally say ‘arrows’ instead of ‘morphisms’ in order to avoid confusion with algebra homomorphisms. All categories which we consider are essentially small, i.e. equivalent to a small category. We mostly speak of ‘tensor categories’ rather than ‘monoidal categories’ but cannot avoid the adjective ‘monoidal’. In view of the coherence theorems for (braided, symmetric) tensor categories we may assume all tensor categories to be strict, satisfying $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ for all X, Y, Z . Following widespread use, we also consider the tensor categories of vector spaces as strict, appealing to the canonical isomorphisms to identify $X \otimes (Y \otimes Z)$ with $(X \otimes Y) \otimes Z$. We often write XY in the place of $X \otimes Y$. By \mathcal{H} we mean the strictification of Hilb . (We also suppress all other canonical isomorphisms in Hilb , identifying $B(H \otimes K) = B(H) \otimes B(K)$.) Note, however, that one cannot assume all tensor functors to be strict without losing generality. Thus we need the following.

DEFINITION 2.1. Let \mathcal{C}, \mathcal{D} be (strict) tensor categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is weakly monoidal if there exist morphisms $d_{X,Y}: F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ for all $X, Y \in \mathcal{C}$ and a morphism $e: \mathbf{1}_{\mathcal{D}} \rightarrow F(\mathbf{1}_{\mathcal{C}})$ such that

- (1) The morphisms $d_{X,Y}$ are natural w.r.t. both arguments.
- (2) For all $X, Y, Z \in \mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(X) \otimes F(Y) \otimes F(Z) & \xrightarrow{d_{X,Y} \otimes \text{id}_{F(Z)}} & F(X \otimes Y) \otimes F(Z) \\ \text{id}_{F(X)} \otimes d_{Y,Z} \downarrow & & \downarrow d_{X \otimes Y, Z} \\ F(X) \otimes F(Y \otimes Z) & \xrightarrow{d_{X, Y \otimes Z}} & F(X \otimes Y \otimes Z) \end{array}$$

- (3) The following compositions are the identity morphisms of $F(X)$

$$F(X) \equiv F(X) \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\text{id}_{F(X)} \otimes e} F(X) \otimes F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{d_{X, \mathbf{1}}} F(X \otimes \mathbf{1}_{\mathcal{C}}) \equiv F(X)$$

$$F(X) \equiv \mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{e \otimes \text{id}_{F(X)}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{d_{\mathbf{1}, X}} F(\mathbf{1}_{\mathcal{C}} \otimes X) \equiv F(X)$$

for all $X \in \mathcal{C}$.

The functor F is monoidal if e and all the $d_{X,Y}$, $X, Y \in \mathcal{C}$ are isomorphisms. If \mathcal{C}, \mathcal{D} are tensor $*$ -categories and F is $*$ -preserving, the isomorphisms $e, d_{X,Y}$ are required to be unitary.

We will mainly be interested in linear categories over the complex field. Viz., for all X, Y , $\text{Mor}(X, Y)$ is a complex vector space, and the compositions \circ, \otimes of morphisms are bilinear. All functors are supposed to be \mathbb{C} -linear. A $*$ -operation on a \mathbb{C} -linear (tensor) category \mathcal{C} is a map which assigns to an arrow $s \in \text{Mor}(X, Y)$ another arrow $s^* \in \text{Mor}(Y, X)$. This map has to be antilinear, involutive ($(s^*)^* = s$), contravariant ($(s \circ t)^* = t^* \circ s^*$) and, if \mathcal{C} is monoidal, monoidal ($(s \times t)^* = s^* \times t^*$). A $*$ -operation is *positive* iff $s^* \circ s = 0$ implies $s = 0$. A (tensor) $*$ -category is a \mathbb{C} -linear (tensor) category with a positive $*$ -operation. A morphism $s: X \rightarrow Y$ is an isometry iff it satisfies $s^* \circ s = \text{id}_X$. A morphism s is unitary iff s and s^* are isometries. A functor F between $*$ -categories is $*$ -preserving if $F(s^*) = F(s)^*$ for every morphism s .

An object Z is a direct sum of X_1, X_2 , denoted $Z \cong X_1 \oplus X_2$, if there are isometries $s_i: X_i \rightarrow Z$, $i = 1, 2$ such that $s_1 \circ s_1^* + s_2 \circ s_2^* = \text{id}_Z$. A $*$ -category ‘has direct sums’ if for every pair X_1, X_2 there exists a direct sum. An object Y is a subobject of X if there exists an isometry $s: Y \rightarrow X$. A $*$ -category ‘has subobjects’ if for every $p = p \circ p = p^* \in \text{End}(X)$ there exist Y and an isometry $s: Y \rightarrow X$ such that $s \circ s^* = p$. An object X in a \mathbb{C} -linear category is irreducible if it is nonzero and $\text{End } X = \mathbb{C} \text{id}_X$.

DEFINITION 2.2. A $*$ -category is semisimple if it has finite-dimensional spaces of morphisms, a zero object, direct sums and subobjects. A tensor $*$ -category is semisimple if, in addition, the tensor unit $\mathbf{1}$ is irreducible.

In a semisimple $*$ -category \mathcal{C} , $\text{End}(X)$ is a finite-dimensional C^* -algebra for every X , and every object is a finite direct sum of irreducible objects. (It is well-known that a category which is semisimple in our sense is Abelian and semisimple in the usual sense, i.e. all exact sequences split.)

Let \mathcal{C} be a tensor $*$ -category and $X \in \mathcal{C}$. A ‘solution of the conjugate equations’ is a triple (\bar{X}, r, \bar{r}) , where $\bar{X} \in \mathcal{C}$ and $r: \mathbf{1} \rightarrow \bar{X} \otimes X$, $\bar{r}: \mathbf{1} \rightarrow X \otimes \bar{X}$ satisfy

$$\begin{aligned} \bar{r}^* \otimes \text{id}_X \circ \text{id}_X \otimes r &= \text{id}_X, \\ r^* \otimes \text{id}_{\bar{X}} \circ \text{id}_{\bar{X}} \otimes \bar{r} &= \text{id}_{\bar{X}}. \end{aligned}$$

A tensor $*$ -category \mathcal{C} ‘has conjugates’ if there is a solution of the conjugate equations for every $X \in \mathcal{C}$. A solution (X, r, \bar{r}) is normalized iff $r^* \circ r = \bar{r}^* \circ \bar{r}$. It is a standard solution iff there are irreducible objects $X_i, i \in I_X$, solutions $(\bar{X}_i, r_i, \bar{r}_i)$ of the conjugate equations and isometries $v_i: X_i \rightarrow X, w_i: \bar{X}_i \rightarrow \bar{X}, i \in I_X$ satisfying

$$v_i^* \circ v_j = \delta_{ij} \text{id}_{X_i}, \quad \sum_i v_i \circ v_i^* = \text{id}_X,$$

similar equations for w_i , and we have

$$r = \sum_i w_i \otimes v_i \circ r_i, \quad \bar{r} = \sum_i v_i \otimes w_i \circ \bar{r}_i.$$

We define the (intrinsic or categorical) dimension $d(X) \in \mathbb{R}_+$ of X by $r^* \circ r = d(X) \text{id}_{\mathbf{1}}$ where (\bar{X}, r, \bar{r}) is a normalized standard solution. One can prove the following facts, cf. [23]. The dimension is additive under direct sums and multiplicative under tensor products. It takes values in the set $\{2 \cos \frac{\pi}{n}, n = 3, 4, \dots\} \cup [2, \infty)$, in particular $d(X) \geq 1$ with $d(X) = 1$ iff $X \otimes \bar{X} \cong \mathbf{1}$ iff X is invertible, i.e. there exists Y such that $X \otimes Y \cong \mathbf{1}$. If $d(X) = 1$ then X is irreducible. In the category \mathcal{H} of Hilbert spaces we have $d_{\mathcal{H}}(H) = \dim_{\mathbb{C}} H$.

We briefly comment on a somewhat more general setting. A C^* -(tensor) category is a (tensor) $*$ -category where $\text{Mor}(X, Y)$ is a Banach space for every pair (X, Y) of objects and the norms satisfy $\|X^* \circ X\| = \|X\|^2$. A W^* -category is a C^* -category where every $\text{Mor}(X, Y)$ is the dual of a Banach space for every pair (X, Y) . In a C^* -category with conjugates and irreducible tensor unit all spaces of morphisms are finite-dimensional. This is useful in applications where this finite-dimensionality is not known a priori, like in quantum field theory. Conversely, every $*$ -category which is semisimple in our sense is a W^* -category, cf. [25]. In a W^* -category, every morphism $s: X \rightarrow Y$ has a polar decomposition $s = p \circ u$, where p is positive and u a partial isometry. As a consequence, whenever $\text{Mor}(X, Y)$ contains a split monic (or isomorphism), it also contains an isometry (respectively, unitary). This shows that most of the above definitions, e.g., of direct sums, are equivalent to the usual ones as given in [24].

If \mathcal{C} is a semisimple tensor $*$ -category we denote the set of isomorphism classes of irreducible objects by $I_{\mathcal{C}}$. Let $(X_i, i \in I_{\mathcal{C}})$ be a complete set of irreducible

objects and write $d_i = d(X_i)$. We have $0 \in I_{\mathcal{C}}$ such that $X_0 \cong \mathbf{1}$. If \mathcal{C} has conjugates, $I_{\mathcal{C}}$ comes with an involution $i \mapsto \bar{i}$ such that $X_{\bar{i}}$ is a conjugate of X_i . For $i, j, k \in I_{\mathcal{C}}$ we define $N_{ij}^k = \dim \text{Mor}(X_k, X_i \otimes X_j)$. These numbers satisfy the following properties.

- (1) For every pair (i, j) there are only finitely many $k \in I_{\mathcal{C}}$ such that $N_{ij}^k \neq 0$. We have $X_i \otimes X_j \cong \bigoplus_{k \in I_{\mathcal{C}}} N_{ij}^k X_k$ and thus $d_i d_j = \sum_k N_{ij}^k d_k$.
- (2) $\sum_l N_{ij}^l N_{lk}^m = \sum_n N_{in}^m N_{jk}^n$ for all $i, j, k, m \in I_{\mathcal{C}}$.
- (3) $N_{ij}^k = N_{k\bar{j}}^i = N_{\bar{i}k}^j = N_{\bar{k}i}^{\bar{j}} = N_{j\bar{k}}^{\bar{i}} = N_{\bar{j}, \bar{i}}^{\bar{k}}$ for all $i, j, k \in I_{\mathcal{C}}$.
- (4) $N_{ij}^0 = \delta_{i, \bar{j}}$.
- (5) If \mathcal{C} is braided, cf. Section 6, then $N_{ij}^k = N_{ji}^k$ for all $i, j, k \in I_{\mathcal{C}}$.

2.2. ALGEBRAIC QUANTUM GROUPS (AQG)

In this subsection we briefly outline those aspects of the theory of AQG that will be needed in the sequel. For the details and proofs, see the original references [35, 37].

Every algebra will be a (not necessarily unital) associative algebra over the complex field \mathbb{C} . The identity map on a set V will be denoted by ι . If V and W are linear spaces, V' denotes the linear space of linear functionals on V and $V \otimes W$ denotes the linear space tensor product of V and W . The *flip map* σ from $V \otimes W$ to $W \otimes V$ is the linear map sending $v \otimes w$ onto $w \otimes v$, for all $v \in V$ and $w \in W$. If V and W are Hilbert spaces, $V \otimes W$ denotes their Hilbert space tensor product; we denote by $B(V)$ and $B_0(V)$ the C^* -algebras of bounded linear operators and compact operators on V , respectively. If V and W are algebras, $V \otimes W$ denotes their algebra tensor product. If V and W are C^* -algebras, then $V \otimes W$ will denote their C^* -tensor product with respect to the minimal C^* -norm.

An algebra A is *nondegenerate* if for any $a \in A$ such that $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, we have $a = 0$. Obviously, all unital algebras are nondegenerate. If A and B are nondegenerate algebras, so is $A \otimes B$. From now on *all* algebras are assumed to be nondegenerate.

Let A be a $*$ -algebra and denote by $\text{End } A$ the unital algebra of linear maps from A to itself. Let

$$M(A) = \{x \in \text{End } A \mid \exists y \in \text{End } A \text{ such that } x(a)^* b = a^* y(b) \forall a, b \in A\}.$$

Then $M(A)$ is a unital subalgebra of $\text{End } A$. The linear map y associated to a given $x \in M(A)$ is uniquely determined by nondegeneracy and we denote it by x^* . The unital algebra $M(A)$ becomes a $*$ -algebra when endowed with the involution $x \mapsto x^*$. This unital $*$ -algebra is called the *multiplier algebra* of A .

Suppose that A is an ideal in a $*$ -algebra B . For $b \in B$, define $L_b \in M(A)$ by $L_b(a) = ba$, for all $a \in A$. Then the map $L: B \rightarrow M(A)$, $b \mapsto L_b$, is a homomorphism. If A is an *essential* ideal in B in the sense that an element b of B is necessarily equal to zero if $ba = 0$, for all $a \in A$, or $ab = 0$, for all $a \in A$, then L is injective. In particular, A is an essential ideal in itself (by nondegeneracy).

and therefore we have an injective homomorphism $L: A \rightarrow M(A)$. We identify the image of A under L with A . Then A is an essential ideal of $M(A)$. (In fact, $M(A)$ is the ‘largest’ algebra containing A as an essential ideal.) Obviously, $M(A) = A$ iff A is unital.

If A and B are $*$ -algebras, then it is easily verified that $A \otimes B$ is an essential self-adjoint ideal in $M(A) \otimes M(B)$. Hence, by the preceding remarks, there exists a canonical injective $*$ -homomorphism from $M(A) \otimes M(B)$ into $M(A \otimes B)$. We use this to identify $M(A) \otimes M(B)$ as a unital $*$ -subalgebra of $M(A \otimes B)$. In general, these algebras are not equal. A linear map $\pi: A \rightarrow B$ is said to be *nondegenerate* if $\pi(A)B = B$ and $B\pi(A) = B$. Here, as elsewhere, $\pi(A)B$ denotes the linear span of $\{\pi(a)b \mid a \in A, b \in B\}$. Whenever π is nondegenerate and multiplicative (resp. nondegenerate and antimultiplicative), there exists a unique extension to a unital homomorphism (resp. antihomomorphism) $\bar{\pi}: M(A) \rightarrow M(B)$. We shall henceforth use the same symbol π to denote the original map and its extension $\bar{\pi}$. A *representation* of a $*$ -algebra is a nondegenerate homomorphism $\pi: A \rightarrow B(K)$, where K is a Hilbert space.

If ω is a linear functional on A and $x \in M(A)$, we define the linear functionals $x\omega$ and ωx on A by setting $(x\omega)(a) = \omega(ax)$ and $(\omega x)(a) = \omega(xa)$, for all $a \in A$. We need the leg numbering notation. Take three $*$ -algebras A, B, C . It can be shown that there exists a nondegenerate $*$ -homomorphism $\theta_{13}: A \otimes C \rightarrow M(A \otimes B \otimes C)$ such that $\theta_{13}(a \otimes c) = a \otimes 1 \otimes c$, for all $a \in A, c \in C$. Thus, it has a unique extension to $M(A \otimes C)$. Set $x_{13} = \theta_{13}(x)$, for all $x \in M(A \otimes C)$. The other variants of the leg numbering notation are defined similarly.

The triple product $A \otimes A \otimes A$ is an essential ideal of both $M(A \otimes A) \otimes A$ and $A \otimes M(A \otimes A)$, thus $M(M(A \otimes A) \otimes A) \subset M(A \otimes A \otimes A)$ and $M(A \otimes M(A \otimes A)) \subset M(A \otimes A \otimes A)$.

DEFINITION 2.3. A multiplier Hopf $*$ -algebra (A, Δ) consists of a $*$ -algebra A and a $*$ -homomorphism Δ from A into $M(A \otimes A)$ such that

- (1) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- (2) The linear mappings T_1, T_2 from $A \otimes A$ into $M(A \otimes A)$ such that

$$T_1(a \otimes b) = \Delta(a)(b \otimes 1), \quad T_2(a \otimes b) = \Delta(a)(1 \otimes b)$$

for all $a, b \in A$, are bijections from $A \otimes A$ to $A \otimes A$.

Here the (unique) extension of $\Delta \otimes \iota: A \otimes A \rightarrow M(A \otimes A \otimes A)$ to $M(A \otimes A)$ is understood and similarly for $\iota \otimes \Delta$. Condition (ii) implies that $\Delta(a)(b \otimes 1), \Delta(a)(1 \otimes b) \in M(A \otimes A)$ lie in fact in $A \otimes A$.

We say that two multiplier Hopf $*$ -algebras (A_1, Δ_1) and (A_2, Δ_2) are *isomorphic*, and write $(A_1, \Delta_1) \cong (A_2, \Delta_2)$, if there exists a bijective $*$ -homomorphism $\theta: A_1 \rightarrow A_2$ such that $(\theta \otimes \theta)\Delta_1 = \Delta_2\theta$.

The following result shows that multiplier Hopf $*$ -algebras share most properties with Hopf algebras. Let $m: A \otimes A \rightarrow A$ denote the linear extension of the multiplication map. Note that m is a $*$ -homomorphism iff A is commutative.

PROPOSITION 2.4. *Let (A, Δ) be a multiplier Hopf $*$ -algebra. There exist a unique $*$ -homomorphism $\varepsilon: A \rightarrow \mathbb{C}$ and a unique invertible antimultiplicative $S \in \text{End } A$ such that*

- (1) $(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab$,
- (2) $(\iota \otimes \varepsilon)(\Delta(a)(b \otimes 1)) = ab$,
- (3) $m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b$,
- (4) $m(\iota \otimes S)(\Delta(a)(b \otimes 1)) = \varepsilon(a)b$

for all $a, b \in A$. We call ε the counit and S the coinverse. Conversely, if $(A \otimes 1)\Delta(A) \subset A \otimes A$ and $(A \otimes 1)\Delta(A) \subset A \otimes A$ and there exist linear maps $\varepsilon: A \rightarrow \mathbb{C}$ and $S: A \rightarrow A$ satisfying properties (1)–(4) with S invertible, then (A, Δ) is a multiplier Hopf $$ -algebra.*

We denote $m_{\text{op}} = m\sigma$ and $\Delta_{\text{op}} = \sigma\Delta$ where $\sigma(a \otimes b) = b \otimes a$ is the flip $*$ -automorphism of $A \otimes A$ and $M(A \otimes A)$. A_{op} denotes the vector space A with multiplication m_{op} . The following result follows easily from the uniqueness of the counit and coinverse.

PROPOSITION 2.5. *The antipode S is invertible and*

- (1) $S^{-1}(a) = S(a^*)^* \forall a \in A$.
- (2) $(S \otimes S)(\Delta(a)(1 \otimes b)) = (1 \otimes S(b))\Delta_{\text{op}}S(a) \forall a, b \in A$.
- (3) $\varepsilon S = \varepsilon$.
- (4) (A_{op}, Δ) and (A, Δ_{op}) are multiplier Hopf $*$ -algebras with counit ε and coinverse S^{-1} .

DEFINITION 2.6. A linear functional ω on a multiplier Hopf $*$ -algebra (A, Δ) is called

- (1) positive iff $\omega(a^*a) \geq 0 \forall a \in A$,
- (2) faithful iff it is positive and $\omega(a^*a) = 0 \Rightarrow a = 0 \forall a \in A$,
- (3) left-invariant iff $(\iota \otimes \omega)(\Delta(a)(b \otimes 1)) = \omega(a)b \forall a, b \in A$,
- (4) right-invariant iff $(\omega \otimes \iota)((1 \otimes b)\Delta(a)) = \omega(a)b \forall a, b \in A$.

Thus ω is left-invariant iff ωS is right-invariant.

DEFINITION 2.7. An AQG is a multiplier Hopf $*$ -algebra (A, Δ) which admits a nonzero left-invariant positive linear functional φ .

THEOREM 2.8. *Let (A, Δ) be an AQG with positive left-invariant functional φ . Then*

- (1) φ is faithful.
- (2) If ω is a left-invariant functional, there exists $c \in \mathbb{C}$ such that $\omega = c\varphi$.
- (3) If ω is a right-invariant functional, there exists $c \in \mathbb{C}$ such that $\omega = c\varphi S$.

An immediate consequence is that the $*$ -operation is positive, viz. $a^*a = 0 \Rightarrow a = 0$, which clearly is stronger than nondegeneracy. Furthermore, there exists a unique complex number μ such that $\varphi S^2 = \mu\varphi$. It can be proved that $|\mu| = 1$, but it has recently been discovered that $\mu \neq 1$ for the quantum group version of $ax + b$ [38]. Every AQG (A, Δ) has the property that to any $a \in A$, there exists $c \in A$ such that $ac = ca = a$. Two AQG are said to be *isomorphic* if they are isomorphic as multiplier Hopf $*$ -algebras and we use the same notation \cong to denote this.

Set $\psi = \varphi S$. Then ψ is a nonzero right-invariant linear functional on A . However, in general, ψ will not be positive. It is known that there exists a nonzero positive right-invariant linear functional on A . Associated to φ there exists a unique automorphism ρ on A such that $\varphi(ab) = \varphi(b\rho(a))$, for every $a, b \in A$. We refer to this as the weak KMS property of φ . Moreover, we have $\Delta\rho = (S^2 \otimes \rho)\Delta$ and $\rho(\rho(a^*)^*) = a$, for every $a \in A$. Also there exists an automorphism ρ' for the right-invariant functional ψ , that is, ρ' is an automorphism on A such that $\psi(ab) = \psi(b\rho'(a))$, for every $a, b \in A$.

It is possible to introduce a modular function for AQG. It is an invertible element δ in $M(A)$ such that $(\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) = \varphi(a)\delta b$, for every $a, b \in A$. This modular function satisfies

$$\Delta(\delta) = \delta \otimes \delta, \quad \varepsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}.$$

As in the classical case we have

$$\varphi(S(a)) = \varphi(a\delta) = \mu\varphi(\delta a) \quad \forall a \in A.$$

We now discuss duality within the category of AQG. Define a subspace \hat{A} of A' by

$$\hat{A} = \{\varphi a \mid a \in A\} = \{a\varphi \mid a \in A\}.$$

Then \hat{A} is a nondegenerate $*$ -algebra under the definitions

- (1) For every $\omega_1, \omega_2 \in \hat{A}$ and $a \in A$, we have $(\omega_1\omega_2)(a) = (\omega_1 \otimes \omega_2)(\Delta(a))$.
- (2) For every $\omega \in \hat{A}$ and $a \in A$, define $\omega^*(a) = \omega(S(a)^*)$.

The comultiplication $\hat{\Delta}$ is defined on \hat{A} by $\hat{\Delta}(\omega)(x \otimes y) = \omega(xy)$, for every $\omega \in \hat{A}$ and $x, y \in A$. For this to make sense, $M(A \otimes A)$ should be embedded in $(A \otimes A)'$ in a proper way. A definition of the comultiplication $\hat{\Delta}$ that does not use such an embedding can be found in [18]. Then $(\hat{A}, \hat{\Delta})$ is a multiplier Hopf $*$ -algebra with counit $\hat{\varepsilon}$ and the antipode \hat{S} given by

- (1) $\hat{\varepsilon}(\omega) = \omega(1)$, for every $\omega \in \hat{A}$.
- (2) $\hat{S}(\omega)(a) = \omega(S(a))$, for every $\omega \in \hat{A}$ and $a \in A$.

Define $\hat{a} = a\varphi \in \hat{A}$, for $a \in A$. The map $A \rightarrow \hat{A}$ sending a to \hat{a} is a bijection, and is referred to as the Fourier transform. Next define the linear functional $\hat{\psi}$ on \hat{A} by setting $\hat{\psi}(\hat{a}) = \varepsilon(a)$, for every $a \in A$. It is possible to prove that $\hat{\psi}$ is

right-invariant, and that $\hat{\psi}(\hat{a}^*\hat{a}) = \varphi(a^*a)$, for every $a \in A$. It follows that $\hat{\psi}$ is a nonzero positive linear functional on \hat{A} , and that the dual $(\hat{A}, \hat{\Delta})$ is an AQG. Let $\hat{\rho}$ denote the automorphism of \hat{A} such that $\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$, for all $a, b \in A$. The AQG version of Pontryagin's duality theorem takes the following form:

THEOREM 2.9. *Let (A, Δ) be an AQG. Then the double dual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ is an AQG isomorphic to (A, Δ) . More precisely, there exists a canonical $*$ -isomorphism $\theta: A \rightarrow \hat{\hat{A}}$ such that $\theta(a)(\omega) = \omega(a)$, for all $a \in \hat{A}$, and that satisfies the equation $(\theta \otimes \theta)\Delta = \hat{\hat{\Delta}}\theta$.*

Note that an AQG (A, Δ) is commutative iff $(\hat{A}, \hat{\Delta})$ is cocommutative.

We shall need to consider an object associated to an AQG called its analytic extension. See [20] for full details. If A is a C^* -algebra, then $M(A)$ denotes the usual multiplier algebra of A , and $*$ -homomorphism between C^* -algebras are assumed to be nondegenerate in the usual operator algebraic sense. We need first to recall the concept of a GNS pair. Suppose given a positive linear functional ω on a $*$ -algebra A . Let H be a Hilbert space, and let $\Lambda: A \rightarrow H$ be a linear map with dense range for which $(\Lambda(a), \Lambda(b)) = \omega(b^*a)$, for all $a, b \in A$. Then we call (H, Λ) a *GNS pair* associated to ω . As is well known, such a pair always exists and is essentially unique.

If φ is a left Haar integral on an AQG (A, Δ) , and (H, Λ) is an associated GNS pair, then it can be shown that there is a unique $*$ -homomorphism $\pi: A \rightarrow B(H)$ such that $\pi(a)\Lambda(b) = \Lambda(ab)$, for all $a, b \in A$. Moreover, π is faithful and nondegenerate. We let A_r denote the norm closure of $\pi(A)$ in $B(H)$. Thus, A_r is a nondegenerate C^* -subalgebra of $B(H)$. There exists a unique nondegenerate $*$ -homomorphism $\Delta_r: A_r \rightarrow M(A_r \otimes A_r)$ such that, for all $a \in A$ and all $x \in A \otimes A$, we have $\Delta_r(\pi(a))(\pi \otimes \pi)(x) = (\pi \otimes \pi)(\Delta(a)x)$ and $(\pi \otimes \pi)(x)\Delta_r(\pi(a)) = (\pi \otimes \pi)(x\Delta(a))$.

Define a linear map $\hat{\Lambda}: \hat{A} \rightarrow H$ by setting $\hat{\Lambda}(\hat{a}) = \Lambda(a)$, for all $a \in \mathcal{A}$. Since $\hat{\psi}(\hat{b}^*\hat{a}) = \varphi(b^*a) = (\Lambda(a), \Lambda(b))$, for all $a, b \in A$, it follows that $(H, \hat{\Lambda})$ is a GNS-pair associated to $\hat{\psi}$. It can be shown that it is unitarily equivalent to the GNS-pair for a left Haar integral $\hat{\varphi}$ of $(\hat{A}, \hat{\Delta})$. Hence, we can use $(H, \hat{\Lambda})$ to define a representation of the analytic extension $(\hat{A}_r, \hat{\Delta}_r)$ of $(\hat{A}, \hat{\Delta})$ on the space H . There is a unique $*$ -homomorphism $\hat{\pi}_r: \hat{A} \rightarrow B(H)$ such that $\hat{\pi}_r(a)\hat{\Lambda}(b) = \hat{\Lambda}(ab)$, for all $a, b \in \hat{A}$. Moreover, $\hat{\pi}_r$ is faithful and nondegenerate. Let \hat{A}_r be the norm closure of $\hat{\pi}_r(A)$ in $B(H)$, so \hat{A}_r is a nondegenerate C^* -subalgebra of $B(H)$. There exist a unique $*$ -homomorphism $\hat{\Delta}_r: \hat{A}_r \rightarrow M(\hat{A}_r \otimes \hat{A}_r)$ such that, for all $a \in \hat{A}$ and $x \in \hat{A} \otimes \hat{A}$, $\hat{\Delta}_r(\hat{\pi}_r(a))(\hat{\pi}_r \otimes \hat{\pi}_r)(x) = (\hat{\pi}_r \otimes \hat{\pi}_r)(\hat{\Delta}(a)x)$ and $(\hat{\pi}_r \otimes \hat{\pi}_r)(x)\hat{\Delta}_r(\hat{\pi}_r(a)) = (\hat{\pi}_r \otimes \hat{\pi}_r)(x\hat{\Delta}(a))$.

2.3. DISCRETE AND COMPACT CASE

The following proposition is well known and easy to prove.

PROPOSITION 2.10. *Let I be a set and $H_i, i \in I$ finite-dimensional Hilbert spaces. Consider the algebraic direct sum $A = \bigoplus_{i \in I} B(H_i)$.*

- (1) *A is a $*$ -algebra with the usual $*$ -operation on $B(H_i)$.*
- (2) *A is unital iff I is finite.*
- (3) *The canonical embedding maps $\iota_i: B(H_i) \hookrightarrow A$ and projections $p_i: A \rightarrow B(H_i)$ are $*$ -homomorphisms.*
- (4) *Let $M(A)$ be the multiplier algebra and $L: A \rightarrow M(A)$ the canonical embedding map. There exists a unique isomorphism $P: M(A) \rightarrow \prod_{i \in I} B(H_i)$ such that $PL = \text{id}$, where we use the obvious identification of the direct sum with a subalgebra of the direct product.*
- (5) *If π is an irreducible $(*)$ -representation of A there is $i \in I$ and an invertible (unitary) $U \in B(H, H_i)$ unique up to a scalar such that $\pi(\cdot) = \pi \iota_i p_i(\cdot) = U p_i(\cdot) U^{-1}$.*
- (6) *Any $(*)$ -representation of A is a direct sum of irreducible $(*)$ -representations.*

From now on the isomorphism P will be used to make an identification $M(A) \equiv \prod_{i \in I} B(H_i)$ and therefore suppressed.

DEFINITION 2.11. A multiplier Hopf $*$ -algebra (A, Δ) is called discrete if $A = \bigoplus_i B(H_i)$. An AQG is called discrete if it is discrete as a multiplier Hopf $*$ -algebra.

The proof of the following result is included, since it is instructive for our purposes.

THEOREM 2.12. *Let (A, Δ) be a discrete multiplier Hopf $*$ -algebra. Then there is a unique $f \in M(A)$ satisfying*

- (1) *f is positive and invertible,*
- (2) *$S^2(a) = f a f^{-1} \forall a \in A,$*
- (3) *$\text{Tr}_i p_i(f) = \text{Tr}_i p_i(f^{-1}),$ where Tr_i is the usual trace on $B(H_i)$.*

It also satisfies $S(f) = f^{-1}$ (where the unique extension of S to $M(A)$ is understood).

Proof. To simplify the formulae we introduce the antilinear map $\chi(a) = S(a^*)$ which satisfies $\chi^2 = \text{id}_A$. For every $i \in I$ we pick an antiunitary $J_i: H_i \rightarrow H_i$. Now, $J_i p_i(\cdot) J_i^{-1}$ is a (linear) irreducible representation of A on H_i for any $i \in I$. By Prop. 2.10 there is a unique $c(i) \in I$ and an invertible $V_i: H_i \rightarrow H_{c(i)}$ such that

$$V_i J_i p_i \chi(\cdot) J_i^{-1} V_i^{-1} = p_{c(i)}. \tag{2.1}$$

Now $U_i \equiv V_i J_i$ is unique up to a scalar. We calculate

$$U_i p_i(a) U_i^{-1} = U_i p_i(\chi \chi(a)) U_i^{-1} = p_{c(i)}(\chi(a)) = U_{c(i)}^{-1} p_{c(c(i))} U_{c(i)}.$$

In the first step we have used $\chi^2 = \text{id}$ and then we have used (2.1) twice. This shows that the irreducible representations p_i and $p_{c(c(i))}$ are equivalent and therefore c is involutive: $c(c(i)) = i$. From now on we write $\bar{i} = c(i)$. We have

$$U_{\bar{i}}U_i p_i(\cdot)U_i^{-1}U_{\bar{i}} = p_i(\cdot),$$

and by irreducibility $U_{\bar{i}}U_i = k_i 1 \in B(H_i)$. Using the freedom (up to a scalar) in the choice of the U_i we can assume that $k_i = 1 \forall i \in I$. (If $i = \bar{i}$ we can absorb k_i in U_i . If $i \neq \bar{i}$, we can achieve $U_{\bar{i}}U_i = 1$, and since the U_i are invertible this implies that automatically $k_{\bar{i}} = 1$.)

From (2.1) we obtain $p_i S(a) = U_i^{-1} p_{\bar{i}}(a^*)U_i$ and applying this twice yields

$$\begin{aligned} p_i S^2(a) &= U_i^{-1} p_{\bar{i}} S(a)^* U_i = U_i^{-1} [U_{\bar{i}}^{-1} p_{\bar{i}}(a^*) U_{\bar{i}}]^* U_i \\ &= U_i^{-1} U_{\bar{i}}^* p_i(a) U_{\bar{i}}^{-1*} U_i. \end{aligned}$$

Thus $p_i S^2(a) = G_i p_i(a) G_i^{-1}$ with $G_i = U_i^{-1} U_{\bar{i}}^* = k_i^{-1} U_{\bar{i}} U_{\bar{i}}^*$ positive and invertible for all $i \in I$. Defining

$$F_i = \sqrt{\frac{\text{Tr}_i G_i^{-1}}{\text{Tr}_i G_i}} G_i,$$

we have

$$\text{Tr}_i F_i = \sqrt{\text{Tr}_i G_i^{-1} \text{Tr}_i G_i} = \text{Tr}_i F_i^{-1}$$

as required. Obviously, also the F_i are positive and invertible and $p_i S^2(a) = F_i p_i(a) F_i^{-1}$. The uniqueness is obvious in view of the normalizations. It remains to prove $S(f) = f^{-1}$, which is clearly equivalent to $p_i S(f) = p_i(f^{-1}) \forall i$. As above, with (2.1) we have $p_i S(f) = U_i^{-1} p_{\bar{i}}(f^*)U_i = U_i^{-1} F_{\bar{i}}^* U_i$. On the other hand, $p_i(f^{-1}) = F_i^{-1}$. Thus we have to show

$$U_i^{-1} F_{\bar{i}}^* U_i = F_i^{-1} \quad \forall i \in I.$$

Inserting the definition of the F_i this is equivalent to

$$U_i^{-1} \sqrt{\frac{\text{Tr}_{\bar{i}} G_{\bar{i}}^{-1}}{\text{Tr}_{\bar{i}} G_{\bar{i}}}} U_i U_{\bar{i}}^{-1*} U_i = \sqrt{\frac{\text{Tr}_i G_i}{\text{Tr}_i G_i^{-1}}} U_{\bar{i}}^{*-1} U_i \quad \forall i \in I.$$

This is clearly equivalent to

$$\text{Tr}_{\bar{i}} G_{\bar{i}}^{-1} \text{Tr}_i G_i^{-1} = \text{Tr}_i G_i \text{Tr}_{\bar{i}} G_{\bar{i}} \quad \forall i \in I.$$

Plugging in $G_i = U_i^{-1} U_{\bar{i}}^*$ this is an easy computation which we omit. (One uses $U_{\bar{i}} = U_i^{-1}$ and cyclicity of the traces.) □

THEOREM 2.13. *A discrete multiplier Hopf $*$ -algebra (A, Δ) admits a nonzero left-invariant positive linear functional φ , thus is an AQG.*

PROPOSITION 2.14. *An AQG (A, Δ) is discrete iff there exists $h \in A$ such that $ah = \varepsilon(a)h$, for all $a \in A$.*

DEFINITION 2.15. An AQG (A, Δ) is called compact if A is unital.

Note that an AQG (A, Δ) is compact iff it is a Hopf $*$ -algebra. Hence the analytical extension of a compact AQG is a compact quantum group in the sense of S. L. Woronowicz. A multiplier Hopf $*$ -algebra (A, Δ) with A unital is not necessarily an AQG, but this is the case whenever A has a C^* -algebra envelope. It is easily seen that (A, Δ) is compact and discrete iff A is finite-dimensional.

THEOREM 2.16. *Let (A, Δ) be an AQG. Then (A, Δ) is compact iff $(\hat{A}, \hat{\Delta})$ is discrete.*

2.4. EXAMPLES

The examples discussed below are treated in depth in [11]. The translation into the framework of AQG is straightforward and is recommended as an exercise.

Throughout this section let Γ denote a discrete group with unit e . Consider the group algebra $\mathbb{C}\Gamma$ and the algebra $C_c(\Gamma)$ of finitely supported complex valued functions with the usual pointwise operations. They are both nondegenerate $*$ -algebras. The group algebra has unit δ_e , whereas $C_c(\Gamma)$ has a unit iff Γ is finite.

In the group algebra case $(\mathbb{C}\Gamma, \Delta)$, the comultiplication is given by $\Delta(\delta_g) = \delta_g \otimes \delta_g$ and yields a Hopf $*$ -algebra with counit $\varepsilon(\delta_g) = 1$ and coinverse $S(\delta_g) = \delta_{g^{-1}}$, for all $g \in \Gamma$. The functional defined by $\varphi(\delta_g) = \delta_{g,e}$, for $g \in \Gamma$, is unital, positive and both left and right invariant, so $(\mathbb{C}\Gamma, \Delta)$ is a cocommutative compact AQG.

Note that $M(C_c(\Gamma)) \cong C(\Gamma)$, where $C(\Gamma)$ is the unital $*$ -algebra of all functions on Γ , and that $C_c(\Gamma \times \Gamma) \cong C_c(\Gamma) \otimes C_c(\Gamma)$ (as $*$ -algebras). Thus the algebra $C_c(\Gamma)$ has comultiplication $\Delta: C_c(\Gamma) \rightarrow M(C_c(\Gamma) \otimes C_c(\Gamma))$ given by $\Delta(a)(g, h) = a(gh)$ for $a \in C_c(\Gamma)$ and $g, h \in \Gamma$. It is easy to see that the support of $\Delta(a)$ for $a \in C_c(\Gamma)$ is infinite whenever Γ is. This shows that the Hopf algebra framework is too restrictive to cover this example. For $(C_c(\Gamma), \Delta)$, we have $\varepsilon(a) = a(e)$ and $S(a)(g) = a(g^{-1})$, where $a \in C_c(\Gamma \times \Gamma)$ and $g \in \Gamma$. The integral $\varphi(a) = \sum_{g \in \Gamma} a(g)$, $a \in \mathbb{C}\Gamma$, w.r.t. the counting measure is positive and both left and right invariant, so $(C_c(\Gamma), \Delta)$ is a commutative discrete AQG.

The following proposition shows that these two examples exhaust the cocommutative compact AQG and the commutative discrete AQG.

PROPOSITION 2.17. *Let (A, Δ) be an AQG. Then*

- (1) *(A, Δ) is discrete and commutative iff there exists a discrete group Γ such that $(A, \Delta) \cong (C_c(\Gamma), \Delta)$.*

- (2) (A, Δ) is compact and co-commutative iff there exists a discrete group Γ such that $(A, \Delta) \cong (\mathbb{C}\Gamma, \Delta)$.

It is also easy to check that $(\widehat{\mathbb{C}\Gamma}, \hat{\Delta}) \cong (C_c(\Gamma), \Delta)$ and $(\widehat{C_c(\Gamma)}, \hat{\Delta}) \cong (\mathbb{C}\Gamma, \Delta)$, so they are dual to each other.

Giving a characterization of the commutative compact and the cocommutative discrete algebraic quantum groups on the algebraic level, is less immediate and requires some (co-)representation theory. For the moment we mention that the commutative-compact case can be described in terms of the algebra of regular functions on a compact group, and we make here an analytic statement concerning both cases.

PROPOSITION 2.18. *Let (A, Δ) be an AQG with analytic extension (A_r, Δ_r) . Then*

- (1) (A, Δ) is compact and commutative iff there exists a compact group G such that $(A_r, \Delta_r) \cong (C(G), \Delta_r)$.
 (2) (A, Δ) is discrete and cocommutative iff there exists a compact group G such that $(A_r, \Delta_r) \cong (C_r^*(G), \Delta_r)$.

Again these two cases are dual to each other. Here $C(G)$ denotes the C^* -algebra of continuous functions on G with uniform norm, whereas $C_r^*(G)$ denotes the reduced group C^* -algebra of G . Since a compact group G is amenable, $C_r^*(G)$ coincides with the universal group C^* -algebra $C^*(G)$, and therefore, as we shall see very explicitly in the next section, the representation theory of a discrete co-commutative AQG coincides with that of G . We shall also give a simple algebraic description of the associated AQG in this case.

3. Representation Theory

3.1. THE UNIVERSAL COREPRESENTATION

Throughout this section (A, Δ) stands for an arbitrary AQG and B denotes a $*$ -algebra. We recall the universal corepresentation U of (A, Δ) and discuss its various properties [18].

DEFINITION 3.1. A corepresentation of (A, Δ) on B is an element $V \in M(A \otimes B)$ such that $(\Delta \otimes \iota)V = V_{13}V_{23}$.

PROPOSITION 3.2. *Suppose V is a corepresentation of (A, Δ) on B . Then the following are equivalent:*

- (1) V is invertible in $M(A \otimes B)$ with $V^{-1} = (S \otimes \iota)V$.
 (2) $(\varepsilon \otimes \iota)V = 1$.
 (3) $V(A \otimes B) = A \otimes B = (A \otimes B)V$.

A corepresentation V satisfying these equivalent conditions is said to be nondegenerate.

DEFINITION 3.3. A unitary corepresentation V of (A, Δ) on B is a corepresentation on B which is a unitary element of $M(A \otimes B)$.

The following result is fundamental for corepresentations of AQG. We regard elements of $A \otimes \hat{A}$ as endomorphisms of A .

THEOREM 3.4. *There exists a unique element $U \in M(A \otimes \hat{A})$ such that*

$$\begin{aligned} [U(x \otimes \omega)](y) &= (\iota \otimes \omega)(\Delta(y)(x \otimes 1)), \\ [(x \otimes \omega)U](y) &= (\omega \otimes \iota)((1 \otimes x)\Delta(y)), \end{aligned}$$

for all $x, y \in A$ and $\omega \in \hat{A}$. Moreover, we have the following properties for U :

- (1) U is unitary in $M(A \otimes \hat{A})$.
- (2) $(\Delta \otimes \iota)U = U_{13}U_{23}$.
- (3) $(\iota \otimes \hat{\Delta})U = U_{12}U_{13}$.
- (4) $(\omega \otimes \iota)U = \omega$ for all $\omega \in \hat{A}$.
- (5) $(\iota \otimes a)U = a$ for all $a \in A$, where a acts on \hat{A} by the identification $A = \hat{\hat{A}}$.

Note that claims (1) and (2) say that U is a unitary corepresentation of (A, Δ) on \hat{A} , whereas claims (1) and (3) say that $\sigma(U)$ is a unitary corepresentation of $(\hat{A}, \hat{\Delta})$ on A . Below we will explain why U is called the universal corepresentation of (A, Δ) . It follows that $\sigma(U)$ is the universal corepresentation of $(\hat{A}, \hat{\Delta})$.

Let $\text{Hom}(A, B)$ denote the set of $*$ -homomorphisms from A to $M(B)$ satisfying $\theta(A)B = B$.

PROPOSITION 3.5. *Let $V \in M(A \otimes B)$ and consider the linear map*

$$\pi_V: \hat{A} \rightarrow M(B), \quad \omega \mapsto (\omega \otimes \iota)V.$$

Then the following equivalences hold:

- (1) V is a corepresentation (of (A, Δ) on B) iff π_V is multiplicative.
- (2) V is a nondegenerate corepresentation iff π_V is multiplicative and nondegenerate.
- (3) V is a unitary corepresentation iff $\pi_V \in \text{Hom}(\hat{A}, B)$.

The universality of U can now be formulated as follows.

THEOREM 3.6. *For any unitary corepresentation V of (A, Δ) on B and any $\theta \in \text{Hom}(\hat{A}, B)$, we have:*

- (1) $(\iota \otimes \pi_V)U = V$.
- (2) $\pi_{(\iota \otimes \theta)U} = \theta$.

The result remains valid for the nondegenerate situation without $*$ -operation.

DEFINITION 3.7. Let B' be nondegenerate $*$ -algebra. For $\theta \in \text{Hom}(A, B)$ and $\theta' \in \text{Hom}(A, B')$ we define $\theta \times \theta' \in \text{Hom}(A, B \otimes B')$ by $\theta \times \theta' = (\theta \otimes \theta')\Delta$. If V and V' are unitary corepresentations of (A, Δ) on B and B' , respectively, we define a unitary corepresentation $V \times V'$ of (A, Δ) on $B \otimes B'$ by $V \times V' = V_{12}V'_{13}$.

THEOREM 3.8. *The map sending unitary corepresentations V of (A, Δ) on B to $\pi_V \in \text{Hom}(\hat{A}, B)$, is a bijection. Let V' be a unitary corepresentation of (A, Δ) on B' . Then $\pi_{V \times V'} = \pi_V \times \pi_{V'}$. For the universal corepresentation U of (A, Δ) on \hat{A} we have $\pi_U = \iota \in \text{Hom}(\hat{A}, \hat{A})$, and finally for the trivial unitary corepresentation $1 \otimes 1 \in A \otimes B(\mathbb{C})$ of (A, Δ) we have $\pi_{1 \otimes 1} = \hat{\varepsilon}$, where $\hat{\varepsilon}$ is the counit of $(\hat{A}, \hat{\Delta})$.*

Proof. The only part which is not immediate from the results above, is the identity $\pi_{V \times V'} = \pi_V \times \pi_{V'}$. To show this, first note that

$$\begin{aligned} V \times V' &= V_{12}V'_{13} = [(\iota \otimes \pi_V)U]_{12}[(\iota \otimes \pi_{V'})U]_{13} \\ &= (\iota \otimes \pi_V \otimes \pi_{V'})(U_{12}U_{13}) \\ &= (\iota \otimes \pi_V \otimes \pi_{V'})(\iota \otimes \hat{\Delta})U = (\iota \otimes (\pi_V \otimes \pi_{V'})\hat{\Delta})U \\ &= (\iota \otimes \pi_V \times \pi_{V'})U. \end{aligned}$$

Thus we may conclude that

$$\begin{aligned} \pi_{V \times V'}(\omega) &= (\omega \otimes \iota \otimes \iota)(V \times V') = (\omega \otimes \iota \otimes \iota)(\iota \otimes \pi_V \times \pi_{V'})U \\ &= (\pi_V \times \pi_{V'})(\omega \otimes \iota)U = (\pi_V \times \pi_{V'})(\omega), \end{aligned}$$

for all $\omega \in \hat{A}$. □

Remark 3.9. The correspondence established in Theorem 3.8 is in fact a functor if we define arrows for representations and corepresentations on $*$ -algebras in the following way. For any objects $\pi \in \text{Hom}(A, B)$ and $\pi' \in \text{Hom}(A, B')$ define an arrow $f: \pi \rightarrow \pi'$ to be $f \in \text{Hom}(B, B')$ such that $f\pi = \pi'$. Similarly, if V and V' are unitary corepresentations of (A, Δ) on B and B' , respectively, we define an arrow $f: V \rightarrow V'$ between these objects to be $f \in \text{Hom}(B, B')$ such that $V' = (\iota \otimes f)V$. Then the correspondence $V \mapsto \pi_V$ in Theorem 3.8 is an equivalence of tensor categories (they are not $*$ -categories).

Remark 3.10. Let us include some remarks on the tensor categories of actions and coactions of an AQG (A, Δ) on $*$ -algebras.

By an action γ of (A, Δ) on a $*$ -algebra B , we mean a surjective linear map $\gamma: A \otimes B \rightarrow B$ such that $\gamma(m \otimes \iota) = \gamma(\iota \otimes \gamma)$. An arrow $f: \gamma \rightarrow \gamma'$ is an element $f \in \text{Hom}(B, B')$ such that $f\gamma = \gamma'f$. We do not define tensor products in the general case.

Given $\pi \in \text{Hom}(A, B)$, we define the action γ_π of (A, Δ) on B by $\gamma_\pi(a \otimes b) = \pi(a)b$, for all $a \in A$ and $b \in B$. Using nondegeneracy of f , we see that for any

$f \in \text{Hom}(B, B')$, we have $f: \pi \rightarrow \pi'$ iff $f: \gamma_\pi \rightarrow \gamma_{\pi'}$. Thus $\pi \mapsto \gamma_\pi$ is an equivalence of the tensor category of nondegenerate $*$ -representations of (A, Δ) on $*$ -algebras to the tensor category of actions of (A, Δ) on $*$ -algebras contained in the image of the functor and with tensor product $\gamma_\pi \times \gamma_{\pi'}$ simply defined to be the action $\gamma_{\pi \times \pi'}$. We have not incorporated the $*$ -preserving property of π in γ_π and the axioms for actions.

A coaction δ of (A, Δ) on a $*$ -algebra B is a $\delta \in \text{Hom}(B, A \otimes B)$ such that $(\iota \otimes \delta)\delta = (\Delta \otimes \iota)\delta$ and $(\varepsilon \otimes \iota)\delta = \iota$. Such coactions form a tensor category, an arrow $f: \delta \rightarrow \delta'$ being an element $f \in \text{Hom}(B, B')$ such that $(\iota \otimes f)\delta = \delta'f$ and the tensor product $\delta \times \delta'$ being the comodule of (A, Δ) on $B \otimes B'$ given by $(\delta \times \delta')(b \otimes b') = \delta(b)_{12}\delta'(b')_{13}$, for all $b \in B$ and $b' \in B'$.

Let V be a unitary corepresentation of (A, Δ) on B . Then the linear map $\delta_V: B \rightarrow M(A \otimes B)$ given by $\delta_V(b) = V(1 \otimes b)V^*$ for all $b \in B$, is indeed a coaction of (A, Δ^{op}) on B . Nondegeneracy of δ_V follows from $V(A \otimes B) = A \otimes B$, and the formula $(\iota \otimes \delta_V)\delta_V = (\Delta^{\text{op}} \otimes \iota)\delta_V$ follows from the calculation

$$\begin{aligned} (\iota \otimes \delta_V)\delta_V(b) &= (\iota \otimes \delta_V)V(1 \otimes \delta_V(b))((\iota \otimes \delta_V)V)^* \\ &= V_{23}V_{13}V_{23}^*(1 \otimes V(1 \otimes b)V^*)(V_{23}V_{13}V_{23}^*)^* \\ &= V_{23}V_{13}(1 \otimes 1 \otimes b)V_{13}^*V_{23}^* \\ &= (\Delta^{\text{op}} \otimes \iota)V(1 \otimes 1 \otimes b)((\Delta^{\text{op}} \otimes \iota)V)^* \\ &= (\Delta^{\text{op}} \otimes \iota)(V(1 \otimes b)V^*) \\ &= (\Delta^{\text{op}} \otimes \iota)\delta_V(b) \end{aligned}$$

for all $b \in B$. Note that any arrow $f: V \rightarrow V'$ will be an arrow $f: \delta_V \rightarrow \delta_{V'}$, but the converse is not true (consider B' commutative). So we have a tensor functor $V \mapsto \delta_V$ from the tensor category of unitary corepresentations of (A, Δ) on $*$ -algebras to the tensor category of coactions of (A, Δ^{op}) on $*$ -algebras, which in general, is not an equivalence.

3.2. REPRESENTATIONS VS. COREPRESENTATIONS AS TENSOR $*$ -CATEGORIES

We now proceed to establish the correspondence between representations of an AQG (A, Δ) and corepresentations of $(\hat{A}, \hat{\Delta})$, restricting ourselves to finite-dimensional Hilbert spaces. For homomorphisms $\pi: A \rightarrow \text{End } K$ there are two different notions of nondegeneracy which, fortunately, are equivalent for K finite-dimensional.

LEMMA 3.11. *Let A be an algebra and K a finite-dimensional vector space. For a homomorphism $\pi: A \rightarrow \text{End } K$ the conditions $\pi(A)\text{End } K = \text{End } K$ and $\pi(A)K = K$ are equivalent.*

Proof. For K finite-dimensional we clearly have $(\text{End } K)K = K$. Assuming $\pi(A)\text{End } K = \text{End } K$ we compute

$$\pi(A)K = \pi(A)((\text{End } K)K) = (\pi(A)\text{End } K)K = (\text{End } K)K = K.$$

Conversely, assume $\pi(A)K = K$. With the isomorphism $\alpha: K \otimes K^* \rightarrow \text{End } K$ determined by $\alpha(x \otimes \phi)(z) = x\phi(z)$ for $x, z \in K, \phi \in K^*$, we have $(s\alpha(x \otimes \phi))(z) = sx\phi(z) = \alpha(sx \otimes \phi)(z)$ and therefore

$$\begin{aligned}\pi(A) \text{End } K &= \pi(A)\alpha(K \otimes K^*) = \alpha((\pi(A)K) \otimes K^*) = \alpha(K \otimes K^*) \\ &= \text{End } K,\end{aligned}$$

as desired. \square

DEFINITION 3.12. Let (A, Δ) be an AQG. Let $\text{Rep}_f(A, \Delta)$ denote the class of (algebraically) nondegenerate $*$ -representations of A on finite-dimensional Hilbert spaces including the zero representation 0 . Let $\pi_1, \pi_2 \in \text{Rep}_f(A, \Delta)$. Then $\pi_i \in \text{Hom}(A, B(K_i))$, so we define $\pi_1 \times \pi_2 \in \text{Hom}(A, B(K_1) \otimes B(K_2)) = \text{Hom}(A, B(K_1 \otimes K_2)) \subset \text{Rep}_f(A, \Delta)$ according to Definition 3.7.

We regard $\text{Rep}_f(A, \Delta)$ as a tensor $*$ -category with representations as objects and intertwiners as arrows. (Recall that a zero representation 0 is not regarded as an irreducible object.) The tensor product is the one given in the definition above. Clearly, $\text{Rep}_f(A, \Delta)$ is a tensor $*$ -category with the counit ε as the irreducible unit. Of course, the $*$ -operation in the category is the usual Hilbert space adjoint of operators.

DEFINITION 3.13. We say V is a finite-dimensional unitary corepresentation of an AQG (A, Δ) on a Hilbert space K if K is finite-dimensional and V is a unitary corepresentation of (A, Δ) on the unital $*$ -algebra $B(K)$ of linear operators on K . Let $\text{Corep}_f(A, \Delta)$ denote the class of all such unitary corepresentations including the zero corepresentation 0 . Let $V, V' \in \text{Corep}_f(A, \Delta)$. Then we define the unitary corepresentation $V \times V'$ of (A, Δ) on $B(K) \otimes B(K') = B(K \otimes K')$ according to Definition 3.7. Clearly, we have $V \times V' \in \text{Corep}_f(A, \Delta)$.

We regard $\text{Corep}_f(A, \Delta)$ as a tensor $*$ -category with corepresentations as objects and intertwiners as arrows. Recall that $T \in B(K, K')$ is an intertwiner between two corepresentations $V, V' \in \text{Corep}_f(A, \Delta)$ iff $T(\omega \otimes \iota)V = (\omega \otimes \iota)V'T$, for all $\omega \in \hat{A}$. The tensor product is the one given in the definition above. Clearly, $\text{Corep}_f(A, \Delta)$ is a tensor $*$ -category with the unit $1 \otimes 1$ as the irreducible unit. Again, the $*$ -operation in the category is the usual Hilbert space adjoint of operators.

The two tensor categories are related in the following way.

THEOREM 3.14. *Let notation be as above. The correspondence*

$$P: \text{Corep}_f(A, \Delta) \rightarrow \text{Rep}_f(\hat{A}, \hat{\Delta}), \quad V \mapsto \pi_V$$

provided by Proposition 3.5 together with the identity map on morphisms gives rise to an isomorphism of tensor $$ -categories.*

Proof. The proof is immediate from Theorem 3.8. □

Remark 3.15. A similar result can be obtained for the infinite-dimensional case. One needs then to talk about $*$ -representations which are nondegenerate in the C^* -algebra sense, so $\pi \in \text{Rep}(A, \Delta)$ iff there exists a (possibly infinite-dimensional) Hilbert space K and a $*$ -homomorphism $\pi: A \rightarrow B(K)$ such that the vector space $\pi(A)K$ is dense in K . The tensor $*$ -category $\text{Corep}(A_r, \Delta_r)$ of unitary corepresentations on Hilbert spaces consists of Hilbert spaces K and unitaries $V \in M(A_r \otimes B_0(K))$ such that $(\Delta_r \otimes \iota)V = V_{13}V_{23}$. For the exact correspondence thus established, see [5].

We will look at another way of obtaining a tensor $*$ -category $\text{Rep}_f(A_s^o, \hat{\Delta})$ from a compact AQG (A, Δ) which is equivalent to $\text{Corep}_f(A, \Delta)$. This can sometimes come in very handy, especially when dealing with representations of quantized universal enveloping Lie algebras $U_q(\mathfrak{g})$.

First recall that (A, Δ) is a Hopf $*$ -algebra with counit ε , coinverse S and unit I , so the vector space A' of all linear functionals on A is a unital $*$ -algebra with unit ε , product $\omega\eta = (\omega \otimes \eta)\Delta \in A'$ and $*$ -operation $\omega^* \in A'$ given by $\omega^*(a) = \omega(S(a)^*)$, for all $a \in A$. Define a linear map $\hat{\Delta}: A' \rightarrow (A \otimes A)'$ by $\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)$, for all $\omega \in A'$ and $a, b \in A$. Consider the subspace A^o of A' given by

$$A^o = \{\omega \in A' \mid \hat{\Delta}(\omega) \in A' \otimes A'\},$$

where we understand the embedding $A' \otimes A' \subset (A \otimes A)'$. Then $(A^o, \hat{\Delta})$ is called the *maximal dual Hopf $*$ -algebra* of (A, Δ) . Note that for $\hat{f} \in M(\hat{A})$ as in Theorem 2.12, we have $\hat{f} \in A^o$ and $\hat{\Delta}(\hat{f}) = \hat{f} \otimes \hat{f}$. Regard A^o as a locally convex topological vector space with pointwise convergence, so $\omega_\lambda \rightarrow \omega$ in A^o if $\omega_\lambda(a) \rightarrow \omega(a)$, for all $a \in A$. The continuous unital $*$ -representations of $(A^o, \hat{\Delta})$ on finite-dimensional Hilbert spaces clearly form a tensor $*$ -category with conjugates given by the same formulas as for $\text{Rep}_f(\hat{A}, \hat{\Delta})$. Note that in general, the maximal dual Hopf $*$ -algebra can be very small!

THEOREM 3.16. *Let (A, Δ) be a compact AQG and let $(A_s^o, \hat{\Delta})$ be a sub Hopf $*$ -algebra of the maximal dual Hopf $*$ -algebra of (A, Δ) which separates the elements of A . (Thus A_s^o is a unital $*$ -subalgebra of A^o and $\hat{\Delta}$ is given by restriction.) Consider the tensor $*$ -category $\text{Rep}_f(A_s^o, \hat{\Delta})$ of continuous unital $*$ -representations of $(A_s^o, \hat{\Delta})$ on finite-dimensional Hilbert spaces with arrows, $*$ -operation and tensor product defined in the obvious way. Then the assignment*

$$\begin{aligned} N: \text{Corep}_f(A, \Delta) &\rightarrow \text{Rep}_f(A_s^o, \hat{\Delta}), \\ V &\mapsto \pi_V \quad \text{with } \pi_V(\omega) = (\omega \otimes \iota)V, \quad \forall \omega \in A_s^o, \end{aligned}$$

together with the identity map on arrows, is an equivalence of tensor $$ -categories.*

Proof. The only part which requires a proof is that any $\pi \in \text{Rep}_f(A_s^o, \hat{\Delta})$ is of the form π_{V_π} for some $V_\pi \in \text{Corep}_f(A, \Delta)$. Thus let $\pi: A_s^o \rightarrow B(K)$ be a continuous unital $*$ -representation of A_s^o on a finite-dimensional Hilbert space K . We will provide a $V_\pi \in A \otimes B(K)$ such that $\pi_{V_\pi} = \pi$. Using the fact that A_s^o separates the elements of A , it is then clear that V_π will be unitary, that $(\Delta \otimes \iota)V_\pi = (V_\pi)_{13}(V_\pi)_{23}$ and that the $*$ -preserving tensor functor N is an equivalence.

Now pick an orthonormal basis (e_i) for K and consider the system m_{ij} of matrix units for $B(K)$ defined as $m_{ij}(e_k) = \delta_{jk}e_i$, for all i, j, k . Consider, for fixed i and j , the linear functional on A_s^o given by $\omega \mapsto (\pi(\omega)e_j, e_i)$ for all $\omega \in A_s^o$. Since π is continuous w.r.t. the pointwise topology on A_s^o , it follows from [27, Proposition 2.4.4], that there exists a unique $a_{ij} \in A$ such that $\omega(a_{ij}) = (\pi(\omega)e_j, e_i)$, for all $\omega \in A_s^o$. Define now $V_\pi \in A \otimes B(K)$ by $V_\pi = \sum_{ij} a_{ij} \otimes m_{ij}$. By construction $\pi_{V_\pi} = \pi$ and we are done. \square

Remark 3.17. It is known that for the usual quantized compact AQG (A_q, Δ) , the associated quantized universal enveloping Lie algebras $U_q(\mathfrak{g})$ are Hopf $*$ -algebras with $A_s^o = U_q(\mathfrak{g})$ which separate the elements of A_q [17, Sect. 7.1.5]. Thus, the tensor $*$ -category of continuous unital $*$ -representations of $(U_q(\mathfrak{g}), \hat{\Delta})$ on finite-dimensional Hilbert spaces is equivalent to $\text{Corep}_f(A_q, \Delta)$. Also $\hat{f} \in U_q(\mathfrak{g})$, so the intrinsic dimension can be read off conveniently for such representations [30]. We return to this issue in the next subsection.

We now recast our results in the language of modules and comodules.

PROPOSITION 3.18. *Any nondegenerate $*$ -representation π of a discrete AQG (A, Δ) on a finite-dimensional Hilbert space K gives rise to an A -module K . Namely, define a linear map $\alpha_\pi: A \otimes K \rightarrow K$ by $\alpha_\pi(a \otimes \xi) = \pi(a)\xi$, for all $a \in A$ and $\xi \in K$. Then the following properties hold:*

- (1) $\alpha_\pi(m \otimes \iota) = \alpha_\pi(\iota \otimes \alpha_\pi)$.
- (2) $\alpha_\pi(A \otimes K) = K$.
- (3) $(\alpha_\pi(a^* \otimes \xi), \eta) = (\xi, \alpha_\pi(a \otimes \eta)) \forall a \in A \forall \xi, \eta \in K$.

Denote by $\text{Mod}_f(A, \Delta)$ the tensor $*$ -category of linear maps $\alpha: A \otimes K \rightarrow K$ satisfying properties (1)–(3) above, with arrows $t: \alpha \rightarrow \alpha'$ being linear maps $t: K \rightarrow K'$ satisfying $\alpha' = t\alpha$ and tensor product $\alpha \times \alpha': A \otimes K \otimes K' \rightarrow K \otimes K'$ uniquely determined by

$$(\alpha \times \alpha')(\iota \otimes \alpha \otimes \alpha)(a \otimes b \otimes \xi \otimes c \otimes \xi') = \sum_k \alpha(a_k \otimes \xi) \otimes \alpha'(b_k \otimes \xi'),$$

for all $a, b \in A, \xi \in K$ and $\xi' \in K'$ and where $\sum_k a_k \otimes b_k = \Delta(a)(b \otimes c)$. Then the assignment $\pi \mapsto \alpha_\pi$ is a monoidal $*$ -preserving equivalence of $\text{Rep}_f(A, \Delta)$ and $\text{Mod}_f(A, \Delta)$.

Proof. Straightforward, once one shows that $\alpha \times \alpha'$ is well-defined. This is most easily done by considering the corresponding well-defined tensor product $\pi \times \pi'$ of the associated nondegenerate $*$ -representations, i.e, of π and π' such that $\alpha = \alpha_\pi$ and $\alpha' = \alpha_{\pi'}$, and then noting that $\alpha \times \alpha' = \alpha_{\pi \times \pi'}$. \square

PROPOSITION 3.19. *Any unitary corepresentation V of a compact AQG (A, Δ) on a finite-dimensional Hilbert space K gives rise to an A -comodule K . Namely, define a linear map $\beta_V: K \rightarrow A \otimes K$ by $\beta_V(\xi) = \sum_k a_k \otimes x_k \xi$, for all $\xi \in K$ and where $V = \sum_k a_k \otimes x_k \in A \otimes B(K)$. Then the following properties hold:*

- (1) $(\Delta^{\text{op}} \otimes \iota)\beta_V = (\iota \otimes \beta_V)\beta_V$.
- (2) $(S \otimes (\cdot, \eta))\beta_V(\xi) = (* \otimes (\xi, \cdot))\beta_V(\eta) \quad \forall \xi, \eta \in K$,

where ε and S are the counit and coinverse of (A, Δ) , respectively. Let $\text{Comod}_f(A, \Delta^{\text{op}})$ denote the tensor $*$ -category of linear maps $\beta: K \rightarrow A \otimes K$ satisfying properties (1)–(2) above, with arrows $t: \beta \rightarrow \beta'$ being linear maps $t: K \rightarrow K'$ satisfying $\beta't = (\iota \otimes t)\beta$ and tensor product $\beta \times \beta': K \otimes K' \rightarrow A \otimes K \otimes K'$ given by

$$(\beta \times \beta')(\xi \otimes \xi') = \beta(\xi)_{12}\beta'(\xi')_{13},$$

for all $\xi \in K$ and $\xi' \in K'$. Then the assignment $V \mapsto \beta_V$ is a monoidal $*$ -preserving equivalence of $\text{Corep}_f(A, \Delta)$ and $\text{Comod}_f(A, \Delta^{\text{op}})$.

Proof. Straightforward. How to produce a unitary corepresentation $V \in A \otimes B(K)$ of (A, Δ) from a prescribed comodule $\beta: K \rightarrow A \otimes K$ of (A, Δ^{op}) perhaps calls for some explanation. Pick an orthonormal basis (e_k) for K and consider the system (m_{ij}) of matrix units of $B(K)$ given by $m_{ij}(e_k) = \delta_{jk} e_i$ for all i, j, k . Define $V \in A \otimes B(K)$ by $V = \sum_{ij} a_{ij} \otimes m_{ij}$, where the elements $a_{ij} \in A$ are given by $a_{ij} = (\iota \otimes (\cdot, e_i))\beta(e_j)$, for all i, j . It is now easily checked that $\beta_V = \beta$. \square

We thus get the following restatement of Theorem 3.14:

THEOREM 3.20. *Let (A, Δ) be a discrete AQG. Then $\text{Mod}_f(A, \Delta)$ and $\text{Comod}_f(\hat{A}, \hat{\Delta}^{\text{op}})$ are equivalent as semisimple tensor $*$ -categories.*

3.3. CONJUGATES FOR REPRESENTATIONS AND COREPRESENTATIONS

Suppose that (A, Δ) is a discrete AQG with counit ε and coinverse S . We now show that the semisimple tensor $*$ -categories $\text{Rep}_f(A, \Delta)$ and $\text{Corep}_f(\hat{A}, \hat{\Delta})$ have conjugates, and provide a conjugate object for every nonzero object.

Suppose π is a nonzero nondegenerate $*$ -representation on the finite-dimensional Hilbert space K . Let $f \in M(A)$ be as in Theorem 2.12 and consider the positive operator $\pi(f^{-1}) \in B(K)$. Pick any Hilbert space \bar{K} and any invertible antilinear

operator $J: K \rightarrow \overline{K}$ such that $J^*J = \pi(f^{-1})$. This can clearly be done. Now pick an orthonormal basis (e_i) for K and define linear maps $r: \mathbb{C} \rightarrow \overline{K} \otimes K$ and $\bar{r}: \mathbb{C} \rightarrow K \otimes \overline{K}$ by

$$r(1) = \sum_i J e_i \otimes e_i \quad \text{and} \quad \bar{r}(1) = \sum_i e_i \otimes J^{*-1} e_i,$$

respectively. Next define

$$\bar{\pi}: A \rightarrow B(\overline{K}), \quad a \mapsto J\pi S(a^*)J^{-1}.$$

As J and J^{-1} both are antilinear, $\bar{\pi}(a)$ is linear and thus $\bar{\pi}(a) \in B(\overline{K})$ for all $a \in A$. By the following proposition, $\bar{\pi}$ is indeed a conjugate to π and r and \bar{r} are normalized solutions of the conjugate equations for π and $\bar{\pi}$.

PROPOSITION 3.21. *Let (A, Δ) be a discrete AQG and π a nonzero object of $\text{Rep}_f(A, \Delta)$. Then $\bar{\pi}$ is a nonzero object of $\text{Rep}_f(A, \Delta)$. Moreover, the following hold:*

- (1) $r \in \text{Hom}(\varepsilon, \bar{\pi} \times \pi)$,
- (2) $\bar{r} \in \text{Hom}(\varepsilon, \pi \times \bar{\pi})$,
- (3) $\bar{r}^* \otimes \text{id}_\pi \circ \text{id}_\pi \otimes r = \text{id}_\pi$,
- (4) $r^* \otimes \text{id}_{\bar{\pi}} \circ \text{id}_{\bar{\pi}} \otimes \bar{r} = \text{id}_{\bar{\pi}}$,
- (5) $r^* \circ r = \bar{r}^* \circ \bar{r}$.

Proof. Claims (3) and (4) hold for any invertible antilinear map J , as is easily verified. Claim (5) is simply a restatement of the fact $\text{Tr } \pi(f) = \text{Tr } \pi(f^{-1})$ stated in Theorem 2.12. To show that $\bar{\pi} \in \text{Rep}_f(A, \Delta)$, we first note that $\bar{\pi}$ is linear (as $\text{Ad } J$ and $*$ are both antilinear), multiplicative and nondegenerate, thus nonzero. To see that $\bar{\pi}$ is $*$ -preserving, first notice that $S(a^*)^* = S^{-1}(a)$ and $S^2(a) = f a f^{-1}$ for all $a \in A$, and then calculate

$$\begin{aligned} \bar{\pi}(a)^* &= J^{*-1} \pi S^{-1}(a) J^* = J \pi(f) \pi S^{-1}(a) \pi(f^{-1}) J^{-1} = J \pi S(a) J^{-1} \\ &= \bar{\pi}(a^*), \end{aligned}$$

for all $a \in A$.

It remains to show relations (1) and (2). We prove only the first, the second being proved similarly. Now $r \in \text{Hom}(\varepsilon, \bar{\pi} \times \pi)$ simply means that $\varepsilon(a)r(1) = (\bar{\pi} \times \pi)(a)r(1)$, for all $a \in A$. By the nondegeneracy of π it thus suffices to show that

$$(\varepsilon(a)r(1), J^{*-1} e_j \otimes \pi(b^*) e_l) = ((\bar{\pi} \times \pi)(a)r(1), J^{*-1} e_j \otimes \pi(b^*) e_l)$$

for all $a, b \in A$ and all j, l . On the l.h.s. we have

$$\begin{aligned} &(\varepsilon(a)r(1), J^{*-1} e_j \otimes \pi(b^*) e_l) \\ &= \varepsilon(a) \sum_i (J e_i, J^{*-1} e_j)(e_i, \pi(b^*) e_l) \\ &= \varepsilon(a) \sum_i \delta_{ij} (e_i, \pi(b^*) e_l) = \varepsilon(a) (e_j, \pi(b^*) e_l) = (\pi(\varepsilon(a)b) e_j, e_l). \end{aligned}$$

To see that the right-hand side coincides with this expression, first write $(1 \otimes b)\Delta(a) = \sum_k a_k \otimes b_k$ and notice that

$$\begin{aligned} \sum_k b_k S(a_k^*)^* &= \sum_k b_k S^{-1}(a_k) = m(\iota \otimes S^{-1})\left(\sum_k b_k \otimes a_k\right) \\ &= m(\iota \otimes S^{-1})((b \otimes 1)\Delta^{\text{op}}(a)) = \varepsilon(a)b. \end{aligned}$$

Hence

$$\begin{aligned} &((\bar{\pi} \times \pi)(a)r(1), J^{*-1}e_j \otimes \pi(b^*)e_l) \\ &= \sum_i ((\bar{\pi} \otimes \pi)\Delta(a)(Je_i \otimes e_i), J^{*-1}e_j \otimes \pi(b^*)e_l) \\ &= \sum_i ((\bar{\pi} \otimes \pi)(1 \otimes b)(\bar{\pi} \otimes \pi)\Delta(a)(Je_i \otimes e_i), J^{*-1}e_j \otimes e_l) \\ &= \sum_i ((\bar{\pi} \otimes \pi)((1 \otimes b)\Delta(a))(Je_i \otimes e_i), J^{*-1}e_j \otimes e_l) \\ &= \sum_{ik} (\bar{\pi}(a_k)Je_i \otimes \pi(b_k)e_i, J^{*-1}e_j \otimes e_l) \\ &= \sum_{ik} (J\pi S(a_k^*)e_i \otimes \pi(b_k)e_i, J^{*-1}e_j \otimes e_l) \\ &= \sum_{ik} (J\pi S(a_k^*)e_i, J^{*-1}e_j)(\pi(b_k)e_i, e_l) = \sum_{ik} (e_j, \pi S(a_k^*)e_i)(\pi(b_k)e_i, e_l) \\ &= \sum_{ik} (\pi S(a_k^*)^*e_j, e_i)(\pi(b_k)e_i, e_l) = \sum_k (\pi(b_k) \sum_i (\pi S(a_k^*)^*e_j, e_i)e_i, e_l) \\ &= \sum_k (\pi(b_k)\pi S(a_k^*)^*e_j, e_l) = \left(\pi\left(\sum_k b_k S(a_k^*)^*\right)e_j, e_l\right) \\ &= (\pi(\varepsilon(a)b)e_j, e_l), \end{aligned}$$

as desired. □

Note that for a discrete AQG (A, Δ) , the intrinsic dimension $d(\pi)$ of an irreducible object $\pi \in \text{Rep}_f(A, \Delta)$ is then given by

$$d(\pi) = r^* \circ r = \bar{r}^* \circ \bar{r} = \text{Tr } \pi(f) = \text{Tr } \pi(f^{-1}).$$

In fact, the latter two expressions can be thought of as the quantum dimension of π [30] and gives the intrinsic dimension for any $\pi \in \text{Rep}_f(A, \Delta)$. By the Schwarz inequality for the inner product given by Tr it follows that if $d(\pi)$ equals the dimension of the Hilbert space on which π acts, then $\pi(f) = 1$. Thus the von Neumann extension of (A, Δ) is a Kac algebra iff the intrinsic dimension of any finite-dimensional representation coincides with the dimension of the Hilbert space on which it acts.

Suppose now that (A, Δ) is a compact AQG with counit ε and coinverse S . We would like to find an expression for the conjugate unitary corepresentation \overline{V} of a nonzero object $V \in \text{Corep}_f(A, \Delta)$. To this end, we will use the correspondence between $\text{Rep}_f(\hat{A}, \hat{\Delta})$ and $\text{Corep}_f(A, \Delta)$ established in Theorem 3.14. Recall that the dual $(\hat{A}, \hat{\Delta})$ of (A, Δ) is a discrete AQG. Let $\hat{\varepsilon}$ denote its counit, \hat{S} its coinverse, and let $\hat{f} \in M(\hat{A})$ be as in Theorem 2.12.

Let $V \in A \otimes B(K)$ be a nonzero unitary corepresentation of (A, Δ) on a finite-dimensional Hilbert space K . Pick a finite-dimensional Hilbert space \overline{K} and a antilinear map $J: K \rightarrow \overline{K}$ such that $J^*J = \pi_V(\hat{f}^{-1}) \in B(K)$. Given any invertible antilinear map J we define a linear map $j: B(K) \rightarrow B(\overline{K})$ by $j(x) = Jx^*J^{-1}$, for all $x \in B(K)$.

PROPOSITION 3.22. *Suppose (A, Δ) is a compact AQG. Let V be a unitary corepresentation of (A, Δ) on a finite-dimensional Hilbert space K with $J^*J = \pi_V(\hat{f}^{-1}) \in B(K)$. Define $\overline{V} \in A \otimes B(\overline{K})$ by $\overline{V} = (S^{-1} \otimes j)V$. Then the following hold:*

- (1) $V(1 \otimes J^*J) = (1 \otimes J^*J)(S^2 \otimes \iota)V$,
- (2) $\overline{V} \in \text{Corep}_f(A, \Delta)$,
- (3) $\overline{\pi_V} = \pi_{\overline{V}}$. (Here it is understood that the same $J: K \rightarrow \overline{K}$ is used in both constructions.)

Proof. To prove (1), observe that

$$\begin{aligned} (\omega \otimes \iota)VJ^*J &= \pi_V(\omega)\pi_V(\hat{f}^{-1}) = \pi_V(\omega\hat{f}^{-1}) = \pi_V(\hat{f}^{-1})\pi_V(\hat{f}\omega\hat{f}^{-1}) \\ &= J^*J\pi_V(\hat{S}^2(\omega)) = J^*J(\hat{S}^2(\omega) \otimes \iota)V \\ &= J^*J(\omega \otimes \iota)(S^2 \otimes \iota)V \end{aligned}$$

for all $\omega \in \hat{A}$.

To prove (3), note first that $\hat{S}(\omega)(a) = \omega S(a)$ and $\omega^*(a) = \overline{\omega(S(a)^*)}$ for all $a \in A$ and $\omega \in \hat{A}$. Writing $V = \sum_k a_k \otimes x_k$ we compute

$$\begin{aligned} \sum_k \hat{S}(\omega^*)(a_k)x_k &= \sum_k \omega^*S(a_k)x_k = (\omega^* \otimes \iota)(S \otimes \iota)V = (\omega^* \otimes \iota)(V^*) \\ &= \sum_k \omega^*(a_k^*)x_k^* = \sum_k \overline{\omega(S(a_k^*)^*)}x_k^* = \sum_k \overline{\omega S^{-1}(a_k)}x_k^*. \end{aligned}$$

Since $(S \otimes \iota)V = V^*$, we thus obtain

$$\begin{aligned} \overline{\pi_V}(\omega) &= J\pi_V\hat{S}(\omega^*)J^{-1} = J(\hat{S}(\omega^*) \otimes \iota)VJ^{-1} = \sum_k J(\hat{S}(\omega^*)(a_k)x_k)J^{-1} \\ &= \sum_k J\overline{\omega S^{-1}(a_k)}x_k^*J^{-1} \\ &= \sum_k \omega S^{-1}(a_k)Jx_k^*J^{-1} = (\omega \otimes \iota)(S^{-1} \otimes j)\left(\sum_k a_k \otimes x_k\right) \\ &= (\omega \otimes \iota)(S^{-1} \otimes j)V = \pi_{\overline{V}}(\omega) \end{aligned}$$

for all $\omega \in \hat{A}$, proving (3). Now (3), together with Proposition 3.5 and Proposition 3.21, imply (2), completing the proof. \square

This suggests the following proposition, which is a formulation more intrinsic to the category $\text{Corep}_f(A, \Delta)$.

PROPOSITION 3.23. *Let (A, Δ) be a compact AQG and $V \in \text{Corep}_f(A, \Delta)$ be nonzero. Pick a finite-dimensional Hilbert space \overline{K} and any antilinear map $J: K \rightarrow \overline{K}$ such that $V(1 \otimes J^*J) = (1 \otimes J^*J)(S^2 \otimes \iota)V$. Then $\overline{V} \in A \otimes B(\overline{K})$ given by $\overline{V} = (S^{-1} \otimes j)V$ belongs to $\text{Corep}_f(A, \Delta)$. It is a conjugate of V with normalized solutions $r \in \text{Hom}(1 \otimes 1, \overline{V} \times V)$, $\overline{r} \in \text{Hom}(1 \otimes 1, V \times \overline{V})$ of the conjugate equations for V and \overline{V} given by*

$$r(1) = \sum_i J e_i \otimes e_i \quad \text{and} \quad \overline{r}(1) = \sum_i e_i \otimes J^{*-1} e_i.$$

Proof. The irreducible case follows from Proposition 3.21, Theorem 3.14 and the fact that J^*J is the unique (up to a positive scalar) positive operator in $B(K)$ with the property $V(1 \otimes J^*J) = (1 \otimes J^*J)(S^2 \otimes \iota)V$ [40, 30]. We content ourselves with proving one of the least obvious parts of the proposition, namely that r , defined above, belongs to $\text{Hom}(1 \otimes 1, \overline{V} \times V)$. In other words we must show that $r(\omega \otimes \iota)(1 \otimes 1) = (\omega \otimes \iota \otimes \iota)(\overline{V} \times V)r$ or $\omega(1)r(1) = (\omega \otimes \iota \otimes \iota)(\overline{V} \times V)r(1)$ for all $\omega \in \hat{A}$. Write $V = \sum_k a_k \otimes x_k \in A \otimes B(K)$, so $\overline{V} = \sum_k S^{-1}(a_k) \otimes Jx_k^*J^{-1}$. For any $\omega \in \hat{A}$ and any elements e_j, e_r in the chosen orthonormal basis for K , we then get

$$\begin{aligned} & ((\omega \otimes \iota \otimes \iota)(\overline{V} \times V)r(1), J^{*-1}e_j \otimes e_r) \\ &= \sum_i ((\omega \otimes \iota \otimes \iota)(\overline{V}_{12}V_{13})(Je_i \otimes e_i), J^{*-1}e_j \otimes e_r) \\ &= \sum_{ikl} \omega(S^{-1}(a_k)a_l)(Jx_k^*J^{-1}Je_i, J^{*-1}e_j)(x_l e_i, e_r) \\ &= \sum_{ikl} \omega(S^{-1}(a_k)a_l)(Jx_k^*e_i, J^{*-1}e_j)(x_l e_i, e_r) \\ &= \sum_{ikl} \omega(S^{-1}(a_k)a_l)(e_j, x_k^*e_i)(x_l e_i, e_r) \\ &= \sum_{ikl} \omega(S^{-1}(a_k)a_l)(x_k e_j, e_i)(x_l e_i, e_r) \\ &= \sum_{kl} \omega(S^{-1}(a_k)a_l)(x_l \sum_i (x_k e_j, e_i)e_i, e_r) \\ &= \sum_{kl} \omega(S^{-1}(a_k)a_l)(x_l x_k e_j, e_r) \\ &= \sum_{kl} \omega S^{-1}(S(a_l)a_k)(x_l x_k e_j, e_r) = \sum_{kl} \omega S^{-1}(a_l^* a_k)(x_l^* x_k e_j, e_r) \end{aligned}$$

$$\begin{aligned}
&= (\omega S^{-1} \otimes \omega_{e_j, e_r})(V^*V) = (\omega \otimes \omega_{e_j, e_r})(1 \otimes 1) = \omega(1)(e_j, e_r) \\
&= \sum_i \omega(1)(J e_i, J^{*-1} e_j)(e_i, e_r) = (\omega(1)r(1), J^{*-1} e_j \otimes e_r),
\end{aligned}$$

where

$$\omega_{e_j, e_r}: B(K) \rightarrow \mathbb{C}, \quad x \mapsto (x e_j, e_r).$$

Thus

$$(\omega \otimes \iota \otimes \iota)(\bar{V} \times V)r(1) = \omega(1)r(1),$$

completing the proof. \square

Remark 3.24. In the last two results, the conjugates can be expressed alternatively in terms of the unitary antipode R and an antiunitary J [5].

4. Cocommutative Algebraic Quantum Groups vs. Compact Groups

DEFINITION 4.1. Let (A, Δ) be an AQG. The intrinsic group G of (A, Δ) is the following subgroup of the unitaries in the multiplier algebra $M(A)$:

$$G = \{g \in M(A) \mid \Delta g = g \otimes g, g^* g = g g^* = 1\}.$$

Since the extended comultiplication $\Delta: M(A) \rightarrow M(A \otimes A)$ is a unital $*$ -homomorphism and the algebra $M(A)$ is associative, the set G is indeed a group. It is easy to see that $\varepsilon(g) = 1$ and $S(g) = g^{-1} = g^*$, for any $g \in G$.

Remark 4.2. It can be shown that any bounded group-like element of $M(A)$ is automatically unitary with $S(g) = g^*$. A proof of this may be found in [20] and [11], but the above definition suffices for our purposes.

LEMMA 4.3. Let (A, Δ) be a discrete AQG. Equipped with the product topology, $M(A) = \prod_{i \in I} B(H_i)$ is a complete locally convex topological vector space and A is dense in $M(A)$.

Proof. Since the blocks $B(H_i)$ are Banach spaces, the functions $\|p_i(\cdot)\|$ form a separating family of seminorms which induces the product topology. Completeness is obvious by semisimplicity of the $B(H_i)$, $i \in I$. To see that A is dense in $M(A)$, let $x \in M(A)$ and consider the net $(x_\lambda)_{\lambda \in \Lambda}$ in A given by $x_\lambda = \bigoplus_{i \in \lambda} p_i(x)$, where Λ is the collection of finite subsets of I directed by inclusions. Then clearly $x_\lambda \rightarrow x$. \square

PROPOSITION 4.4. If (A, Δ) is a discrete AQG, its intrinsic group G is a compact topological group w.r.t. the product topology on $M(A)$.

Proof. Note that a net (g_λ) converges to g in G iff $p_i(g_\lambda) \rightarrow p_i(g)$ in norm, for all $i \in I$. By Tychonov's theorem it suffices to show that G is closed in $\prod_{i \in I} U(B(H_i))$. Given a net (g_λ) in G which converges to $a \in M(A)$, we must show that $\Delta a = a \otimes a$. Let $i, j \in I$ and consider the finite-dimensional nondegenerate $*$ -representation $(p_i \otimes p_j)\Delta: A \rightarrow B(H_i \otimes H_j)$. By Proposition 2.10, we may write $(p_i \otimes p_j)\Delta \simeq \bigoplus_{k \in I} N_{ij}^k p_k$. Since both these expressions are nondegenerate $*$ -representations, they extend to equivalent unital $*$ -representations of $M(A)$, which we now identify. Therefore

$$\begin{aligned} (p_i \otimes p_j)\Delta a &= \bigoplus_{k \in I} N_{ij}^k p_k(a) = \lim_{\lambda} \bigoplus_{k \in I} N_{ij}^k p_k(g_\lambda) = \lim_{\lambda} (p_i \otimes p_j)\Delta g_\lambda \\ &= \lim_{\lambda} p_i(g_\lambda) \otimes p_j(g_\lambda) = p_i(a) \otimes p_j(a) = (p_i \otimes p_j)(a \otimes a), \end{aligned}$$

thus $\Delta a = a \otimes a$ and G is closed. □

Remark 4.5. Note furthermore that in the proof of the proposition we have shown that

$$\Delta: M(A) = \prod_{i \in I} B(H_i) \rightarrow M(A \otimes A) = \prod_{i, j \in I} B(H_i \otimes H_j)$$

is continuous w.r.t. the product topologies.

PROPOSITION 4.6. *Suppose (A, Δ) is a discrete AQG with intrinsic group G . Let π be a nondegenerate $*$ -representation of A on a finite-dimensional Hilbert space K . Define a map $u_\pi: G \rightarrow B(K)$ by $u_\pi: g \mapsto \pi(g)$, where the extension of π to $M(A)$ is understood. Then u_π is strongly continuous and $D: (K, \pi) \mapsto (K, u_\pi)$ together with the identity map on the morphisms is a faithful and tensor functor from $\text{Rep}_f(A, \Delta)$ to the category $\text{Rep}_f G$ of finite-dimensional continuous representations of G .*

Proof. Continuity of u_π w.r.t. the topology on G is obvious. As $\pi: M(A) \rightarrow B(K)$ is a unital $*$ -homomorphism, clearly $(K, u_\pi) \in \text{Rep } G$. Functoriality and faithfulness of D are obvious. Monoidality follows from the calculation

$$\begin{aligned} D(\pi_1 \times \pi_2)(g) &= (\pi_1 \otimes \pi_2)\Delta g = \pi_1(g) \otimes \pi_2(g) = u_{\pi_1}(g) \otimes u_{\pi_2}(g) \\ &= (u_{\pi_1} \times u_{\pi_2})(g) \end{aligned}$$

for all $g \in G$ and $\pi_i \in \text{Rep}(A, \Delta)$. Every $\pi \in \text{Rep}(A, \Delta)$ is equivalent to a direct sum of the representations p_i by Proposition 2.10. Each u_{p_i} is (strongly) continuous and therefore the direct sum is strongly continuous. □

Thus far we have not assumed (A, Δ) to be cocommutative. The following characterization of cocommutativity will be crucial for proving that D gives rise to an equivalence of categories.

THEOREM 4.7. *A discrete AQG (A, Δ) is cocommutative iff*

$$\overline{\text{span}_{\mathbb{C}}\{g \mid g \in G\}} = M(A).$$

Remark 4.8. This theorem implies that

$$\text{span}_{\mathbb{C}}\{p_i(g) \mid g \in G\} = B(H_i)$$

for every $i \in I$ and in particular that $A = \text{span}_{\mathbb{C}}\{gI_i \mid i \in I, g \in G\}$. These results are rigorous formulations of the heuristic idea that a cocommutative AQG is ‘spanned by its grouplike elements’. Before we give the proof of Theorem 4.7 we show that it leads to the desired equivalence of categories.

In what follows we fix a cocommutative discrete AQG (A, Δ) where $A = \bigoplus_{i \in I} B(H_i)$ and let G denote its intrinsic group. For every $i \in I$, $u_i: g \mapsto p_i(g) \in B(H_i)$ is a continuous unitary representation of G . By Theorem 4.7, the span of $p_i(g)$, $g \in G$ is dense in $B(H_i)$, thus u_i is irreducible. This defines a map $\gamma: I \rightarrow I_G$.

PROPOSITION 4.9. *The map γ is a bijection.*

Proof. If there is a unitary $V: H_i \rightarrow H_j$ such that $Vp_i(g) = p_j(g)V$ for all $g \in G$ then $Vp_i(x) = p_j(x)V$ for all $x \in \text{span}_{\mathbb{C}}\{g \mid g \in G\}$. Since these x are dense in $M(A)$ by Theorem 4.7 and the p_i are continuous we conclude that $Vp_i(x) = p_j(x)V$ for all $x \in M(A)$, thus $i = j$ so that $\gamma: I \rightarrow I_G$ is injective.

Obviously if $g \neq e$ there is an $i \in I$ such that $p_i(g) \neq 1_{B(H_i)}$. Since the category $\text{Rep}_f(A, \Delta)$ is monoidal and has conjugates, $\gamma(I) \subset I_G$ is closed w.r.t. conjugation and tensor products and reduction. The surjectivity of γ now follows from the following well known group theoretical fact. \square

LEMMA 4.10. *Let G be a compact group and let $J \subset I_G$ be closed w.r.t. conjugation and tensor products and reduction. If J separates points on G then $J = I_G$.*

Proof. For every equivalence class $i \in I_G$ pick a representative u_i . The assumptions on J imply that the span of the matrix elements $(u_j)_{nm}$, $j \in J$ is a unital $*$ -subalgebra of $C(G)$. Since it separates the points of G it is dense in $C(G)$ by the Stone–Weierstrass theorem and therefore dense in $L^2(G, \mu)$, where μ is the Haar measure. If there were a $k \in I_G \setminus J$ then by the Peter–Weyl theorem the matrix elements of u_k would be orthogonal to the dense subspace of $L^2(G, \mu)$ generated by the $(u_j)_{nm}$, $j \in J$, which is a contradiction. \square

THEOREM 4.11. *Let (A, Δ) be a cocommutative discrete AQG and G its intrinsic group. Then the functor $D: \text{Rep}(A, \Delta) \rightarrow \text{Rep } G$ induces a canonical equivalence of tensor $*$ -categories: $\text{Rep}_f(A, \Delta) \overset{\otimes}{\simeq} \text{Rep}_f G$.*

Proof. In view of Proposition 4.6 it only remains to prove that the functor D is full and essentially surjective. By Proposition 2.10 the category $\text{Rep}_f(A, \Delta)$

is semisimple, and for a compact group G the semisimplicity of $\text{Rep}_f G$ is well known. Recall from Section 2.1 that a faithful functor between semisimple categories is full if and only if it maps simple objects to simple objects and nonisomorphic simple objects have nonisomorphic images. The first property was used to define the map γ of Proposition 4.9, and the second is the injectivity of γ . Finally, essential surjectivity of D is expressed by surjectivity of γ . Since D is monoidal it gives rise to an equivalence of tensor categories by [31]. \square

Now we prove the characterization of cocommutative discrete AQG used above.

Proof of Theorem 4.7. Clearly, if the closure of $\text{span}_{\mathbb{C}}\{g \mid g \in G\}$ is $M(A)$, then by linearity and continuity of $\Delta: M(A) \rightarrow M(A \otimes A)$, we see that (A, Δ) is cocommutative. We prove the converse direction. By Lemma 4.3 it suffices to show that any $x \in A$ is the limit of linear combinations of elements of G . Fix $x \in A$. For any AQG (A, Δ) there exists a linear inclusion $Q: A \hookrightarrow \hat{A}_r^*$ determined by

$$Q(b)\hat{\pi}_r(\hat{a}) = \hat{a}(b) = \varphi(ba)$$

for all $a, b \in A$. To see this, to any $a, b \in A$ we can choose an element $c \in A$ such that $\hat{c}\hat{b}^* = \hat{b}^*$. (Such c can be obtained as inverse Fourier transform of a local unit for \hat{b}^* in \hat{A} .) Now we observe that

$$\begin{aligned} Q(b)\hat{\pi}_r(\hat{a}) &= \varphi((b^*)^*a) = \hat{\psi}(\hat{b}^*\hat{a}) = \hat{\psi}(\hat{c}\hat{b}^*\hat{a}) = \hat{\psi}(\hat{b}^*\hat{a}\hat{\rho}(\hat{c})) \\ &= (\hat{\pi}_r(\hat{a})\hat{\Lambda}(\hat{\rho}(\hat{c})), \hat{\Lambda}(\hat{b}^*)), \end{aligned}$$

so $Q(b) = (\cdot\hat{\Lambda}(\hat{\rho}(\hat{c})), \hat{\Lambda}(\hat{b}^*))$. Now \hat{A}_r is a unital commutative C^* -algebra. Let Y be the set of $*$ -characters on \hat{A}_r . Gelfand's theorem tells us that Y is a compact Hausdorff space and that the map $\Gamma: \hat{A}_r \rightarrow C(Y)$ given by $\Gamma(a)(y) = y(a)$, for all $a \in \hat{A}_r$ and $y \in Y$, is a unital $*$ -isomorphism from \hat{A}_r to the C^* -algebra $C(Y)$ of continuous functions on Y . Thus $Q(x)\Gamma^{-1} \in C(Y)^*$. By the Krein–Milman theorem for probability measures [27, Theorem 2.5.4] any element of $C(Y)^*$ is a w^* -limit of linear combinations of Dirac measures $\delta_y \in C(Y)^*$. Hence we may write

$$Q(x)\Gamma^{-1} = \lim_{\lambda} \sum_k c_{\lambda k} \delta_{y_{\lambda k}}$$

for some $c_{\lambda k} \in \mathbb{C}$ and $y_{\lambda k} \in Y$. Thus we get

$$\begin{aligned} \hat{a}(x) &= Q(x)\hat{\pi}_r(\hat{a}) = Q(x)\Gamma^{-1}\Gamma\hat{\pi}_r(\hat{a}) = \lim_{\lambda} \sum_k c_{\lambda k} \delta_{y_{\lambda k}} \Gamma\hat{\pi}_r(\hat{a}) \\ &= \lim_{\lambda} \sum_k c_{\lambda k} \Gamma\hat{\pi}_r(\hat{a})(y_{\lambda k}) = \lim_{\lambda} \sum_k c_{\lambda k} y_{\lambda k} \hat{\pi}_r(\hat{a}) \end{aligned}$$

for all $a \in A$. Now the $y_{\lambda k} \hat{\pi}_r: \hat{A} \rightarrow \mathbb{C}$ are unital $*$ -homomorphisms, so we may define $g_{\lambda k} \in M(A)$ by

$$g_{\lambda k} = (\iota \otimes y_{\lambda k} \hat{\pi}_r)U,$$

where $U \in M(A \otimes \hat{A})$ is the universal corepresentation of (A, Δ) . Since U is unitary, so are the elements $g_{\lambda k}$ and moreover,

$$\begin{aligned} \Delta g_{\lambda k} &= \Delta(\iota \otimes y_{\lambda k} \hat{\pi}_r)U = (\iota \otimes \iota \otimes y_{\lambda k} \hat{\pi}_r)(\Delta \otimes \iota)U = (\iota \otimes y_{\lambda k} \hat{\pi}_r)(U_{13}U_{23}) \\ &= (\iota \otimes y_{\lambda k} \hat{\pi}_r)U \otimes (\iota \otimes y_{\lambda k} \hat{\pi}_r)U = g_{\lambda k} \otimes g_{\lambda k}, \end{aligned}$$

thus all $g_{\lambda k} \in G$.

Now for any $\xi \in B(H_i)'$ and $i \in I$ observe that $\xi p_i \in \hat{A}$ since φ is, up to a factor d_i , the trace on $B(H_i)$. Thus by Theorem 3.4 we have $(\xi p_i \otimes \iota)U = \xi p_i$. Hence by the previous formulae we get

$$\begin{aligned} \xi p_i(x) &= \lim_{\lambda} \sum_k c_{\lambda k} y_{\lambda k} \hat{\pi}_r(\xi p_i) = \lim_{\lambda} \sum_k c_{\lambda k} y_{\lambda k} \hat{\pi}_r(\xi p_i \otimes \iota)U \\ &= \lim_{\lambda} \sum_k c_{\lambda k} \xi p_i (\iota \otimes y_{\lambda k} \hat{\pi}_r)U = \lim_{\lambda} \sum_k c_{\lambda k} \xi p_i(g_{\lambda k}) \\ &= \lim_{\lambda} \xi p_i \left(\sum_k c_{\lambda k} g_{\lambda k} \right). \end{aligned}$$

Since this is true for all $i \in I$ and $\xi \in B(H_i)'$, we thus get $x = \lim_{\lambda} \sum_k c_{\lambda k} g_{\lambda k}$ as desired. This concludes the proof of Theorem 4.7. \square

COROLLARY 4.12. *For every $i \in I$ we have $\text{span}_{\mathbb{C}}\{p_i(g) \mid g \in G\} = B(H_i)$.*

Proof. First we show that for $0 \neq a \in A$, there exist $g \in G$ such that $\varphi(ga) \neq 0$. Thus let $0 \neq a \in A$. Then there exists $x \in A$ such that $\hat{a}(x) = \varphi(xa) \neq 0$, thus $\hat{a} = \varphi(\cdot a) \neq 0$. Furthermore, there exists a finite set $\Lambda \subset I$ such that

$$\hat{a}(b) = \varphi(ba) = \sum_{i \in \Lambda} d_i \text{Tr}_i(p_i(ba)),$$

for all $b \in M(A)$. Since the product in $M(A)$ is continuous, we thus see that $\hat{a}: M(A) \rightarrow \mathbb{C}$ is continuous. Thus by Theorem 4.7 there exists $g \in G$ such that $\varphi(ga) \neq 0$.

Assume now that $\text{span}_{\mathbb{C}}\{p_i(g) \mid g \in G\} \neq B(H_i)$. Then there exists a nonzero $a \in B(H_i)$ such that $\text{Tr}_{B(H_i)}(p_i(g)a) = 0$ for all $g \in G$. Since $\varphi \upharpoonright B(H_i) = d_i \text{Tr}_{B(H_i)}$, this is a contradiction. \square

Let G be a compact group and consider a unitary representation $u: G \rightarrow B(K)$ of G on some finite-dimensional Hilbert space K . Since u is continuous (here with respect to the norm), we have $u \in C(G, B(K))$. Since $C(G, B(K)) \cong C(G) \otimes B(K)$, we may thus regard u as a unitary element \tilde{V} of the unital $*$ -algebra $C(G) \otimes B(K)$. Now $u: G \rightarrow B(K)$ being a representation simply means that $(\Delta \otimes \iota)\tilde{V} = \tilde{V}_{13}\tilde{V}_{23}$, where $\Delta: C(G) \rightarrow C(G \times G) \cong C(G) \otimes C(G)$ is given by $\Delta(a)(g, g') = a(gg')$, for all $a \in C(G)$ and $g, g' \in G$. Thus \tilde{V} is a unitary corep-

representation of the commutative compact quantum group $(C(G), \Delta)$ in the sense of Woronowicz. It is easy to see that one may recover u from \tilde{V} by $u(g) = (\delta_g \otimes \iota)\tilde{V}$, for $g \in G$.

THEOREM 4.13. *Let (A, Δ) be a cocommutative discrete AQG, so $(\hat{A}, \hat{\Delta})$ is a commutative compact AQG. Let G be the compact group such that $(C(G), \Delta) \cong (\hat{A}_r, \hat{\Delta}_r)$. For any $V \in \text{Corep}_f(\hat{A}, \hat{\Delta})$, let $\tilde{V} \in C(G) \otimes B(K)$ be the unitary corepresentation of $(C(G), \Delta)$ given by $\tilde{V} = (\hat{\pi}_r \otimes \iota)V$. Then there exists an isomorphism from G to the intrinsic group of (A, Δ) given by $g \mapsto \tilde{g} = (\iota \otimes \delta_g \hat{\pi}_r)U \in M(A)$, where U is the universal corepresentation of (A, Δ) . Moreover, the equivalence*

$$\text{Corep}_f(\hat{A}, \hat{\Delta}) \xrightarrow{P} \text{Rep}_f(A, \Delta) \xrightarrow{D_f} \text{Rep}_f G$$

given by the composition $V \mapsto \pi_V \mapsto u_{\pi_V} =: u_V$, with P and D_f established in Theorem 3.14 and Theorem 4.11, is described more directly by $u_V(\tilde{g}) = (\delta_g \otimes \iota)\tilde{V}$, for all $g \in G$.

Proof. From the proof of Theorem 4.7, it is easily seen that $g \mapsto \tilde{g}$ is indeed a homeomorphism from G to the intrinsic group of (A, Δ) which also preserves the products. Let $g \in G$. We proceed to show that $u_V(\tilde{g}) = (\delta_g \otimes \iota)\tilde{V}$. Since $V \in \hat{A} \otimes B(K)$ we may write $V = \sum_k \omega_k \otimes x_k$ for $\omega_k \in \hat{A}$ and $x_k \in B(K)$. Then

$$\begin{aligned} u_V(\tilde{g}) &= u_V((\iota \otimes \delta_g \hat{\pi}_r)U) = \pi_V((\iota \otimes \delta_g \hat{\pi}_r)U) \\ &= [(\iota \otimes \delta_g \hat{\pi}_r)U] \otimes \iota V = \sum_k [(\iota \otimes \delta_g \hat{\pi}_r)U](\omega_k)x_k \\ &= \sum_k (\omega_k \otimes \delta_g \hat{\pi}_r)U x_k \\ &= \sum_k \delta_g \hat{\pi}_r((\omega_k \otimes \iota)U)x_k = \sum_k \delta_g \hat{\pi}_r(\omega_k)x_k \\ &= (\delta_g \hat{\pi}_r \otimes \iota) \left(\sum_k \omega_k \otimes x_k \right) = (\delta_g \hat{\pi}_r \otimes \iota)V = (\delta_g \otimes \iota)\tilde{V}, \end{aligned}$$

as claimed. □

5. From Concrete Tensor Categories to Discrete Quantum Groups

The strategy we adopt in this section was been pursued by J. E. Roberts [29] in the discrete Kac algebra case, and has then been adapted to the more algebraic setting of discrete AQG (A, Δ) by the authors.

5.1. NATURAL TRANSFORMATIONS OF *-PRESERVING FUNCTORS

PROPOSITION 5.1. *Let \mathcal{D}, \mathcal{K} be *-categories and $F: \mathcal{D} \rightarrow \mathcal{K}$ a *-preserving functor. Let $\text{Nat } F$ be the set of all natural transformations from F to itself, viz.*

$$\text{Nat } F = \{b = (b_X \in \text{End } F(X), X \in \mathcal{D}) \mid F(s) \circ b_X = b_Y \circ F(s) \\ \forall s: X \rightarrow Y\}.$$

*Then with the following pointwise operations $\text{Nat } F$ is a unital *-algebra:*

$$\begin{aligned} (\lambda b + \lambda' b')_X &= \lambda b_X + \lambda' b'_X, \\ (bb')_X &= b_X \circ b'_X, \\ (b^*)_X &= (b_X)^*, \\ \mathbf{1}_X &= \text{id}_{F(X)}, \end{aligned}$$

for $b, b' \in \text{Nat } F$, $\lambda, \lambda' \in \mathbb{C}$ and $X \in \mathcal{D}$.

Proof. If $s \in \text{Mor}(X, Y)$, then

$$F(s) \circ (b^*)_X = (b_X \circ F(s^*))^* = (F(s^*) \circ b_Y)^* = (b^*)_Y \circ F(s),$$

where we have used $F(s^*) = F(s)^*$. Thus $b^* \in \text{Nat } F$. Similarly, we get $bb', \lambda b + \lambda' b' \in \text{Nat } F$, so with these operations $\text{Nat } F$ is a *-algebra. Obviously $\mathbf{1}$ is a unit for $\text{Nat } F$. \square

DEFINITION 5.2. Let B be a *-algebra and \mathcal{K} a *-category. Then the category $\text{Rep}_{\mathcal{K}} B$ of ‘representations of B in \mathcal{K} ’ has as objects the pairs (X, π_X) , where $X \in \text{Obj } \mathcal{K}$ and π_X is a nondegenerate *-homomorphism of B into $\text{End } X$. The morphisms are given by

$$\begin{aligned} \text{Mor}_{\text{Rep}_{\mathcal{K}} B}((X, \pi_X), (Y, \pi_Y)) \\ = \{s \in \text{Mor}_{\mathcal{K}}(X, Y) \mid s \circ \pi_X(b) = \pi_Y(b) \circ s \forall b \in B\}. \end{aligned}$$

Note that if $\mathcal{K} = \mathcal{H}$, the category of finite-dimensional Hilbert spaces, we have $\text{End } H = B(H)$. Thus our definition reduces to the usual notion of finite-dimensional nondegenerate *-representations of B : $\text{Rep}_{\mathcal{H}} B = \text{Rep}_f B$.

PROPOSITION 5.3. *Let \mathcal{C}, \mathcal{K} be *-categories and $E: \mathcal{C} \rightarrow \mathcal{K}$ a *-preserving functor. Define a unital *-algebra $B = \text{Nat } E$. There is a *-preserving functor $F: \mathcal{C} \rightarrow \text{Rep}_{\mathcal{K}} B$ satisfying $K \circ F = E$, where $K: \text{Rep}_{\mathcal{K}} B \rightarrow \mathcal{K}$, $(X, \pi_X) \mapsto X$ is the forgetful functor.*

Proof. We set $F(X) = (E(X), \pi_X)$ for $X \in \text{Obj } \mathcal{C}$ and $F(s) = E(s)$ for the morphisms of \mathcal{C} . Here the *-representation π_X is defined by $\pi_X: B \rightarrow \text{End } X$, $b \mapsto b_X$. That F really is a functor follows from

$$\begin{aligned} F(s) \circ \pi_X(b) &= E(s) \circ b_X = b_Y \circ E(s) \\ &= \pi_Y(b) \circ F(s) \quad \forall s: X \rightarrow Y \forall b \in B, \end{aligned}$$

where we used $b \in B = \text{Nat } E$. \square

PROPOSITION 5.4. *Let \mathcal{D} be a semisimple $*$ -category, \mathcal{K} a $*$ -category and $F: \mathcal{D} \rightarrow \mathcal{K}$ a $*$ -preserving functor. Then there is an isomorphism $\psi_F: \text{Nat } F \rightarrow \prod_{i \in I_{\mathcal{D}}} \text{End } F(X_i)$ of $*$ -algebras.*

Proof. We define a map $\psi_F: \text{Nat } F \rightarrow \prod_{i \in I_{\mathcal{D}}} \text{End } F(X_i)$ by $\psi_F(b) = \prod_{i \in I_{\mathcal{D}}} b_{X_i}$ for all $b \in \text{Nat } F$. Clearly ψ_F is a unital $*$ -algebra homomorphism. We first show that it is injective. Suppose $\psi_F(b) = 0$, so $b_{X_i} = 0$ for all $i \in I_{\mathcal{D}}$. Let $X \in \mathcal{D}$. We must show that $b_X = 0$. Recall that, for any $i \in I_{\mathcal{D}}$, $\text{Mor}(X_i, X)$ is a Hilbert space with the inner product $(s, t)\text{id}_{X_i} = t^*s$ and that every $s \in \text{Mor}(X_i, X)$ is a multiple of an isometry. Let $s_{i\alpha} \in \text{Mor}(X_i, X)$, $i \in I_{\mathcal{D}}$, $\alpha = 1, \dots, \dim \text{Mor}(X_i, X)$ be an orthonormal basis w.r.t. this inner product satisfying $\sum_{i\alpha} s_{i\alpha} \circ s_{i\alpha}^* = \text{id}_X$. Hence

$$b_X = b_X \circ F(\text{id}_X) = \sum_{i\alpha} b_X \circ F(s_{i\alpha} \circ s_{i\alpha}^*) = \sum_{i\alpha} F(s_{i\alpha}) \circ b_{X_i} \circ F(s_{i\alpha}^*) = 0,$$

and injectivity of ψ_F follows.

Next we prove surjectivity of ψ_F . Given $(b_i \in \text{End } F(X_i), i \in I_{\mathcal{D}})$ we need to construct $b \in \text{Nat } F$ such that $\psi_F(b) = \prod_{i \in I_{\mathcal{D}}} b_i$. Thus let $X \in \mathcal{D}$. As above, pick a basis of isometries $s_{i\alpha} \in \text{Mor}(X_i, X)$, $i \in I_{\mathcal{D}}$, with $\sum_{i\alpha} s_{i\alpha} \circ s_{i\alpha}^* = \text{id}_X$. Define $b_X \in \text{End } F(X)$ by

$$b_X = \sum_{i\alpha} F(s_{i\alpha}) \circ b_i \circ F(s_{i\alpha}^*).$$

Another choice of orthonormal bases $(s_{i\alpha})$ in $\text{Mor}(X_i, X)$, $i \in I_{\mathcal{D}}$ does not affect b_X since any two such bases are related by an orthogonal transformation. It remains to show that $b \in \text{Nat } F$, viz. $F(s) \circ b_X = b_Y \circ F(s) \forall s: X \rightarrow Y$. To this purpose, consider $u \in \text{Mor}(X, Y)$. Pick isometries $t_{j\beta} \in \text{Mor}(Y_j, Y)$, with $\sum_{j\beta} t_{j\beta} \circ t_{j\beta}^* = \text{id}_Y$, and define $b_Y \in \text{End } F(Y)$ by

$$b_Y = \sum_{j\beta} F(t_{j\beta}) \circ b_j \circ F(t_{j\beta}^*).$$

Then

$$\begin{aligned} F(u) \circ b_X &= \sum_{i\alpha} F(u \circ s_{i\alpha}) \circ b_i \circ F(s_{i\alpha}^*) \\ &= \sum_{i\alpha, j\beta} F(t_{j\beta} \circ t_{j\beta}^* \circ u \circ s_{i\alpha}) \circ b_i \circ F(s_{i\alpha}^*). \end{aligned}$$

But $t_{j\beta}^* \circ u \circ s_{i\alpha} \in \text{Mor}(X_i, X_j)$ is zero unless $i = j$ and otherwise a scalar multiple of id_{X_i} . Hence

$$\begin{aligned} F(u) \circ b_X &= \sum_{i\alpha\beta} F(t_{i\beta} \circ (t_{i\beta}^* \circ u \circ s_{i\alpha})) \circ b_i \circ F(s_{i\alpha}^*) \\ &= \sum_{i\alpha, j\beta} F(t_{j\beta}) \circ b_j \circ F(t_{j\beta}^* \circ u \circ s_{i\alpha} \circ s_{i\alpha}^*) = b_Y \circ F(u). \end{aligned}$$

The last identity in this equation follows by precisely the same sort of argument as used before. This proves the naturality of the family $b \in \text{Nat } F$. Obviously we have $b_{X_i} = b_i$ for all $i \in I_{\mathcal{D}}$. Thus $\psi_F(b) = \prod_{i \in I_{\mathcal{D}}} b_i$, which proves surjectivity of ψ_F . \square

The isomorphism ψ_F depends on the chosen family $X_i, i \in I$ of irreducibles in \mathcal{C} , but is canonical once the latter is fixed. Thus there is no risk in suppressing it and writing $\text{Nat } F = \prod_{i \in I_{\mathcal{D}}} \text{End } F(X_i)$, as will be done in the sequel.

EXAMPLE 5.5. The simplest application of this proposition is to the identity $\mathcal{D} \rightarrow \mathcal{D}$, when we get $\text{Nat}(\text{Id}) \cong \prod_{i \in I_{\mathcal{D}}} \mathbb{C}$. Examples of natural transformations of the identity are the ‘twists’ appearing in the context of braided tensor categories.

EXAMPLE 5.6. Suppose \mathcal{D} is a tensor $*$ -category with tensor product \otimes . A less trivial application of the proposition is the case of the tensor functor $F = \otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$, where $\mathcal{D} \times \mathcal{D}$ is the product $*$ -category. Note that the isomorphism classes of irreducible objects in $\mathcal{D} \times \mathcal{D}$ are precisely $I_{\mathcal{D}} \times I_{\mathcal{D}}$. By semisimplicity of \mathcal{D} we have a finite decomposition $X_i \otimes X_j \cong \bigoplus_{k \in I_{\mathcal{D}}} N_{ij}^k X_k$, for any $i, j \in I_{\mathcal{D}}$ and thus $\text{End}(X_i \otimes X_j) \cong \bigoplus_{k \in I_{\mathcal{D}}} M_{N_{ij}^k}(\mathbb{C})$. We therefore obtain

$$\text{Nat } \otimes \cong \prod_{\substack{i, j, k \in I_{\mathcal{D}} \\ N_{ij}^k \neq 0}} M_{N_{ij}^k}(\mathbb{C}).$$

The following example is a continuation of Example 5.5.

EXAMPLE 5.7. Suppose \mathcal{C} is a semisimple $*$ -category. Consider the $*$ -algebra $A = \bigoplus_{i \in I_{\mathcal{C}}} \mathbb{C}$. Then Proposition 5.3 provides an equivalence $F: \mathcal{C} \rightarrow \text{Rep}_f A$ of categories such that $F(X_i) \cong p_i$ for all $i \in I_{\mathcal{C}}$. Let $K: \text{Rep}_f A \rightarrow \mathcal{H}$ be the forgetful functor. Then $K \circ F$ is an embedding functor for \mathcal{C} as a $*$ -category. By Proposition 5.4 and Example 5.5 we also conclude that

$$\text{Nat}(K \circ F) \cong \text{Nat Id} \cong \prod_{i \in I_{\mathcal{C}}} \mathbb{C}.$$

(Here, of course, it does not make sense to say that the functors Id and $K \circ F$ are unitarily equivalent.) Suppose now that \mathcal{C} is in addition a tensor category. Then the faithful functor $K \circ F$ cannot in general be monoidal. To see this let X_i, X_j with $i, j \in I_{\mathcal{C}}$. Then $K \circ F(X_i) \cong K(p_i) \cong \mathbb{C}$, so $K \circ F(X_i) \otimes K \circ F(X_j) \cong \mathbb{C}$, whereas

$$K \circ F(X_i \otimes X_j) \cong \bigoplus_{k \in I_{\mathcal{C}}} N_{ij}^k K \circ F(X_k) \cong \bigoplus_{k \in I_{\mathcal{C}}} N_{ij}^k \mathbb{C},$$

where we used the decomposition $X_i \otimes X_j \cong \bigoplus_{k \in I_{\mathcal{C}}} N_{ij}^k X_k$.

Given a $*$ -preserving functor $E: \mathcal{C} \rightarrow \mathcal{K}$ and writing $B = \text{Nat } E$ it is natural to ask under which conditions the functor $F: \mathcal{C} \rightarrow \text{Rep}_{\mathcal{K}} B$ defined in Proposition 5.3 is an equivalence of categories. Instead of pursuing this line further we restrict ourselves to the case where \mathcal{C} is semisimple and $\mathcal{K} = \mathcal{H}$. Then $\text{End } E(X)$ is a finite-dimensional matrix algebra for every $X \in \mathcal{C}$ and Proposition 5.4 shows that $\text{Nat } E$ is a direct product of matrix algebras. The representation theory of such algebras is quite intricate, cf. [13]. There it is shown, among other results, that every irreducible representation of $B = \prod_{i \in I} B(H_i)$ is equivalent to a projection p_i iff the index set I has less than measurable cardinality. In practice, this is no serious restriction but for reasons which will become clear in the next subsection we prefer to work with direct sums instead of direct products.

THEOREM 5.8. *Let \mathcal{C} be a semisimple $*$ -category and $E: \mathcal{C} \rightarrow \mathcal{H}$ a $*$ -preserving functor. The $*$ -algebra $A = \bigoplus_{i \in I_{\mathcal{C}}} B(E(X_i))$ embeds canonically into $\text{Nat } E \cong \prod_{i \in I_{\mathcal{C}}} B(E(X_i))$. The functor*

$$F: \mathcal{C} \rightarrow \text{Rep}_f A,$$

$$F(X) = (E(X), \pi_X), \quad \pi_X(a) = a_X \quad \forall a \in A, \quad F(s) = E(s)$$

satisfies $K \circ F = E$. It gives rise to an equivalence of categories iff E is faithful.

Proof. F as defined above differs from the functor defined in Proposition 5.3 only in the restriction of the representations π_X to $A \subset \text{Nat } E$. Thus it clearly is a $*$ -preserving functor. By definition it is clear that $K \circ F = E$. Since F coincides with E on morphisms, E is faithful if F is an equivalence. Assume now that E is faithful. Let $X \in \mathcal{C}$ be irreducible. Then $F(X)$ is a representation $\pi_X: a \mapsto a_X$ of A on the Hilbert space $E(X)$. As is clear from the proof of Proposition 5.4, for $X = X_i$ we have $\pi_X = p_i$, which clearly is irreducible. Since $a \in A$ is a natural transformation, $\pi_X: a \mapsto a_X$ is an irreducible $*$ -representation whenever X is irreducible. Consider $i \neq j, i, j \in I_{\mathcal{C}}$. By Proposition 2.10, the projections $p_i = F(X_i), p_j = F(X_j)$ are inequivalent $*$ -representations. By semisimplicity of $\text{Rep}_f A$ this implies that F is full. Since the $p_i, i \in I_{\mathcal{C}}$ exhaust the equivalence classes of irreducible $*$ -representations of A it is clear that F is essentially surjective, completing the proof. \square

Remark 5.9. The theorem remains true if one replaces A by the direct product algebra $\text{Nat } E$ provided one makes the additional assumption that the set $I_{\mathcal{C}}$ of isomorphism classes of \mathcal{C} has less than measurable cardinality.

5.2. CONCRETE TENSOR $*$ -CATEGORIES AND DISCRETE QUANTUM SEMIGROUPS

From now on, \mathcal{C} will be a semisimple strict tensor $*$ -category. (As discussed in the Introduction, the strictness assumption does not entail a loss of generality.) We will

introduce other requirements as we proceed. By Theorem 5.8 we have associated a $*$ -algebra A to a $*$ -preserving functor $E: \mathcal{C} \rightarrow \mathcal{H}$. The aim of this subsection is to endow A with the structure of a discrete AQG under suitable conditions on \mathcal{C} and E . To bring out the special rôle of the target category \mathcal{H} , we work in a more general setting as far as seems justifiable.

Let $E: \mathcal{C} \rightarrow \mathcal{D}$ be a $*$ -preserving tensor functor into some tensor $*$ -category \mathcal{D} . Recall that by Proposition 5.1, $\text{Nat } E$ is a unital $*$ -algebra for every $*$ -preserving functor $E: \mathcal{C} \rightarrow \mathcal{D}$.

PROPOSITION 5.10. *Let \mathcal{C}, \mathcal{D} be tensor $*$ -categories, \mathcal{C} being strict, and let $E: \mathcal{C} \rightarrow \mathcal{D}$ be a $*$ -preserving tensor functor. Then*

$$\tilde{\Delta}: \text{Nat } E \rightarrow \text{Nat } E \otimes, \quad b = (b_X) \mapsto \tilde{\Delta}(b) = (\tilde{\Delta}(b)_{Y,Z}), \quad \tilde{\Delta}(b)_{Y,Z} = b_{Y \otimes Z}$$

defines a unital $*$ -homomorphism.

Proof. The morphisms in $\mathcal{C} \times \mathcal{C}$ are of the form $s \times t$ where $s: Y \rightarrow Y'$, $t: Z \rightarrow Z'$. Then $s \otimes t \in \text{Mor}(Y \otimes Z, Y' \otimes Z')$. We compute

$$\begin{aligned} (E \otimes)(s \times t) \circ \tilde{\Delta}(b)_{Y,Z} &= E(s \otimes t) \circ b_{Y \otimes Z} = b_{Y' \otimes Z'} \circ E(s \otimes t) \\ &= \tilde{\Delta}(b)_{Y',Z'} \circ (E \otimes)(s \times t), \end{aligned}$$

where we used $b = (b_X) \in \text{Nat } E$. We conclude that $\tilde{\Delta}(b) \in \text{Nat } E \otimes$. That $b \mapsto \tilde{\Delta}(b)$ is a unital $*$ -homomorphism follows immediately from Proposition 5.1. \square

We now consider the analogues of $(\Delta \otimes \iota)$, \mathcal{C} being strict, and $(\iota \otimes \Delta)$ in the present general setting. \otimes^2 will denote the functor $\otimes \circ (\otimes \times \text{Id}) = \otimes \circ (\text{Id} \times \otimes): \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, where we have used the strictness of \mathcal{C} .

PROPOSITION 5.11. *Let \mathcal{C}, \mathcal{D} be tensor $*$ -categories and let $E: \mathcal{C} \rightarrow \mathcal{D}$ be a $*$ -preserving tensor functor. There are unital $*$ -homomorphisms $\widetilde{\Delta \otimes \iota}, \widetilde{\iota \otimes \Delta}: \text{Nat } E \otimes \rightarrow \text{Nat } E \otimes^2$ defined by*

$$\begin{aligned} \widetilde{\Delta \otimes \iota}: \text{Nat } E \otimes &\rightarrow \text{Nat } E \otimes^2, \\ b = (b_{X,Y}) &\mapsto (\widetilde{\Delta \otimes \iota})(b) = ((\widetilde{\Delta \otimes \iota})(b)_{U,V,W}), \\ (\widetilde{\Delta \otimes \iota})(b)_{U,V,W} &= b_{U \otimes V, W}, \\ \widetilde{\iota \otimes \Delta}: \text{Nat } E \otimes &\rightarrow \text{Nat } E \otimes^2, \\ b = (b_{X,Y}) &\mapsto (\widetilde{\iota \otimes \Delta})(b) = ((\widetilde{\iota \otimes \Delta})(b)_{U,V,W}), \\ (\widetilde{\iota \otimes \Delta})(b)_{U,V,W} &= b_{U, V \otimes W}. \end{aligned}$$

These morphisms satisfy $(\widetilde{\Delta \otimes \iota})\tilde{\Delta} = (\widetilde{\iota \otimes \Delta})\tilde{\Delta}$.

Proof. That $(\widetilde{\Delta \otimes \iota})(b) \in \text{Nat } E \otimes^2$ for $b \in \text{Nat } E \otimes$ follows from the following computation:

$$\begin{aligned} (E \otimes^2)(s \times t \times u) \circ (\widetilde{\Delta \otimes \iota})(b)_{Y,Z,W} &= E(s \otimes t \otimes u) \circ b_{Y \otimes Z, W} \\ &= b_{Y' \otimes Z', W'} \circ E(s \otimes t \otimes u) = (\widetilde{\Delta \otimes \iota})(b)_{Y', Z', W'} \circ (E \otimes^2)(s \times t \times u), \end{aligned}$$

whenever $s: U \rightarrow U', t: V \rightarrow V', u: W \rightarrow W'$. The argument for $(\widetilde{\iota \otimes \Delta})(b)$ is completely analogous. Clearly, both maps $(\widetilde{\Delta \otimes \iota}), (\widetilde{\iota \otimes \Delta})$ are unital $*$ -homomorphisms. The coassociativity property follows from

$$\begin{aligned} (\widetilde{\Delta \otimes \iota})\tilde{\Delta}(b)_{U,V,W} &= \tilde{\Delta}(b)_{U \otimes V, W} = b_{(U \otimes V) \otimes W} = b_{U \otimes (V \otimes W)} = \tilde{\Delta}(b)_{U, V \otimes W} \\ &= (\widetilde{\Delta \otimes \iota})\tilde{\Delta}(b)_{U, V, W} \end{aligned}$$

with $b \in \text{End } E$, where we have used the strictness of \mathcal{C} . □

PROPOSITION 5.12. *There are unital $*$ -homomorphisms $\tilde{\varepsilon}: \text{Nat } E \rightarrow \text{End}(E(\mathbf{1}))$ and $\widetilde{\varepsilon \otimes \iota}, \widetilde{\iota \otimes \varepsilon}: \text{Nat } E \otimes \rightarrow \text{Nat } E$ defined by*

$$\begin{aligned} \tilde{\varepsilon}: \text{Nat } E &\rightarrow \text{End}(E(\mathbf{1})), & b &= (b_X) \mapsto b_{\mathbf{1}}, \\ \widetilde{\varepsilon \otimes \iota}: \text{Nat } E \otimes &\rightarrow \text{Nat } E, & b &= (b_{X,Y}) \mapsto (\widetilde{\varepsilon \otimes \iota})(b) = ((\widetilde{\varepsilon \otimes \iota})(b)_X), \\ (\widetilde{\varepsilon \otimes \iota})(b)_X &= b_{\mathbf{1}, X}, \\ \widetilde{\iota \otimes \varepsilon}: \text{Nat } E \otimes &\rightarrow \text{Nat } E, & b &= (b_{X,Y}) \mapsto (\widetilde{\iota \otimes \varepsilon})(b) = ((\widetilde{\iota \otimes \varepsilon})(b)_X), \\ (\widetilde{\iota \otimes \varepsilon})(b)_X &= b_{X, \mathbf{1}}. \end{aligned}$$

They satisfy $(\widetilde{\varepsilon \otimes \iota})\tilde{\Delta} = (\widetilde{\iota \otimes \varepsilon})\tilde{\Delta} = \iota$.

Proof. Clearly, the map $\tilde{\varepsilon}: b = (b_X) \mapsto b_{\mathbf{1}} \in \text{End } E(\mathbf{1})$ is a unital $*$ -homomorphism. The proofs that $(\widetilde{\varepsilon \otimes \iota})(b), (\widetilde{\iota \otimes \varepsilon})(b)$ are in fact in $\text{Nat } E$ and that the maps $b \mapsto (\widetilde{\varepsilon \otimes \iota})(b), b \mapsto (\widetilde{\iota \otimes \varepsilon})(b)$ are $*$ -homomorphisms proceed along the same lines as for coproducts and are therefore omitted. For $b \in \text{Nat } E$ the computation

$$(\widetilde{\varepsilon \otimes \iota})\tilde{\Delta}(b) = \tilde{\Delta}(b)_{\mathbf{1}, X} = b_{\mathbf{1} \otimes X} = b_X = b_{X \otimes \mathbf{1}} = \tilde{\Delta}(b)_{X, \mathbf{1}} = (\widetilde{\iota \otimes \varepsilon})\tilde{\Delta}(b),$$

where we used the strictness of the unit $\mathbf{1}$, concludes the proof. □

We have thus seen that we can define a ‘coproduct’ $\tilde{\Delta}: \text{Nat } E \rightarrow \text{Nat } E \otimes$ and a ‘counit’ $\tilde{\varepsilon}: \text{Nat } E \rightarrow \text{End } E(\mathbf{1})$ satisfying analogues of the usual properties. A priori, however, $\text{Nat } E \otimes$ has nothing to do with $\text{Nat } E \otimes \text{Nat } E$, regardless of how we interpret the tensor product. The following result indicates how to proceed. We now assume that \mathcal{C} is semisimple. As before, by $I_{\mathcal{C}}$ we denote the set of isomorphism classes of irreducible objects of \mathcal{C} . For every $i \in I_{\mathcal{C}}$ we pick a representative X_i .

DEFINITION 5.13. Given a tensor $*$ -category \mathcal{C} , an embedding functor (for \mathcal{C}) is a faithful $*$ -preserving tensor functor $E: \mathcal{C} \rightarrow \mathcal{H}$. A concrete tensor $*$ -category is a tensor $*$ -category together with an embedding functor.

PROPOSITION 5.14. Let \mathcal{C} be a semisimple tensor $*$ -category and $E: \mathcal{C} \rightarrow \mathcal{H}$ an embedding functor. With the $*$ -algebra $A = \bigoplus_{i \in I_{\mathcal{C}}} B(E(X_i))$ we have the following canonical $*$ -isomorphisms:

$$\begin{aligned} \text{Nat } E &\xrightarrow{\psi_1} M(A) = \prod_{i \in I_{\mathcal{C}}} B(E(X_i)), \\ \text{Nat } E \otimes &\xrightarrow{\psi_2} M(A \otimes A) = \prod_{i, j \in I_{\mathcal{C}}} B(E(X_i)) \otimes B(E(X_j)), \\ \text{Nat } E \otimes^2 &\xrightarrow{\psi_3} M(A \otimes A \otimes A) \\ &= \prod_{i, j, k \in I_{\mathcal{C}}} B(E(X_i)) \otimes B(E(X_j)) \otimes B(E(X_k)). \end{aligned}$$

Proof. The isomorphism ψ_1 was established in Propositions 5.4 and 2.10 and suppressed subsequently. As to ψ_2 , we have

$$\begin{aligned} \text{Nat } E \otimes &= \prod_{i, j \in I_{\mathcal{C}}} \text{End } E(X_i \otimes X_j) \cong \prod_{i, j \in I_{\mathcal{C}}} \text{End}(E(X_i) \otimes E(X_j)) \\ &= \prod_{i, j \in I_{\mathcal{C}}} B(E(X_i) \otimes E(X_j)) = \prod_{i, j \in I_{\mathcal{C}}} B(E(X_i)) \otimes B(E(X_j)) \\ &= M(A \otimes A). \end{aligned}$$

Here we used the identification $B(H \otimes K) \cong B(H) \otimes B(K)$ for finite-dimensional Hilbert spaces and the identifications made after Propositions 2.10 and 5.4. The isomorphism between $\text{End } E(X_i \otimes X_j)$ and $\text{End } E(X_i) \otimes E(X_j)$ is induced in the canonical way from the isomorphism $d_{X_i, X_j}^E: E(X_i) \otimes E(X_j) \rightarrow E(X_i \otimes X_j)$ which makes E monoidal. Unitarity of the d_{X_i, X_j}^E implies that ψ_2 is $*$ -preserving. The argument for ψ_3 is completely analogous. \square

Remark 5.15. In the above proof it was crucial that the canonical $*$ -homomorphisms $\otimes: \text{End } X \otimes_{\mathbb{C}} \text{End } Y \rightarrow \text{End}(X \otimes Y)$ are $*$ -isomorphisms for all $X, Y \in \mathcal{H}$. A semisimple tensor $*$ -category has this property iff every object is a finite direct sum of copies of the tensor unit $\mathbf{1}$. [*Proof.* The ‘if’ clause is easy. As to the ‘only if’ part, let Z be irreducible and consider $X \cong Y \cong \mathbf{1} \oplus Z$, thus $X \otimes Y \cong \mathbf{1} \oplus Z \oplus Z \oplus Z^2$. If $Z \not\cong \mathbf{1}$ then $\text{End } X \otimes_{\mathbb{C}} \text{End } Y \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ is commutative whereas $\text{End}(X \otimes Y)$ is noncommutative since $X \otimes Y$ contains the irreducible Z with multiplicity at least two. This contradicts $\text{End } X \otimes_{\mathbb{C}} \text{End } Y \cong \text{End}(X \otimes Y)$.] This property is known to characterize the (semisimple) tensor $*$ -category \mathcal{H} uniquely up to equivalence. We thus see that with $E = \text{Id}: \mathcal{C} \rightarrow \mathcal{C}$,

we cannot conclude that $\text{Nat } E \otimes \cong M(A \otimes A)$. This is consistent with the computation of $\text{Nat } \otimes$ in Example 5.6.

From now on we restrict ourselves to the case we are ultimately interested in.

DEFINITION 5.16. Let \mathcal{C} be a semisimple tensor $*$ -category and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be an embedding functor. Using the preceding results we define unital $*$ -homomorphisms as follows.

$$\begin{aligned} \bar{\varepsilon}: M(A) &\rightarrow \mathbb{C}, & a &\mapsto \tilde{\varepsilon} \circ \psi_1^{-1}(a), \\ \bar{\Delta}: M(A) &\rightarrow M(A \otimes A), & a &\mapsto \psi_2 \circ \tilde{\Delta} \circ \psi_1^{-1}(a). \end{aligned}$$

Here we have implicitly used the irreducibility of $E(\mathbf{1}_{\mathcal{C}}) \cong \mathbf{1}_{\mathcal{H}} \cong \mathbb{C}$, giving rise to the isomorphism $\psi_0: \text{End } E(\mathbf{1}) \rightarrow \mathbb{C}$, $\text{cid}_{E(\mathbf{1})} \mapsto c$.

Remark 5.17. (1) In the sequel we will usually use the isomorphisms ψ_1, ψ_2 to identify $M(A) \equiv \text{Nat } E$, $M(A \otimes A) \equiv \text{Nat } E \otimes$ and suppress the symbols ψ_1, ψ_2 . Thus for $a \in M(A), i \in I$, a_i will denote both the i -component of a in $\prod_{i \in I} B(E(X_i))$ and the action a_{X_i} on $E(X_i)$ of the natural transformation $\psi_1^{-1}(a) \in \text{Nat } E$.

(2) Let $a \in M(A)$. As $\Delta(a)_{ij} = a_{X_i \otimes X_j}$ and $(a \otimes a)_{ij} = a_i \otimes a_j$ we find

$$\Delta(a) = a \otimes a \iff a_{X_i \otimes X_j} = a_{X_i} \otimes a_{X_j} \quad \forall i, j \in I.$$

Thus the grouplike elements of $M(A)$ are in one-to-one correspondence with those natural transformations of the embedding functor which respect the tensor structure.

LEMMA 5.18. *The restriction $\Delta = \bar{\Delta} \upharpoonright A$ is a nondegenerate $*$ -homomorphism from A to $M(A \otimes A)$, i.e. $\Delta(A)(A \otimes A) = (A \otimes A)\Delta(A) = (A \otimes A)$.*

Proof. To simplify the notation we write $H_i = B(E(X_i))$ and use the isomorphisms ψ_1, ψ_2 to identify $M(A)$ with $\text{Nat } E$ and $M(A \otimes A)$ with $\text{Nat } E \otimes$. We will only prove $\Delta(A)(A \otimes A) = (A \otimes A)$, the proof of the other identity being completely analogous. As $A = \bigoplus_i B(H_i)$, it suffices to show that for every $i, j \in I_{\mathcal{C}}$ there is a $c \in A$ with $\Delta(c)(I_i \otimes I_j) = I_i \otimes I_j$, since this implies $a_i \otimes b_j = \Delta(c)(I_i \otimes I_j)(a_i \otimes b_j) \in \Delta(A)(A \otimes A)$ for all $a_i \in B(H_i), b_j \in B(H_j)$. Recall from the Propositions 5.14 and 5.4 that $M(A)$ is isomorphic to the $*$ -algebra of natural transformations of E , and these are uniquely determined by their actions on the irreducibles: $M(A) \ni b \leftrightarrow (b_i)$, where $b_i \equiv b_{X_i}$. Similarly, $M(A \otimes A) \ni b \leftrightarrow (b_{ij})$ with $b_{ij} \equiv b_{X_i, X_j}$. By the definition of $\tilde{\Delta}$ we have $\tilde{\Delta}(b)_{ij} = b_{X_i \otimes X_j}$. As usual, let $(v_{ij}^{k\alpha}, \alpha = 1, \dots, N_{ij}^k)$ be an orthonormal basis in $\text{Mor}(X_k, X_i \otimes X_j)$ for all $i, j, k \in I_{\mathcal{C}}$. Since $b \in M(A) \equiv \text{End } E$ is a natural transformation we have

$$\begin{aligned} \tilde{\Delta}(b)_{ij} &= b_{X_i \otimes X_j} = b_{X_i \otimes X_j} \circ \text{id}_{E(X_i \otimes X_j)} \\ &= b_{X_i \otimes X_j} \circ E\left(\sum_{k\alpha} v_{ij}^{k\alpha} \circ v_{ij}^{k\alpha*}\right) = \sum_{k\alpha} E(v_{ij}^{k\alpha}) \circ b_k \circ E(v_{ij}^{k\alpha*}). \end{aligned}$$

Now fix $i_0, j_0 \in I_{\mathcal{C}}$ and let $b_k = \text{id}_{E(X_k)}$ if X_k is contained in $X_{i_0} \otimes X_{j_0}$ and $b_k = 0$ otherwise. Since $X_k \prec X_{i_0} \otimes X_{j_0}$ only for finitely many k we have $b = (b_k) \in A$ and

$$\begin{aligned} \tilde{\Delta}(b)_{i_0 j_0} &= \sum_{k\alpha} E(v_{i_0 j_0}^{k\alpha}) \circ \text{id}_{E(X_k)} \circ E(v_{i_0 j_0}^{k\alpha*}) \\ &= \sum_{k\alpha} E(v_{i_0 j_0}^{k\alpha} \circ v_{i_0 j_0}^{k\alpha*}) = E(\text{id}_{X_{i_0} \otimes X_{j_0}}) = \text{id}_{E(X_{i_0} \otimes X_{j_0})}. \end{aligned}$$

Having identified $\text{Nat } E \otimes \equiv M(A \otimes A)$ this precisely means that the (i_0, j_0) -component of $\Delta(b) \in M(A \otimes A) = \prod_{ij} B(H_i) \otimes B(H_j)$ is the identity $I_{i_0} \otimes I_{j_0}$ and therefore $\Delta(b)(I_{i_0} \otimes I_{j_0}) = (I_{i_0} \otimes I_{j_0})$ as desired. \square

COROLLARY 5.19. *Defining ε -homomorphisms $\varepsilon: A \rightarrow \mathbb{C}$ and $\Delta: A \rightarrow M(A \otimes A)$ by $\varepsilon = \bar{\varepsilon} \upharpoonright A$ and $\Delta = \bar{\Delta} \upharpoonright A$, $\bar{\varepsilon}$ and $\bar{\Delta}$ are the unique extensions of ε and Δ , respectively, to $M(A)$.*

Proof. It is clear that ε is not identically zero, thus nondegenerate. Nondegeneracy of Δ has been proven in Lemma 5.18. Now the claim follows from the fact [35] that a nondegenerate \ast -homomorphism $\phi: A \rightarrow M(B)$ has a unique extension to a unital \ast -homomorphism $\bar{\phi}: M(A) \rightarrow M(B)$. \square

PROPOSITION 5.20. *Let \mathcal{C} be a semisimple tensor \ast -category and $E: \mathcal{C} \rightarrow \mathcal{H}$ an embedding functor. Then the \ast -algebra $A = \bigoplus_{i \in I_{\mathcal{C}}} B(E(X_i))$ is a discrete quantum semigroup, i.e. has a nondegenerate coproduct $\Delta: A \rightarrow M(A \otimes A)$ in the sense of [35] and a counit $\varepsilon: A \rightarrow \mathbb{C}$.*

Proof. By Lemma 5.18, Δ is nondegenerate, thus also $\Delta \otimes \iota: A \otimes A \rightarrow M(A \otimes A) \otimes A \subset M(A \otimes A \otimes A)$ is nondegenerate and extends uniquely to $\bar{\Delta} \otimes \iota: M(A \otimes A) \rightarrow M(A \otimes A \otimes A)$. Another unital \ast -homomorphism $M(A \otimes A) \rightarrow M(A \otimes A \otimes A)$ is given by $\psi_3 \circ (\widetilde{\bar{\Delta} \otimes \iota}) \circ \psi_2^{-1}$. The latter clearly restricts to $\Delta \otimes \iota$ on $A \otimes A$, thus it coincides with $\bar{\Delta} \otimes \iota$ on all of $M(A \otimes A)$. In the same way we make sense of $\bar{\iota} \otimes \bar{\Delta}: M(A \otimes A) \rightarrow M(A \otimes A \otimes A)$ and see that it coincides with $\psi_3 \circ (\widetilde{\bar{\iota} \otimes \Delta}) \circ \psi_2^{-1}$. Now, the coassociativity property proved in Proposition 5.11 implies

$$\overline{\bar{\Delta} \otimes \iota} \circ \bar{\Delta} = \overline{\bar{\iota} \otimes \Delta} \circ \bar{\Delta}: M(A) \rightarrow M(A \otimes A \otimes A),$$

which is the proper formulation of coassociativity for (A, Δ) .

As observed earlier, the \ast -homomorphism $\varepsilon: A \rightarrow \mathbb{C}$ is nondegenerate, thus $\varepsilon \otimes \iota, \iota \otimes \varepsilon: A \otimes A \rightarrow A$ are nondegenerate too and have unique extensions $\overline{\varepsilon \otimes \iota}, \overline{\iota \otimes \varepsilon}$ to $M(A \otimes A)$. Again, it is clear that $\overline{\varepsilon \otimes \iota} = \psi_1 \circ (\widetilde{\bar{\varepsilon} \otimes \iota}) \circ \psi_2^{-1}$ and similarly for $\overline{\iota \otimes \varepsilon}$. Thus with Proposition 5.12 we conclude

$$\overline{\varepsilon \otimes \iota} \circ \bar{\Delta} = \overline{\iota \otimes \varepsilon} \circ \bar{\Delta} = \text{id}: M(A) \rightarrow M(A).$$

We conclude that (A, Δ) is a discrete quantum semigroup. \square

THEOREM 5.21. *Let \mathcal{C} be a semisimple tensor $*$ -category and $E: \mathcal{C} \rightarrow \mathcal{H}$ an embedding functor. Let (A, Δ) be the corresponding discrete quantum semigroup. Then the functor $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ defined by*

$$F(X) = (E(X), \pi_X), \quad \pi_X(a) = a_X \quad \forall a \in A, \quad F(s) = E(s)$$

provides an equivalence of tensor $$ -categories such that $K \circ F = E$.*

Proof. With A and $F: \mathcal{C} \rightarrow \text{Rep}_f A$ as defined in Theorem 5.8 we have an equivalence of categories. It remains to show that $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ is monoidal for the monoidal structure of $\text{Rep}_f(A, \Delta)$ in Definition 3.12. Thus we must exhibit isomorphisms $d_{X,Y}^F: F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ satisfying the conditions of Definition 2.1. In view of $F(X) = (E(X), \pi_X)$, an obvious candidate is the isomorphism $d_{X,Y}^E: E(X) \otimes E(Y) \rightarrow E(X \otimes Y)$ which we have by virtue of the assumed monoidality of E . If we set $d_{X,Y}^F = d_{X,Y}^E$ for all $X, Y \in \mathcal{C}$, the equations in Definition 2.1 are clearly satisfied. It only remains to prove that $d_{X,Y}^F$ is in $\text{Mor}_{\text{Rep}_f(A, \Delta)}(F(X) \otimes F(Y), F(X \otimes Y))$, i.e.

$$d_{X,Y}^F \circ (\pi_X \times \pi_Y)(a) = \pi_{X \otimes Y}(a) \circ d_{X,Y}^F \quad \forall a \in A.$$

Keeping in mind the various identifications we have made so far, we get

$$\begin{aligned} d_{X,Y}^F \circ (\pi_X \times \pi_Y)(a) &= d_{X,Y}^F \circ \overline{(\pi_X \otimes \pi_Y)} \Delta(a) = d_{X,Y}^F \circ \overline{(\pi_X \otimes \pi_Y)} \tilde{\Delta}(a) \\ &= d_{X,Y}^F \circ \overline{(\pi_X \otimes \pi_Y)} \psi_2 \tilde{\Delta}(a) = \pi_{X,Y} \tilde{\Delta}(a) \circ d_{X,Y}^F \\ &= a_{X \otimes Y} \circ d_{X,Y}^F = \pi_{X \otimes Y}(a) \circ d_{X,Y}^F \end{aligned}$$

for all $a \in A$. (The reader is invited to check the required identity explicitly for $X = X_i, Y = X_j$, as suffices by naturality of the isomorphisms d^F .) \square

5.3. CONJUGATES AND ANTIPODES: DISCRETE QUANTUM GROUPS

We have already seen that the concrete semisimple tensor $*$ -categories $\text{Rep}_f(A, \Delta)$ and $\text{Corep}_f(\hat{A}, \hat{\Delta})$ come with conjugates whenever (A, Δ) is a discrete AQG. In this section we prove the converse. We start off with a couple of preparatory results.

LEMMA 5.22. *Let \mathcal{C} be a semisimple tensor $*$ -category with conjugates and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be an embedding functor. Then the coproduct $\Delta: A \rightarrow M(A \otimes A)$ for the discrete quantum semigroup (A, Δ) given by Proposition 5.20 has the following inclusion properties*

$$(1 \otimes A)\Delta(A) \subset A \otimes A, \quad (A \otimes 1)\Delta(A) \subset A \otimes A.$$

Proof. We prove only the inclusion $(A \otimes 1)\Delta(A) \subset A \otimes A$, the proof of the inclusion $(1 \otimes A)\Delta(A) \subset A \otimes A$ being completely analogous. We need to show

that $(a \otimes 1)\Delta(b) \in A \otimes A$ for all $a, b \in A$. It is clearly sufficient to show this for $b = (b_i)$ where all components except $i = m$ vanish and for $a = I_n$. By the calculation done in the proof of Lemma 5.18 we have

$$\Delta(b)_{ij} = b_{X_i \otimes X_j} = \sum_{k \in I_{\mathcal{C}}} \sum_{\alpha=1}^{N_{ij}^k} E(V_{ij}^{k\alpha}) \circ b_k \circ E(V_{ij}^{k\alpha*}) \in \text{End } E(X_i \otimes X_j).$$

In view of $a = I_n$ and $b_i = 0$ for $i \neq m$, we obtain

$$((a \otimes 1)\Delta(b))_{ij} = \delta_{i,n} \sum_{\alpha=1}^{N_{nj}^m} E(V_{nj}^{m\alpha}) \circ b_m \circ E(V_{nj}^{m\alpha*}).$$

Now, since \mathcal{C} has conjugates we have isomorphisms $\text{Mor}(X_n \otimes X_j, X_m) \cong \text{Mor}(X_j, \overline{X}_n \otimes X_m)$. By semisimplicity, for fixed n, m the latter space is nonzero only for finitely many $j \in I_{\mathcal{C}}$. Thus $((a \otimes 1)\Delta(b))_{ij}$ is nonzero only for $i = n$ and finitely many j and therefore $(a \otimes 1)\Delta(b) \in A \otimes A$. \square

PROPOSITION 5.23. *Let \mathcal{C} be a semisimple tensor $*$ -category with conjugates and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be an embedding functor. Let $b = (b_{X,Y}) \in \text{Nat } E \otimes$. Using the identifications*

$$B(E(X \otimes Y)) = B(E(X)) \otimes B(E(Y)),$$

for any objects X and Y in \mathcal{C} , we may write

$$b_{X,X} = \sum_k a_X^k \otimes b_X^k \in B(E(X)) \otimes B(E(X)),$$

and then define $\tilde{m}(b)_X \in B(E(X))$ by $\tilde{m}(b)_X = \sum_k a_X^k \circ b_X^k$, for any object $X \in \mathcal{C}$. This gives a unital linear map $\tilde{m}: \text{Nat } E \otimes \rightarrow \text{Nat } E$ which sends $b \in \text{Nat } E \otimes$ to $\tilde{m}(b)$. Furthermore, it restricts to the linearized multiplication $m: A \otimes A \rightarrow A$ on $A \subset \text{Nat } E$.

Proof. The bilinear multiplication map from $B(E(X)) \otimes B(E(X))$ to $B(E(X))$ exists since $E(X)$ is finite-dimensional. The rest is obvious. \square

PROPOSITION 5.24. *Let \mathcal{C} be a semisimple tensor $*$ -category with conjugates and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be an embedding functor. Then the discrete quantum semigroup (A, Δ) given by Proposition 5.20 is a discrete AQG. The coinverse $S: A \rightarrow A$ is given by*

$$S(a)_X = E(\text{id}_X \otimes r_X^*) \circ \text{id}_{E(X)} \otimes a_{\overline{X}} \otimes \text{id}_{E(X)} \circ E(\overline{r}_X \otimes \text{id}_X)$$

for all $a \in A \subset \text{Nat } E$ and objects X in \mathcal{C} , and where $(r_X, \overline{r}_X, \overline{X})$ is any solution of the conjugate equations associated to X .

Proof. In view of Proposition 2.4 it is sufficient to show that $(A \otimes 1)\Delta(A) \subset A \otimes A$ and $(1 \otimes A)\Delta(A) \subset A \otimes A$ and to produce an invertible coinverse S . We thus get an AQG since Theorem 2.13 provides a (nonzero) left-invariant positive functional for discrete multiplier Hopf $*$ -algebras. Now the inclusions $(A \otimes 1)\Delta(A) \subset A \otimes A$ and $(1 \otimes A)\Delta(A) \subset A \otimes A$ are the content of Lemma 5.22.

We show that the map $S: A \rightarrow A$ given by the formula in the theorem is well-defined. We first show that for $a \in A$ and $X \in \mathcal{C}$, the formula for $S(a)_X \in \text{End } E(X)$ does not depend on the choice of solution $(r_X, \bar{r}_X, \bar{X})$ of the conjugate equations associated to X . Suppose therefore that $(r'_X, \bar{r}'_X, \bar{X}')$ is another solution of the conjugate equations associated to X . By [23] there exists a unique invertible $t \in \text{Hom}(\bar{X}, \bar{X}')$ such that $r'_X = t \circ \text{id}_X \circ r_X$ and $\bar{r}'_X = \text{id}_X \otimes t^{*-1} \circ \bar{r}_X$. Thus

$$\begin{aligned} & E(\text{id}_X \otimes r'^*_X) \circ \text{id}_{E(X)} \otimes a_{\bar{X}'} \otimes \text{id}_{E(X)} \circ E(\bar{r}'_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes (r^*_X \circ t^* \otimes \text{id}_X)) \circ \text{id}_{E(X)} \otimes a_{\bar{X}'} \otimes \text{id}_{E(X)} \\ &\quad \circ E((\text{id}_X \otimes t^{*-1} \circ \bar{r}_X) \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r^*_X) \circ \text{id}_{E(t)} \otimes (E(t^*) \circ a_{\bar{X}'} \circ E(t^{*-1})) \otimes \text{id}_{E(X)} \\ &\quad \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r^*_X) \circ \text{id}_{E(X)} \otimes (E(t^* \circ t^{*-1} \circ a_{\bar{X}'}) \otimes \text{id}_{E(X)}) \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r^*_X) \circ (\text{id}_{E(X)} \otimes a_{\bar{X}'} \otimes \text{id}_{E(X)}) \circ E(\bar{r}_X \otimes \text{id}_X), \end{aligned}$$

and $S(a)_X \in \text{End } E(X)$ does not depend on the choice of solution $(r_X, \bar{r}_X, \bar{X})$ of the conjugate equations associated with X .

Next we show that $S(a) = (S(a)_X) \in \text{Nat } E$ for $a \in A$, i.e., we must show that $E(t) \circ S(a)_X = S(a)_Y \circ E(t)$, for all $a \in A$ and $t \in \text{Hom}(X, Y)$. By the existence of the transpose $t^{\bullet*} \in \text{Hom}(Y, X)$ given in [23], we get

$$\begin{aligned} & E(t) \circ S(a)_X \\ &= E(\text{id}_Y \otimes r^*_X) \circ E(t) \otimes a_{\bar{X}} \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes r^*_X) \circ \text{id}_{E(Y)} \otimes a_{\bar{X}} \otimes \text{id}_{E(X)} \circ E((t \otimes \text{id}_{\bar{X}} \circ \bar{r}_X) \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes r^*_X) \circ \text{id}_{E(Y)} \otimes a_{\bar{X}} \otimes \text{id}_{E(X)} \circ E((\text{id}_Y \otimes t^{\bullet*} \circ \bar{r}_Y) \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes r^*_X) \circ \text{id}_{E(Y)} \otimes (a_{\bar{X}} \circ E(t^{\bullet*})) \otimes \text{id}_{E(X)} \circ E(\bar{r}_Y \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes r^*_X) \circ \text{id}_{E(Y)} \otimes (E(t^{\bullet*}) \circ a_{\bar{Y}}) \otimes \text{id}_{E(X)} \circ E(\bar{r}_Y \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes (r^*_X \circ t^{\bullet*} \otimes \text{id}_X)) \circ \text{id}_{E(Y)} \otimes a_{\bar{Y}} \otimes \text{id}_{E(X)} \circ E(\bar{r}_Y \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes (r^*_Y \circ \text{id}_{\bar{Y}} \otimes t)) \circ \text{id}_{E(Y)} \otimes a_{\bar{Y}} \otimes \text{id}_{E(X)} \circ E(\bar{r}_Y \otimes \text{id}_X) \\ &= E(\text{id}_Y \otimes r^*_Y) \circ \text{id}_{E(Y)} \otimes a_{\bar{Y}} \otimes E(t) \circ E(\bar{r}_Y \otimes \text{id}_Y) \\ &= S(a)_Y \circ E(t), \end{aligned}$$

as desired.

Clearly, we have $S(a) \in A$ for all $a \in A$ and it is also clear that the map S is linear. We must prove the two identities

$$m(t \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a, \quad m(S \otimes t)((1 \otimes a)\Delta(b)) = \varepsilon(b)a,$$

for all $a, b \in A$. We prove only the first one. Thus let $a, b \in A$ and write $(a \otimes 1)\Delta(b) = \sum_k a^k \otimes b^k$ for some $a^k, b^k \in A$. This means that

$$(a_X \otimes 1_Y) \circ b_{X \otimes Y} = \sum_k a_X^k \otimes b_Y^k$$

for all objects X and Y of \mathcal{C} . Thus we get

$$\begin{aligned} & m(\iota \otimes S)((a \otimes 1)\Delta(b))_X \\ &= \sum_k m(a^k \otimes S(b^k))_X = \sum_k a_X^k \circ S(b^k)_X \\ &= \sum_k a_X^k \circ E(\text{id}_X \otimes r_X^*) \circ \text{id}_X \otimes b_X^k \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^*) \circ \left(\sum_k a_X^k \otimes b_X^k \right) \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^*) \circ (a_X \otimes \text{id}_{E(\bar{X})} \circ b_{X \otimes \bar{X}}) \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= a_X \circ E(\text{id}_X \otimes r_X^*) \circ b_{X \otimes \bar{X}} \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= a_X \circ E(\text{id}_X \otimes r_X^*) \circ E(\bar{r}_X \otimes \text{id}_X) b_1 \\ &= a_X \circ E(\text{id}_X \otimes r_X^* \circ \bar{r}_X \otimes \text{id}_X) b_1 \\ &= a_X \circ E(\text{id}_X) b_1 = a_X b_1 = a_X \varepsilon(b) = (\varepsilon(b)a)_X \end{aligned}$$

for all objects X in \mathcal{C} , so $m(\iota \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a$ as natural transformations.

It remains to show that S is invertible. We do that by showing $S(S(a)^*) = a^*$ for all $a \in A$. Now with $X \in \mathcal{C}$, we calculate

$$\begin{aligned} S(S(a)^*)_X &= E(\text{id}_X \otimes r_X^*) \circ \text{id}_{E(X)} \otimes S(a)_X^* \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^*) \circ \text{id}_{E(X)} \otimes [E(\text{id}_{\bar{X}} \otimes \bar{r}_X^*) \circ \text{id}_{E(\bar{X})} \otimes a_X \\ &\quad \otimes \text{id}_{E(\bar{X})} \circ E(r_X \otimes \text{id}_{\bar{X}})]^* \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X), \end{aligned}$$

where we used (\bar{r}_X, r_X, X) as a solution of the conjugate equations for $\bar{X} \in \mathcal{C}$. Thus

$$\begin{aligned} & S(S(a)^*)_X \\ &= E(\text{id}_X \otimes r_X^*) \circ \text{id}_{E(X)} \otimes [E(r_X^* \otimes \text{id}_{\bar{X}}) \circ \text{id}_{E(\bar{X})} \otimes a_X^* \otimes \text{id}_{E(\bar{X})} \\ &\quad \circ E(\text{id}_{\bar{X}} \otimes \bar{r}_X)] \otimes \text{id}_{E(X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^* \circ \text{id}_X \otimes r_X^* \otimes \text{id}_{\bar{X} \otimes X}) \circ \text{id}_{E(X \otimes \bar{X})} \otimes a_X^* \otimes \text{id}_{E(\bar{X} \otimes X)} \\ &\quad \circ E(\text{id}_{X \otimes \bar{X}} \otimes \bar{r}_X \otimes \text{id}_X \circ \bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^* \otimes r_X^*) \circ \text{id}_{E(X \otimes \bar{X})} \otimes a_X^* \otimes \text{id}_{E(\bar{X} \otimes X)} \circ E(\bar{r}_X \otimes \bar{r}_X \otimes \text{id}_X) \\ &= E(\text{id}_X \otimes r_X^* \otimes r_X^* \circ \bar{r}_X \otimes \text{id}_{X \otimes \bar{X} \otimes X}) \circ a_X^* \otimes \text{id}_{E(\bar{X} \otimes X)} \circ E(\bar{r}_X \otimes \text{id}_X) \\ &= E((\text{id}_X \otimes r_X^*) \circ (\bar{r}_X \otimes \text{id}_X) \otimes \text{id}_{\bar{X} \otimes X} \circ \text{id}_X \otimes r_X^*) \circ a_X^* \otimes \text{id}_{E(\bar{X} \otimes X)} \\ &\quad \circ E(\bar{r}_X \otimes \text{id}_X) \end{aligned}$$

$$\begin{aligned}
 &= E(\text{id}_X \otimes r_X^*) \circ a_X^* \otimes \text{id}_{E(\overline{X} \otimes X)} \circ E(\overline{r}_X \otimes \text{id}_X) \\
 &= a_X^* \circ E(\text{id}_X \otimes r_X^* \circ \overline{r}_X \otimes \text{id}_X) \\
 &= a_X^* \circ \text{id}_{E(X)} = a_X^* = (a^*)_X. \quad \square
 \end{aligned}$$

After these preparatory computations, the following Main Result is essentially a restatement of Theorem 5.21 and Proposition 5.24.

THEOREM 5.25. *Let \mathcal{C} be a semisimple tensor $*$ -category with conjugates and $E: \mathcal{C} \rightarrow \mathcal{H}$ an embedding functor. Let (A, Δ) be the corresponding discrete AQG. Then the functor $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ defined by*

$$F(X) = (E(X), \pi_X), \quad \pi_X(a) = a_X \forall a \in A, \quad F(s) = E(s)$$

provides an equivalence of tensor $$ -categories such that $K \circ F = E$.*

Remark 5.26. (1) In combination with Theorem 3.14 this gives Woronowicz’ generalized Tannaka–Krein result. Woronowicz constructs $\text{Corep}_f(\hat{A}, \hat{\Delta})$ for a compact AQG $(\hat{A}, \hat{\Delta})$ more directly and uses the universal corepresentation U of (A, Δ) implicitly. In his proof Woronowicz constructs a compact matrix pseudo-group from a category with a generator. Our result is more general in that this assumption is redundant.

(2) In view of Theorem 3.20, an alternative reconstruction theorem for a concrete semisimple tensor $*$ -category \mathcal{C} with conjugates would be that $\mathcal{C} \cong_{\otimes} \text{Comod}_f(\hat{A}, \hat{\Delta}^{\text{op}})$, where $(\hat{A}, \hat{\Delta}^{\text{op}})$ is a Hopf $*$ -algebra with a positive invariant functional. We refer to [34, 15] for similar constructions, where concrete non-semisimple tensor categories are shown to be equivalent to tensor categories of finite-dimensional comodules over (infinite-dimensional) Hopf algebras (with no $*$ -operation).

We now discuss how the AQG (A, Δ) depends on the choice of the embedding functor E . This discussion is only preliminary and will be taken up again in part II of this series. We need the following

DEFINITION 5.27. Let \mathcal{C}, \mathcal{D} be strict tensor categories and $E, E': \mathcal{C} \rightarrow \mathcal{D}$ be tensor functors (with structural isomorphisms $e, e', d_{X,Y}, d'_{X,Y}$). Then E, E' are said to be isomorphic as tensor functors, $E \cong_{\otimes} E'$, if the diagram

$$\begin{array}{ccc}
 E(X) \otimes E(Y) & \xrightarrow{u_X \otimes u_Y} & E'(X) \otimes E'(Y) \\
 \downarrow d_{X,Y} & & \downarrow d'_{X,Y} \\
 E(X \otimes Y) & \xrightarrow{u_{X \otimes Y}} & E'(X \otimes Y)
 \end{array} \tag{5.1}$$

commutes for all $X, Y \in \mathcal{C}$, where $u: E \rightarrow E'$ is a natural transformation whose components $u_X: E(X) \rightarrow E'(X)$ are isomorphisms. A similar and obvious commutative diagram involves the structural isomorphisms e, e' . If \mathcal{C}, \mathcal{D} are tensor $*$ -categories and E, E' are $*$ -preserving then all u_X must be unitary.

PROPOSITION 5.28. *Let \mathcal{C} be a semisimple tensor $*$ -category \mathcal{C} with conjugates. Let $E, E': \mathcal{C} \rightarrow \mathcal{H}$ be embedding functors, and let $(A, \Delta), (A', \Delta')$ be the corresponding discrete AQG. An isomorphism $u: E \xrightarrow{\cong} E'$ gives rise to an isomorphism $\theta: (A, \Delta) \xrightarrow{\cong} (A', \Delta')$ with $\pi'_X \theta = \pi_X$ for all $X \in \mathcal{C}$.*

Proof. Let $u: E \rightarrow E'$ be a unitary equivalence of tensor $*$ -functors. Then $\theta: \text{Nat } E \rightarrow \text{Nat } E'$ given by $\theta(a)_X = u_X a_X u_X^*$ for $a \in \text{Nat } E$ and $X \in \mathcal{C}$ restricts to a unital $*$ -homomorphism from A to A' . Suppressing as usual the natural transformations d and d' , we clearly have $\Delta'(\theta(a))_{(X,Y)} = u_{X \otimes Y} a_{X \otimes Y} u_{X \otimes Y}^*$, whereas $(\theta \otimes \theta)\Delta(a)_{(X,Y)} = (u_X \otimes u_Y) a_{X \otimes Y} (u_X^* \otimes u_Y^*)$, for all $a \in \text{Nat } E$ and $X, Y \in \mathcal{C}$. Since $u_{X \otimes Y} = u_X \otimes u_Y$, for all $X, Y \in \mathcal{C}$, we get $(\theta \otimes \theta)\Delta = \Delta'\theta$, and $(A, \Delta) \xrightarrow{\cong} (A', \Delta')$. \square

The following definition will be useful in our discussion of embedded symmetric tensor categories.

DEFINITION 5.29. Let \mathcal{C}, \mathcal{D} be tensor categories and $E: \mathcal{C} \rightarrow \mathcal{D}$ a tensor functor. Then the automorphism group $\text{Aut}^{\otimes} E$ consists of the natural isomorphisms of E for which the diagram (5.1) commutes (with $E' = E$), together with the obvious componentwise group structure. If \mathcal{C} is a tensor $*$ -category and E is a $*$ -preserving embedding functor then the intrinsic group G_E is the subgroup of $\text{Aut}^{\otimes} E$ where all u_X are unitaries.

5.4. THE NON- $*$ CASE

Our starting point in this paper is a tensor $*$ -category with direct sums, subobjects, conjugates and irreducible unit. From it we (re)construct a multiplier Hopf $*$ -algebra with positive left invariant functional. Such algebras were called algebraic quantum groups (AQG) by Van Daele, who now prefers to speak of ‘multiplier Hopf $*$ -algebras with integrals’. We stick to ‘algebraic quantum groups’ mainly for the sake of definiteness and brevity. When A is no longer a $*$ -algebra and the invariant functional is required to be faithful rather than positive ($\varphi(ab) = 0$ for all $a \in A$ implies $b = 0$) we arrive at the notion of a regular multiplier Hopf algebra with left invariant functional. Essentially the entire theory works just as well in this setting, provided one adopts a modified notion of dual. We say a tensor category is rigid if every object X comes with a chosen left dual \overline{X} and morphisms $e_X: \overline{X} \otimes X \rightarrow \mathbf{1}$ and $d_X: \mathbf{1} \rightarrow X \otimes \overline{X}$ satisfying the triangular equations instead of the conjugate equations. In a semisimple category, left duals are automatically two-sided duals since dual morphisms $e'_X: \mathbf{1} \rightarrow \overline{X} \otimes X$ and $d'_X: X \otimes \overline{X} \rightarrow \mathbf{1}$ exist. Adopting the standard definition of semisimple categories as abelian categories with all exact sequences splitting, one can prove the following:

THEOREM 5.30. *Let \mathbb{F} be an algebraically closed field (of arbitrary characteristic). Let \mathcal{C} be a \mathbb{F} -linear semisimple rigid tensor category with $\text{End } \mathbf{1} \cong \mathbb{F}$ and*

let $E: \mathcal{C} \rightarrow \text{Vect}_{\mathbb{F}}$ be a faithful tensor functor. Then there is a discrete regular multiplier Hopf algebra (A, Δ) with faithful left invariant functional φ and an equivalence $F: \mathcal{C} \rightarrow \text{Rep}(A, \Delta)$ such that $K \circ F = E$.

The proof proceeds exactly as for $*$ -categories, ignoring all references to the $*$ -preserving property of E and the $*$ -involution of A . Again, existence, uniqueness and faithfulness of the invariant functional are proven using results due to Van Daele. Only the construction of the coinverse requires careful examination since the notion of dual is different.

We close with some remarks on the two different notions of discreteness applying to multiplier Hopf algebras. In analogy to the $*$ -case, a regular multiplier Hopf algebra (A, Δ) with left invariant functional is called *discrete* if A is a direct sum of finite-dimensional matrix algebras over \mathbb{F} . In the non- $*$ case Proposition 2.14 fails, and (A, Δ) is said to be of *discrete type* iff there exists $0 \neq h \in A$ such that $ah = \varepsilon(a)h$ for all $a \in A$. As in the $*$ -case, (A, Δ) is of *compact type* or, equivalently, *compact* if A has a unit. It is well known [37] that (A, Δ) is of discrete type iff $(\hat{A}, \hat{\Delta})$ is of compact type.

Every discrete multiplier Hopf algebra has a copy of \mathbb{F} as a direct summand, corresponding to the one-dimensional representation ε . The unit I_0 of this algebra has the property $aI_0 = I_0a = \varepsilon(a)I_0$, thus a discrete multiplier Hopf algebra is of discrete type. The following characterizes discrete multiplier Hopf algebras, generalizing a well known result for finite-dimensional Hopf algebras (which automatically possess integrals).

PROPOSITION 5.31. *Let (A, Δ) and $(\hat{A}, \hat{\Delta})$ be mutually dual regular multiplier Hopf algebras with faithful left invariant functionals. Then the following are equivalent:*

- (1) (A, Δ) is discrete.
- (2) (A, Δ) is of discrete type and h can be chosen to be idempotent.
- (3) $(\hat{A}, \hat{\Delta})$ is of compact type and $\hat{\varphi}(1) \neq 0$.

Proof. The implication (1) \Rightarrow (2) has been discussed before. Assuming (2), we have $h = hh = \varepsilon(h)h$, thus $\varepsilon(h) = 1$. Conversely, if $\varepsilon(h) \neq 0$ then $h/\varepsilon(h)$ satisfies (2). Now, (2) implies that $(\hat{A}, \hat{\Delta})$ is compact, and compactness implies the existence of $0 \neq h \in A$ satisfying $xh = \varepsilon(x)h$ for all $x \in A$. By definition of the dual $(\hat{A}, \hat{\Delta})$ and assuming the normalization $\varphi(h) = 1$, we have $\hat{\varphi}(1) = \varepsilon(h)$, yielding the equivalence (2) \Leftrightarrow (3). Now assume (3) holds. Then $(\hat{A}, \hat{\Delta})$ is a Hopf algebra with left invariant functional $\hat{\varphi}$ satisfying $\hat{\varphi}(1) \neq 0$. By classical results, cf., e.g., [32, 14.0.3] or [1, Theorem 3.3.2], it follows that $(\hat{A}, \hat{\Delta})$ is cosemisimple. This means that the \mathbb{F} -coalgebra $(\hat{A}, \hat{\Delta}, \hat{\varepsilon})$ is a direct sum of finite-dimensional coalgebras. Since \mathbb{F} is algebraically closed, every finite-dimensional \mathbb{F} -coalgebra is a matrix coalgebra, i.e. it has a basis $\{a_{ij}, 1 \leq i, j \leq n\}$ such that $\Delta(a_{ij}) = \sum_l a_{il} \otimes a_{lj}$. Since $A = \{\hat{\varphi}(\cdot a), a \in \hat{A}\} \subset \hat{A}^*$ and $\hat{\varphi}$ is faithful, A has a basis

$\{b_{ij}^\alpha = \hat{\varphi}(\cdot a_{ij}^\alpha), \alpha \in I, 1 \leq i, j \leq n_\alpha \in \mathbb{N}\}$. These elements multiply according to $b_{ij}^\alpha b_{i'j'}^{\alpha'} = \delta_{\alpha\alpha'} \delta_{ji'} b_{ij}^\alpha$, thus $A = \bigoplus_{\alpha \in I} M_{n_\alpha}(\mathbb{F})$, as desired. \square

Remark 5.32. (1) In the $*$ -case, $\hat{\varphi}$ is positive and $1 > 0$, thus discrete type implies discreteness.

(2) We indicate how Proposition 5.31 can be used to prove Theorem 5.30. Given a semisimple tensor category \mathcal{C} with embedding functor E , by [34] there exists a Hopf algebra H such that $\mathcal{C} \simeq \text{Comod } H$. By [32, 14.0.3], semisimplicity of \mathcal{C} implies the existence of a left invariant functional $\hat{\varphi}$ on H , satisfying $\hat{\varphi}(1) \neq 0$. Thus $(H, \hat{\Delta})$ is a compact multiplier Hopf algebra with left invariant functional. Its dual (A, Δ) then is a discrete multiplier Hopf algebra with left invariant functional, and by Theorem 3.14 we have $\mathcal{C} \simeq \text{Rep}_f(A, \Delta)$. However, the proof of Theorem 5.30 analogous to the proof of Theorem 5.25 is more direct and instructive.

6. Concrete Braided and Symmetric Categories

In this section we prove that a braiding for the concrete tensor $*$ -category \mathcal{C} gives rise to an R -matrix for the associated AQG (A, Δ) . If \mathcal{C} is symmetric and the embedding functor E is *symmetric* monoidal then we may conclude that (A, Δ) is cocommutative.

The flip automorphism σ of $A \otimes A$ defined by $\sigma(a \otimes b) = b \otimes a$ is a nondegenerate $*$ -homomorphism. Thus there exists a unique extension to an involutive $*$ -automorphism of $M(A \otimes A)$ which we denote by the same symbol.

DEFINITION 6.1. An AQG (A, Δ) is said to be quasitriangular w.r.t. R iff R is an invertible element of $M(A \otimes A)$ and satisfies

- (1) $(\Delta \otimes \iota)(R) = R_{13}R_{23}$,
- (2) $(\iota \otimes \Delta)(R) = R_{13}R_{12}$,
- (3) $\Delta^{\text{op}} = R\Delta(\cdot)R^{-1}$,

where $\Delta^{\text{op}} \equiv \sigma \Delta$. We sometimes refer to such an element R as an R -matrix. If, in addition, $\sigma(R) = R^{-1}$ we call (A, Δ) triangular.

The first equation means that the invertible element $R \in M(A \otimes A)$ is a nondegenerate corepresentation of (A, Δ) on A . It is easy to see that (A, Δ) is quasitriangular w.r.t. R iff (A, Δ^{op}) is quasitriangular w.r.t. $\sigma(R)$.

DEFINITION 6.2. A braiding for a (strict) tensor category \mathcal{C} is a family of isomorphisms $c_{X,Y}: X \otimes Y \rightarrow Y \otimes X$, for all $X, Y \in \mathcal{C}$ natural in both arguments

$$t \otimes s \circ c_{X,Y} = c_{X',Y'} \circ s \otimes t \quad \forall s: X \rightarrow X', t: Y \rightarrow Y',$$

satisfying

$$\begin{aligned} c_{X,Y \otimes Z} &= \text{id}_Y \otimes c_{X,Z} \circ c_{X,Y} \otimes \text{id}_Z, \\ c_{X \otimes Y,Z} &= c_{X,Z} \otimes \text{id}_Y \circ \text{id}_X \otimes c_{Y,Z}, \\ c_{X,1} &= c_{1,X} = \text{id}_X \end{aligned}$$

for all $X, Y, Z \in \mathcal{C}$. A braiding is called a symmetry if, in addition, it satisfies

$$c_{X,Y} \circ c_{Y,X} = \text{id}_{Y \otimes X} \quad \forall X, Y \in \mathcal{C}.$$

A tensor category with a (chosen) braiding or symmetry is called a braided or symmetric tensor category, respectively. A braided functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor between strict braided tensor categories making

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{d_{X,Y}^F} & F(X \otimes Y) \\ c_{F(X),F(Y)} \downarrow & & \downarrow F(c_{X,Y}) \\ F(Y) \otimes F(X) & \xrightarrow{d_{Y,X}^F} & F(Y \otimes X) \end{array} \tag{6.1}$$

commutative for all $X, Y \in \mathcal{C}$.

The following result establishes the relation between R -matrices for (A, Δ) and braidings for $\text{Rep}_f(A, \Delta)$.

PROPOSITION 6.3. *Let (A, Δ) be a discrete AQG. Let $R \in M(A \otimes A)$ and consider*

$$c_{(H,\pi),(H',\pi')} \equiv \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(R), \quad (H, \pi), (H', \pi') \in \text{Rep}_f(A, \Delta).$$

Here $\Sigma_{H,K}$ is the canonical unitary flip from $H \otimes K \rightarrow K \otimes H$. Then the family $(c_{(H,\pi),(H',\pi')})$ is a braiding for $\text{Rep}_f(A, \Delta)$ iff R is an R -matrix. Also $c_{(H,\pi),(H',\pi')}$ is unitary for all $(H, \pi), (H', \pi')$ iff R is unitary. If these equivalent conditions are satisfied then (A, Δ) is triangular iff the corresponding braiding for $\text{Rep}_f(A, \Delta)$ is symmetric.

Proof. Throughout the proof we write $c_{\pi,\pi'}$ for $c_{(H,\pi),(H',\pi')}$ and note that $c_{\pi,\pi'}: H \otimes H' \rightarrow H' \otimes H$, for all $(H, \pi), (H', \pi') \in \text{Rep}_f(A, \Delta)$. Also note that

$$\begin{aligned} \overline{(\pi \otimes \pi')\sigma(a)} &= \Sigma_{H',H}(\overline{\pi' \otimes \pi})(a)\Sigma_{H,H'} \\ \forall (H, \pi), (H', \pi') \in \text{Rep}_f(A, \Delta), \forall a \in M(A \otimes A). \end{aligned}$$

Now suppose $R \in M(A \otimes A)$ satisfies $\Delta^{\text{op}}(\cdot)R = R\Delta(\cdot)$. Then

$$\begin{aligned} c_{\pi,\pi'}(\pi \times \pi')(a) &= \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(R) \overline{(\pi \otimes \pi')\Delta(a)} = \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(R\Delta(a)) \\ &= \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(\Delta^{\text{op}}(a)R) \\ &= \Sigma_{H,H'}(\overline{\pi \otimes \pi'})\Delta^{\text{op}}(a)\overline{(\pi \otimes \pi')(R)} \\ &= \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(\sigma \Delta(a))\Sigma_{H',H} \Sigma_{H,H'}(\overline{\pi \otimes \pi'})(R) \\ &= \overline{(\pi' \otimes \pi)\Delta(a)}\Sigma_{H,H'}(\overline{\pi \otimes \pi'})(R) = (\pi' \times \pi)(a) c_{\pi,\pi'}, \end{aligned}$$

so we have seen that $c_{\pi,\pi'}: \pi \times \pi' \rightarrow \pi' \times \pi$ for all $\pi, \pi' \in \text{Rep}_f(A, \Delta)$. Conversely, by reversing the above calculations, we see that $\Delta^{\text{op}}(\cdot)R = R\Delta(\cdot)$ if $c_{\pi,\pi'}: \pi \times \pi' \rightarrow \pi' \times \pi$ for all $\pi, \pi' \in \text{Rep}_f(A, \Delta)$.

It is straightforward to see that R is invertible iff $c_{\pi,\pi'}: \pi \times \pi' \rightarrow \pi' \times \pi$ is an isomorphism for all $\pi, \pi' \in \text{Rep}_f(A, \Delta)$, that R is unitary iff $c_{\pi,\pi'}$ is unitary for all $\pi, \pi' \in \text{Rep}_f(A, \Delta)$, and that $R^{-1} = \sigma(R)$ iff $c_{\pi',\pi} c_{\pi,\pi'} = \text{id}_{\pi \times \pi'}$ for all $\pi, \pi' \in \text{Rep}_f(A, \Delta)$.

Pick $\pi, \pi', \pi'' \in \text{Rep}_f(A, \Delta)$. Then

$$\begin{aligned} & \overline{(\pi \otimes \pi' \otimes \pi'')}(t \otimes \Delta)(R) \\ &= \overline{(\pi \otimes (\pi' \times \pi''))}(R) \\ &= \Sigma_{H' \otimes H'', H} \Sigma_{H, H' \otimes H''} \overline{(\pi \otimes (\pi' \times \pi''))}(R) = \Sigma_{H' \otimes H'', H} c_{\pi, \pi' \times \pi''}, \end{aligned}$$

whereas

$$\begin{aligned} & \overline{(\pi \otimes \pi' \otimes \pi'')}(R_{13}R_{12}) \\ &= (\Sigma_{H'', H} \Sigma_{H, H''} \overline{(\pi \otimes \pi'')}(R))_{13} (\Sigma_{H', H} \Sigma_{H, H'} \overline{(\pi \otimes \pi')}(R))_{12} \\ &= (\Sigma_{H'', H} c_{\pi, \pi''})_{13} (\Sigma_{H', H} c_{\pi, \pi'})_{12} = (\Sigma_{H'', H})_{13} (c_{\pi, \pi''})_{13} (\Sigma_{H', H})_{12} (c_{\pi, \pi'})_{12} \\ &= \Sigma_{H' \otimes H'', H} (c_{\pi, \pi''})_{23} (c_{\pi, \pi'})_{12}. \end{aligned}$$

Hence $(t \otimes \Delta)(R) = R_{13}R_{12}$ iff

$$c_{\pi, \pi' \times \pi''} = (c_{\pi, \pi''})_{23} (c_{\pi, \pi'})_{12} = \text{id}_{\pi'} \otimes c_{\pi, \pi''} \circ c_{\pi, \pi'} \otimes \text{id}_{\pi''},$$

for all $\pi, \pi', \pi'' \in \text{Rep}_f(A, \Delta)$. Similarly, one shows that $(\Delta \otimes t)R = R_{13}R_{23}$ iff

$$c_{\pi \times \pi', \pi''} = (c_{\pi, \pi''})_{12} (c_{\pi', \pi''})_{23} = c_{\pi, \pi''} \otimes \text{id}_{\pi'} \circ \text{id}_{\pi} \otimes c_{\pi', \pi''},$$

for all $\pi, \pi', \pi'' \in \text{Rep}_f(A, \Delta)$. □

THEOREM 6.4. *Let \mathcal{C} be a semisimple braided tensor $*$ -category with conjugates and $E: \mathcal{C} \rightarrow \mathcal{H}$ a faithful $*$ -preserving tensor functor. Let (A, Δ) and $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ be as constructed in Theorem 5.25. Then (A, Δ) is quasitriangular w.r.t. a unique $R \in M(A \otimes A)$ such that the functor $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ is an equivalence of braided tensor categories w.r.t. the braiding for $\text{Rep}_f(A, \Delta)$ provided by R . R is unitary iff $c_{X,Y}$ is unitary for all X, Y . (A, Δ) is triangular iff the corresponding braiding for $\text{Rep}_f(A, \Delta)$ is symmetric.*

Proof. For $i, j \in I_e$ we have

$$\begin{aligned} & F(c_{X_i, X_j}) \in \text{Mor}_{\mathcal{H}}(E(X_i) \otimes E(X_j), E(X_j) \otimes E(X_i)) \\ &= B(E(X_i) \otimes E(X_j), E(X_j) \otimes E(X_i)). \end{aligned}$$

Thus $\Sigma_{E(X_j), E(X_i)} F(c_{X_i, X_j}) \in B(E(X_i) \otimes E(X_j))$ and we can define $R \in M(A \otimes A) = \prod_{ij} B(E(X_i) \otimes E(X_j))$ by $R_{ij} = \Sigma_{E(X_j), E(X_i)} F(c_{X_i, X_j})$. Naturality of the braiding and functoriality of F now imply that we have

$$F(c_{X,Y}) = \Sigma_{E(X), E(Y)} \overline{(\pi_X \otimes \pi_Y)}(R) \quad \forall X, Y \in \mathcal{C}.$$

Since F is an equivalence, we see that the family $F(c_{X,Y})$ is a braiding for $\text{Rep}_f(A, \Delta)$. Denote this family by $(c_{(E(X),\pi_X),(E(Y),\pi_Y)})$, so

$$c_{(E(X),\pi_X),(E(Y),\pi_Y)} = \Sigma_{E(X),E(Y)}(\overline{\pi_X \otimes \pi_Y})(R) \quad \forall X, Y \in \mathcal{C}.$$

By Proposition 6.3, the element $R \in M(A \otimes A)$ is therefore an R -matrix which is unitary iff $c_{X,Y}$ is unitary for all X, Y and satisfies $\sigma(R) = R^{-1}$ iff $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all X, Y . With the braiding on $\text{Rep}_f(A, \Delta)$ given by R according to Proposition 6.3, we get by definition $F(c_{X,Y}) = c_{F(X),F(Y)}$, for all X, Y , as $F(X) = (E(X), \pi_X)$ for all X . Thus F is a braided functor. \square

Remark 6.5. It seems worthwhile comparing our result with [16, Proposition XIII.1.4], where it is shown that the tensor category of left modules over a bialgebra $(H, m, \eta, \Delta, \varepsilon)$ is braided iff H is quasitriangular w.r.t. some R -matrix. The proof uses the left regular representation of H and therefore would not apply to the category of finite-dimensional H -modules if H is infinite-dimensional. Our setting differs too in that A is nonunital whenever it is infinite-dimensional.

Remark 6.6. As is well known, the category \mathcal{H} of Hilbert spaces has a unique braiding, the flip $\Sigma_{H,K}: H \otimes K \rightarrow K \otimes H$. It is thus natural to ask when the functor E respects this braiding. If so, then

$$\begin{aligned} E(c_{X,Y} \circ c_{Y,X}) &= E(c_{X,Y}) \circ E(c_{Y,X}) = \Sigma_{E(X),E(Y)} \circ \Sigma_{E(Y),E(X)} \\ &= \text{id}_{E(Y) \otimes E(X)} = \text{id}_{E(Y \otimes X)} = E(\text{id}_{Y \otimes X}), \end{aligned}$$

where we suppressed the isomorphisms $d_{X,Y}^E$ as is our policy throughout. If E is faithful this implies $c_{X,Y} \circ c_{Y,X} = \text{id}_{Y \otimes X}$, thus \mathcal{C} is symmetric. In other words, for nonsymmetric categories \mathcal{C} there is no embedding functor satisfying $E(c_{X,Y}) = \Sigma_{E(X),E(Y)}$. For symmetric \mathcal{C} we have the following easy yet important result.

COROLLARY 6.7. *Let \mathcal{C} be a semisimple symmetric tensor $*$ -category and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be a symmetric embedding functor, i.e., $E(c_{X,Y}) = \Sigma_{E(X),E(Y)}$ for all $X, Y \in \mathcal{C}$. Then the corresponding discrete AQG (A, Δ) is cocommutative and F is a $*$ -preserving symmetric monoidal equivalence.*

Proof. It is clear from the definition of the R -matrix R in the proof of Theorem 6.4, that $R = 1$ in this case, so $\Delta^{\text{op}} = \Delta$. \square

We are now in a position to re-prove the classical Tannaka–Krein duality theorem for compact groups.

THEOREM 6.8. *Let \mathcal{C} be a semisimple tensor $*$ -category with conjugates and unitary symmetry, and let $E: \mathcal{C} \rightarrow \mathcal{H}$ be a $*$ -preserving faithful symmetric tensor functor. Then there exists a compact group G and an equivalence $F': \mathcal{C} \rightarrow \text{Rep}_f G$ of symmetric tensor $*$ -categories such that $K' \circ F' = E$, where $K': \text{Rep}_f G \rightarrow \mathcal{H}$ is the forgetful functor. The group G is canonically isomorphic to the intrinsic group G_E of the embedding functor.*

Proof. Under the given assumptions, Theorem 6.4 provides us with a discrete AQG (A, Δ) and an equivalence $F: \mathcal{C} \rightarrow \text{Rep}_f(A, \Delta)$ of tensor $*$ -categories such that $K \circ F = E$. By assumption, E preserves the symmetries, i.e. $E(c_{X,Y}) = \Sigma_{E(X),E(Y)}$ for all $X, Y \in \mathcal{C}$. Then by Corollary 6.7 (A, Δ) is cocommutative. Let G be the intrinsic group of (A, Δ) as defined in Definition 4.1. By Theorem 4.11 we have a (concretely given) equivalence $D: \text{Rep}_f(A, \Delta) \rightarrow \text{Rep}_f G$ of tensor $*$ -categories. Thus the composition $F' = D \circ F: \mathcal{C} \rightarrow \text{Rep}_f G$ is the desired equivalence of tensor $*$ -categories. It satisfies $K' \circ F' = K' \circ D \circ F = K \circ F = E$. The final claim is obvious, since the unitary grouplike elements of (A, Δ) are by definition of (A, Δ) precisely the unitary natural isomorphisms of E that are compatible with the tensor structures, in the sense of (5.1). \square

We end this section with some general facts on R -matrices for quasitriangular AQG.

PROPOSITION 6.9. *Let (A, Δ) be a quasitriangular AQG w.r.t. R . Then the following statements hold:*

- (1) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.
- (2) $(\varepsilon \otimes \iota)R = 1 = (\iota \otimes \varepsilon)R$.
- (3) $(S \otimes \iota)R = R^{-1} = (\iota \otimes S^{-1})(R)$.
- (4) $(S \otimes S)R = R$.

The first equation is called the ‘quantum Yang–Baxter equation’.

Proof. R is a nondegenerate corepresentation of (A, Δ) on A , so (2) and the first equation in (3) follow immediately. The second equation in (3) thus follows by noting that (A, Δ^{op}) is quasitriangular w.r.t. $\sigma(R)$ with coinverse S^{-1} . Claim (4) follows from (3) by $(S \otimes S)R = (\iota \otimes S)(S \otimes \iota)R = (\iota \otimes S)(R^{-1}) = (\iota \otimes S)(\iota \otimes S^{-1})R = R$. Finally, to prove (1) first note that both the maps $\Delta^{\text{op}} \otimes \iota$ and $R_{12}(\Delta \otimes \iota)(\cdot)R_{12}^{-1}$ are nondegenerate homomorphisms from $A \otimes A$ to $M(A \otimes A \otimes A)$, and thus both have unique extensions to $M(A \otimes A)$. Because $\Delta^{\text{op}} = R\Delta(\cdot)R^{-1}$ these maps clearly coincide on $A \otimes A$, and thus on $M(A \otimes A)$, extensions being unique. Hence

$$\begin{aligned} R_{12}R_{13}R_{23}R_{12}^{-1} &= R_{12}(\Delta \otimes \iota)(R)R_{12}^{-1} = (\Delta^{\text{op}} \otimes \iota)(R) \\ &= (\sigma \otimes \iota)(\Delta \otimes \iota)(R) = (\sigma \otimes \iota)(R_{13}R_{23}) = R_{23}R_{13}, \end{aligned}$$

as desired. \square

PROPOSITION 6.10. *Suppose (A, Δ) is a quasitriangular AQG w.r.t. R . Let U denote the universal corepresentation of (A, Δ) as defined in Theorem 3.4. Define $\pi_R: \hat{A} \rightarrow M(A)$ as in Proposition 3.5. Then the following hold:*

- (1) π_R is a nondegenerate homomorphism.
- (2) $(\pi_R \otimes \pi_R)\hat{\Delta} = \Delta^{\text{op}}\pi_R$.
- (3) $\Delta^{\text{op}}(\cdot)(\iota \otimes \pi_R)U = (\iota \otimes \pi_R)U\Delta(\cdot)$.

Conversely, any map $\pi: \hat{A} \rightarrow M(A)$ satisfying these three properties arises from a unique $R \in M(A \otimes A)$ with $\pi = \pi_R$ making (A, Δ) into a quasitriangular AQG. Moreover, the map π_R is $*$ -preserving iff R is unitary.

Proof. It suffices to show that statement (2) is equivalent to $(\iota \otimes \Delta)(R) = R_{13}R_{12}$, this being the only nontrivial step not covered by Proposition 3.5 and Theorem 3.6. But by Theorem 3.4, we get

$$\begin{aligned} & (\pi_R \otimes \pi_R)\hat{\Delta}\omega \\ &= (\pi_R \otimes \pi_R)\hat{\Delta}(\omega \otimes \iota)U = (\pi_R \otimes \pi_R)(\omega \otimes \iota \otimes \iota)(\iota \otimes \hat{\Delta})U \\ &= (\pi_R \otimes \pi_R)(\omega \otimes \iota \otimes \iota)(U_{12}U_{13}) = (\omega \otimes \iota \otimes \iota)(\iota \otimes \pi_R \otimes \pi_R)(U_{12}U_{13}) \\ &= (\omega \otimes \iota \otimes \iota)[((\iota \otimes \pi_R)U)_{12}((\iota \otimes \pi_R)U)_{13}] = (\omega \otimes \iota \otimes \iota)(R_{12}R_{13}) \end{aligned}$$

for all $\omega \in \hat{A}$, whereas

$$\Delta^{\text{op}}\pi_R(\omega) = \Delta^{\text{op}}(\omega \otimes \iota)R = (\omega \otimes \iota \otimes \iota)(\iota \otimes \Delta^{\text{op}})R$$

for all $\omega \in \hat{A}$. □

Note that (1) and (2) just mean that π_R is a morphism of multiplier Hopf algebras from $(\hat{A}, \hat{\Delta})$ to (A, Δ^{op}) . It is even an AQG morphism iff R is unitary by the last statement of the proposition.

In Definition 3.13 we introduced the tensor product $V \times V' = V_{12}V'_{13} \in M(A \otimes B \otimes B')$ of two unitary corepresentations V and V' on B and B' , respectively. Another choice of a tensor product which works equally well, is given by $V \times_{\text{op}} V' = V'_{13}V_{12} \in M(A \otimes B \otimes B')$. In general there is no relation between these two tensor products. In view of the relation between representations of an AQG and corepresentations of its dual AQG (and the relation between the identity representation and the universal corepresentation), the following result should not come as a big surprise.

PROPOSITION 6.11. *Let (A, Δ) be an AQG with universal corepresentation U , so $\hat{U} = \sigma(U)$ is the universal corepresentation of $(\hat{A}, \hat{\Delta})$ on A . Let R be any element of $M(A \otimes A)$. Then $\Delta^{\text{op}}(\cdot)R = R\Delta(\cdot)$ iff*

$$(\hat{U} \times_{\text{op}} \hat{U})R_{23} = R_{23}(\hat{U} \times \hat{U}).$$

Proof. By Theorem 3.4, we get

$$\begin{aligned} \Delta^{\text{op}}(a)R &= (\Delta^{\text{op}}(\iota \otimes a)U)R = (\sigma \Delta(\iota \otimes a)U)R \\ &= ((\sigma \otimes a)(\Delta \otimes \iota)U)R = ((\iota \otimes \iota \otimes a)(\sigma \otimes \iota)(U_{13}U_{23}))R \\ &= (\iota \otimes \iota \otimes a)(U_{23}U_{13}R_{12}) \end{aligned}$$

for all $a \in A$, whereas

$$\begin{aligned} R\Delta(a) &= R\Delta(\iota \otimes a)U = R(\iota \otimes \iota \otimes a)(\Delta \otimes \iota)U = R(\iota \otimes \iota \otimes a)(U_{13}U_{23}) \\ &= (\iota \otimes \iota \otimes a)(R_{12}U_{13}U_{23}) \end{aligned}$$

for all $a \in A$. Hence, $\Delta^{\text{op}}(\cdot)R = R\Delta(\cdot)$ iff $U_{23}U_{13}R_{12} = R_{12}U_{13}U_{23}$. Permuting indices, we see that the latter equation is equivalent to $U_{31}U_{21}R_{23} = R_{23}U_{21}U_{31}$, which again is equivalent to $\hat{U}_{13}\hat{U}_{12}R_{23} = R_{23}\hat{U}_{12}\hat{U}_{13}$ as desired. \square

So given any unitary R -matrix R , the map $f \in \text{Hom}(A \otimes A, A \otimes A)$ given by $f(b) = RbR^*$, for all $b \in A \otimes A$, is an arrow from Δ to Δ^{op} and from $\hat{U} \times \hat{U}$ to $\hat{U} \times_{\text{op}} \hat{U}$ in the sense of Remark 3.9. Also $\delta_R(a) = f(1 \otimes a)$ for $a \in A$, is a coaction in the sense of Remark 3.10. Similarly, the linear map $\delta'_R: A \rightarrow M(A \otimes A)$ given by $\delta'_R(a) = f(a \otimes 1)$, for all $a \in A$, is a nondegenerate $*$ -homomorphism such that $(\delta'_R \otimes \iota)\delta'_R = (\iota \otimes \Delta^{\text{op}})\delta'_R$ and $(\iota \otimes \varepsilon)\delta'_R = \iota$.

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