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MONOIDS, EMBEDDING FUNCTORS AND QUANTUM GROUPS

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We show that the left regular representation π_l of a discrete quantum group (A, Δ) has the absorbing property and forms a monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ in the representation category $\mathrm{Rep}(A, \Delta)$.

Next we show that an absorbing monoid in an abstract tensor $*$-category $\mathcal{C}$ gives rise to an embedding functor (or fiber functor) $E : \mathcal{C} \to \mathrm{Vect}_\mathbb{C}$, and we identify conditions on the monoid, satisfied by $(\pi_l, \tilde{m}, \tilde{\eta})$, implying that E is $*$-preserving.

As is well-known, from an embedding functor $E : \mathcal{C} \to \mathrm{Hilb}$ the generalized Tannaka theorem produces a discrete quantum group (A, Δ) such that $\mathcal{C} \simeq \mathrm{Rep}_f(A, \Delta)$. Thus, for a C^*-tensor category $\mathcal{C}$ with conjugates and irreducible unit the following are equivalent: (1) $\mathcal{C}$ is equivalent to the representation category of a discrete quantum group (A, Δ), (2) $\mathcal{C}$ admits an absorbing monoid, (3) there exists a $*$-preserving embedding functor $E : \mathcal{C} \to \mathrm{Hilb}$.

Keywords: Tensor category; quantum group; regular representation; fiber functor; monoid.

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1. Introduction and Related Work

1.1. *Our approach*

As is well-known, see, for example [22, Secs. 2–3], the finite-dimensional representations of a discrete quantum group form a C^*-tensor category with conjugates and irreducible unit. It is therefore natural to ask for a characterization of representation categories of discrete quantum groups among the C^*-tensor categories. A partial solution is provided by the generalized Tannaka theorem, cf. [25, 13], according to which a C^*-tensor category is such a representation category whenever it comes

equipped with an embedding functor, i.e. a faithful $*$-preserving tensor functor into the category $\mathcal{H}$ of finite-dimensional Hilbert spaces. In this case, the category is called concrete as opposed to abstract. The most transparent approach to the Tannaka theorem defines the quantum group as the algebra of natural transformations of the embedding functor to itself. The monoidal structures of the category and of the embedding functor then give rise to the coproduct of the quantum group. For this approach and further references, cf. [22].

The generalized Tannaka theorem reduces the characterization problem to that of producing an embedding functor. Since the representation category of a quantum group comes with an obvious embedding functor, the existence of such a functor clearly is a necessary condition. However, there exist C^*-tensor categories with conjugates and irreducible unit that do not admit an embedding functor: infinitely many examples (which are even braided) are provided by the categories associated with quantum groups at roots of unity, cf. [24]. This shows that additional assumptions on an abstract C^*-tensor category are needed in order to identify it as the representation category of a quantum group. For example, in [15], it is proven that any C^*-tensor category with conjugates, irreducible unit and with fusion ring isomorphic to that of $SU(N)$ is equivalent to the representation category of the discrete quantum group dual to $SU_q(N)$ for some $q \in \mathbb{R}$. Analogous results have been proven for the other classical groups, assuming in addition that the category is braided.

The case of abstract symmetric tensor categories was settled already in the late 1980s. By a remarkable result of Doplicher and Roberts [6], any symmetric C^*-tensor category with conjugates and irreducible unit is equivalent as a C^*-tensor category to the representation category of a unique compact group. If one wishes an equivalence of symmetric categories, one must also allow super groups. This result has applications [7] to algebraic quantum field theory, where symmetric C^*-tensor categories arise without an *a priori* given embedding functor. The proof in [6], however, does not follow the strategy outlined above of constructing an embedding functor and then applying the Tannaka theorem.

Independently and at about the same time, motivated by applications to algebraic geometry, Deligne proved [3] that a rigid abelian symmetric tensor category with irreducible unit is equivalent to the representation category of a proalgebraic group, provided that the intrinsic dimension of every object is a positive integer. His proof consists of constructing an embedding functor and applying the algebraic Tannaka theorem of Saavedra-Rivano.
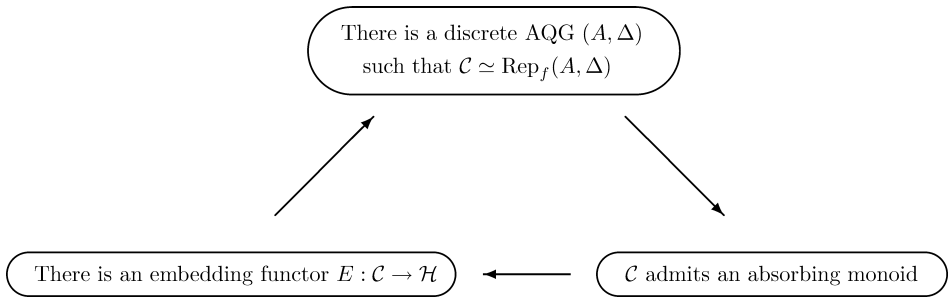
The crucial notion in Deligne's construction of the embedding functor is that of an absorbing commutative monoid. Recall that a monoid in a tensor category is a triple (Q, m, η), where $m : Q \otimes Q \to Q$ and $\eta : \mathbf{1} \to Q$ are morphisms such that $(m \otimes \mathrm{id}_Q) \circ m = (\mathrm{id}_Q \otimes m) \circ m$ and $m \circ (\eta \otimes \mathrm{id}_Q) = \mathrm{id}_Q = m \circ (\mathrm{id}_Q \otimes \eta)$. An object Q is called absorbing if the Q-module $Q \otimes X$ is isomorphic to some multiple of Q for any object X. Deligne obtained the absorbing commutative monoid using categorical generalizations of results from commutative algebra — it is here that the symmetry

plays a central role. His proof was simplified considerably in [2]. Note, however, that the monoid of [3, 2] fails to satisfy hypothesis (1) of Proposition 3.2 below, which complicates the construction of an embedding functor. For a construction of a monoid satisfying all assumptions of Proposition 3.2, cf. [21].

The aim of this paper is to demonstrate the usefulness of the monoid approach in the general non-symmetric case. This is done in two steps. On the one hand, we prove that the passage from an absorbing monoid to an embedding functor works in the general case. We also identify conditions on the monoid guaranteeing that the functor is ∗-preserving. Whereas the existence of an embedding functor refers to $\mathcal{H}$ and thus is an external condition on the category, the existence of an absorbing monoid is an internal property. As such it is more amenable to proof, as Deligne's result in the symmetric case illustrates. A technical aspect should be pointed out though: a category $\mathcal{C}$ with conjugates can contain an absorbing object only if it has finitely many equivalence classes of objects. Otherwise it needs to be suitably enlarged, which is done using the category $\hat{\mathcal{C}}$ of inductive limits. We say that $\mathcal{C}$ *admits* an absorbing object if there exists a monoid (Q, m, η) in $\hat{\mathcal{C}}$ such that the Q-module $(Q \otimes X, m \otimes \mathrm{id}_X)$ is isomorphic to a multiple of the Q-module (Q, m), for every $X \in \mathcal{C}$.

On the other hand, starting with a discrete quantum group (A, Δ), we explicitly construct an absorbing monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ in the representation category. Here π_l is the regular representation of the algebra A on the vector space A given by multiplication from the left. In order to define the morphisms $\tilde{m}, \tilde{\eta}$, let $(\hat{A}, \hat{\Delta})$ denote the dual compact quantum group with multiplication $\hat{m}$ and unit $1_{\hat{A}}$, and let $\mathcal{F} : A \to \hat{A}$, $a \mapsto \varphi(\cdot\, a)$ denote the Fourier transform, where $\varphi : A \to \mathbb{C}$ is the left invariant positive functional of (A, Δ). The linear maps $\tilde{m} : A \otimes A \to A$ and $\tilde{\eta} : \mathbb{C} \to A$ are then given by $\tilde{m} = \mathcal{F}^{-1}\hat{m}(\mathcal{F} \otimes \mathcal{F})$ and $\tilde{\eta}(1) = \mathcal{F}^{-1}(1_{\hat{A}})$. We call this absorbing monoid the regular monoid of (A, Δ).

Our main result then is that, for a C^*-tensor category $\mathcal{C}$ with conjugates and irreducible unit, we have three equivalent statements illustrated by the following diagram:

There is a discrete AQG (A, Δ)
such that $\mathcal{C} \simeq \mathrm{Rep}_f(A, \Delta)$

There is an embedding functor $E : \mathcal{C} \to \mathcal{H}$ ← $\mathcal{C}$ admits an absorbing monoid

We summarize some further results. Our construction actually provides an absorbing semigroup $(\pi_l, \tilde{m})$ for any algebraic quantum group, and we show that this semigroup has a unit $\tilde{\eta}$ if and only if the quantum group is discrete. Dually, there

exists a regular comonoid if and only if the quantum group is compact. In the finite-dimensional case, the regular monoid and comonoid combine to a Frobenius algebra. We identify the intrinsic group of a discrete quantum group with the intrinsic group of its regular monoid.

We also show that an abstract C^*-tensor category $\mathcal{C}$ with conjugates and irreducible unit admits an absorbing object Q in $\hat{\mathcal{C}}$ if and only if $\mathcal{C}$ admits an integer valued dimension function, i.e. a map $\mathrm{Obj}\,\mathcal{C} \to \mathbb{N}$ that is additive and multiplicative. While this clearly is a necessary condition for $\mathcal{C}$ to admit an absorbing monoid, to proceed further in the opposite direction, one also needs an associative morphism $m : Q \otimes Q \to Q$, but the existence of such a morphism remains to be proven.

1.2. *Related work*

We would like to point out several earlier references that are related to the present work. The fact that a C^*-tensor category with finitely many simple objects and an absorbing monoid is the representation category of a finite-dimensional C^*-Hopf algebra was obtained in [18, Theorem 6.7]. The proof basically proceeds by showing that a finite C^*-tensor category can be faithfully realized by endomorphisms of a von Neumann algebra and then appealing to [17, Theorem 6.2]. This approach can in principle (this has never been done) be extended to C^*-tensor categories with countably many simple objects by combining [29], which realizes such categories as categories of bimodules over a von Neumann algebra N, and the extension of Longo's result [17] to infinite index subfactors of depth two obtained in [11]. Concerning this generalization, we observe that putting the above-mentioned results together as indicated would require a non-trivial amount of work since they use different frameworks (type II vs. III algebras, endomorphisms vs. bimodules), and dropping the countability assumption on the category made in [29] seems very difficult. Furthermore, the above approaches (in the finite and countable cases) use very heavy operator algebraic machinery, whereas the approach outlined in Sec. 1.1 is essentially purely algebraic and quite elementary and has the added benefit of working without any assumption on the cardinality of the category.

More recently, the relationship between the absorbing property and embedding functors has been studied in [28, Appendices A–B], though with different emphasis and results. Finally, we would like to point out the papers [8, 9], which provide a study, in the context of C^*-tensor categories, of multiplicative unitaries, which are a convenient tool for the study of the regular representation in the theory of locally compact groups and quantum groups.

2. From Algebraic Quantum Groups to Absorbing Monoids

2.1. *Three representation categories of AQG*

For the general theory of algebraic quantum groups (AQG), we refer to [27] and to, e.g. [10] for the basics of representation theory, as well as to the survey [22],

where both subjects are covered in considerable detail. For the standard categorical notions of (braided/symmetric) tensor categories, natural transformations etc. our standing reference is [19], but most of the relevant notions can also be found in [22]. We will always denote AQG by (A, Δ), where A is a non-degenerate $*$-algebra and $\Delta : A \to M(A \otimes A)$ is the comultiplication. As usual, we denote the multiplication, counit and coinverse by $m : A \otimes A \to A$, $\varepsilon : A \to \mathbb{C}$ and $S : A \to A$, respectively. The left invariant positive functional is denoted by φ.

Definition 2.1. Let (A, Δ) be an AQG. A homomorphism $\pi : A \to \text{End } K$, where K is a complex vector space, is called a representation of A on K if $\pi(A)K = K$. A $*$-representation is a representation π on a pre-Hilbert space K, that is $*$-preserving in the sense that $(\pi(a)u, v) = (u, \pi(a^*)v)$ for all $a \in A$ and $u, v \in K$. By $\text{Rep}(A, \Delta)$ we denote the category whose objects are $*$-representations and whose arrows are the intertwining linear maps, i.e. if π' is another $*$-representation of A on K', then

$$\text{Hom}(\pi, \pi') = \{s \in \text{Hom}(K, K') \mid s\pi(a)v = \pi'(a)sv \ \forall v \in K, \ a \in A\}.$$

Remark 2.2. Recall that a homomorphism $\pi : A \to B$ of non-degenerate algebras is called non-degenerate if $\pi(A)B = B = B\pi(A)$. It would therefore seem natural to define a representation of A on K to be a homomorphism $\pi : A \to \text{End}K$ that satisfies $\pi(A)\text{End}K = \text{End}K = \text{End}K\pi(A)$. However, this notion is too restrictive since it is never satisfied by the usual left regular representation π_l, to be introduced shortly, if (A, Δ) is discrete and non-unital. So see this it suffices to notice that the image of $\pi_l(a)e \in \text{End}A$ is finite-dimensional for all $a \in A$ and $e \in \text{End}A$.

We define the left regular representation $\pi_l : A \to \text{End } A$ of an AQG by $\pi_l(a)(x) = ax$ for $a, x \in A$. This terminology is justified, since the non-degeneracy condition in Definition 2.1 holds because $A^2 = A$, which again follows from the existence of local units for A. Furthermore, π_l is a $*$-representation with respect to the inner product $(\cdot, \cdot)$ on A given by $(x, y) = \varphi(y^*x)$. Thus $\pi_l \in \text{Rep}(A, \Delta)$. Similarly, one defines the right regular representation $\pi_r \in \text{Rep}(A_{\text{op}}, \Delta)$ by the formula $\pi_r(a)(x) = xa$ for $a, x \in A$. It is a $*$-representation with respect to the inner product on the opposite algebra A_{op} given by $(x, y) = \varphi(xy^*)$.

Recall that the left multiplier algebra $L(A)$ of a non-degenerate algebra A is the vector space $L(A) = \{\psi \in \text{End } A \mid \psi(ab) = \psi(a)b \ \forall a, b \in A\}$ with product $\psi_1\psi_2 = \psi_1 \circ \psi_2$, i.e. composition of maps. Note that $\pi_l : A \to L(A)$ is an injective algebra homomorphism. Similarly, the right multiplier algebra $R(A)$ of a non-degenerate algebra A is the vector space $R(A) = \{\phi \in \text{End } A \mid \phi(ab) = a\phi(b) \ \forall a, b \in A\}$ together with the product given by opposite composition: $\phi_1\phi_2 = \phi_2 \circ \phi_1$. Again $\pi_r : A \to R(A)$ is an injective algebra homomorphism. Further, note that by definition the identity map is a linear antimultiplicative map from $\text{Hom}(\pi_l, \pi_l)$ to $R(A)$ and a linear multiplicative map from $\text{Hom}(\pi_r, \pi_r)$ to $L(A)$. Assume A is a non-degenerate $*$-algebra and let $\psi \in L(A)$. Define $\psi^* \in R(A)$ by $\psi^*(a) = \psi(a^*)^*$ for $a \in A$. The assignment $\psi \mapsto \psi^*$ is an antilinear and antimultiplicative bijection from $L(A)$ to $R(A)$. The multiplier algebra $M(A)$ of a non-degenerate algebra A is the vector

space $M(A) = \{(\psi, \phi) \in L(A) \times R(A) \mid \phi(a)b = a\psi(b) \; \forall a, b \in A\}$ with pointwise multiplication, i.e. $(\psi_1, \phi_1)(\psi_2, \phi_2) = (\psi_1\psi_2, \phi_1\phi_2) = (\psi_1 \circ \psi_2, \phi_2 \circ \phi_1)$. Now the map $\pi_{lr} : a \mapsto (\pi_l(a), \pi_r(a))$ embeds A into $M(A)$ as an algebra. Whenever A is a $*$-algebra, so is $M(A)$ and the embedding is $*$-preserving. If A is unital then we have the algebra isomorphisms $M(A) \cong L(A) \cong R(A) \cong A$.

Any homomorphism $\pi : A \to \text{End} K$ of a non-degenerate algebra A such that $\pi(A)K = K$ and such that $\pi(A)v = 0$ implies $v = 0$ has a unique extension to a unital homomorphism $\tilde{\pi} : M(A) \to \text{End} K$ given by the formula $\tilde{\pi}(x)\pi(a)v = \pi(xa)v$, for $x \in M(A)$, $a \in A$ and $v \in K$. Whenever A has local units, the property $\pi(A)v = 0 \Rightarrow v = 0$ follows immediately from $\pi(A)K = K$, see [10] for more details. If $\pi, \pi' \in \text{Rep}(A, \Delta)$, then clearly $\pi \otimes \pi' : A \otimes A \to \text{End} K \otimes \text{End} K' \subset \text{End}(K \otimes K')$ determined by $(\pi \otimes \pi')(a \otimes a') = \pi(a) \otimes \pi(a')$ for $a, a' \in A$ satisfies $(\pi \otimes \pi')(A \otimes A)(K \otimes K') = K \otimes K'$. It therefore has a unique extension to a unital $*$-homomorphism from $M(A \otimes A)$ to $\text{End}(K \otimes K')$, which we again denote by $\pi \otimes \pi'$. It is obvious that $\pi \times \pi' = (\pi \otimes \pi') \circ \Delta$ is non-degenerate, and therefore belongs to $\text{Rep}(A, \Delta)$. Hence $\text{Rep}(A, \Delta)$ is a tensor category with irreducible unit ε. Suppressing the totally canonical associativity constraint, we treat the tensor category $\text{Rep}(A, \Delta)$ as strict. Note that $(\pi_l \times \pi_l)(a)x = \Delta(a)x$ for $a \in A$ and $x \in A \otimes A$. By $\text{Rep}_f(A, \Delta)$ we mean the full tensor subcategory of $\text{Rep}(A, \Delta)$ consisting of finite-dimensional representations, i.e. those $\pi \in \text{Rep}(A, \Delta)$ for which $\dim K < \infty$.

Clearly, $\text{Rep}_f(A, \Delta)$ is a tensor $*$-category with respect to the adjoint operation for bounded linear maps between Hilbert spaces, but we are not aware of a method to turn $\text{Rep}(A, \Delta)$ into a tensor $*$-category which works for any $\text{AQG}(A, \Delta)$. Yet, we have the following.

Proposition 2.3. *Let (A, Δ) be an AQG and define $\text{Rep}_*(A, \Delta)$ to be the full subcategory of $\text{Rep}(A, \Delta)$ consisting of representations that are direct sums of finite-dimensional irreducible $*$-representations with finite multiplicities. Then there exists a $*$-operation on $\text{Rep}_*(A, \Delta)$ extending that of $\text{Rep}_f(A, \Delta)$. This $*$-operation is compatible with the scalar products in the sense that*

$$(su, v)_{K'} = (u, s^*v)_K$$

for $u \in K, v \in K'$ and $s \in \text{Hom}(\pi, \pi')$, where π, π' are representations on K, K' with inner products $(\cdot, \cdot)_K$ and $(\cdot, \cdot)_{K'}$, respectively. For $\pi \cong \oplus_i n_i \pi_i$ and $\pi' \cong \oplus_i n'_i \pi_i$, where the representations $\pi_i \in \text{Rep}_f(A, \Delta)$ are irreducible and pairwise non-isomorphic, we use the isomorphisms

$$\text{Hom}(\pi, \pi') \cong \prod_i \text{Hom}(n_i\pi_i, n'_i\pi_i) \cong \prod_i M_{n_i, n'_i}(\mathbb{C})$$

to equip the spaces $\text{Hom}(\pi, \pi')$, where $\pi, \pi' \in \text{Rep}_*(A, \Delta)$, with the product topology. With respect to these topologies the composition $\circ$ is continuous.

Proof. Let I be the set of unitary equivalence classes of finite-dimensional irreducible $*$-representations and let π_i be a representation in the class $i \in I$ acting on the Hilbert space H_i. Consider two representations $\pi \cong \oplus_i \pi_i \otimes I_{K_i}$ and $\pi' \cong \oplus_i \pi_i \otimes I_{K_i'}$, where K_i, K_i' are finite-dimensional multiplicity spaces. Here it is understood that the scalar products on the finite-dimensional spaces $H_i \otimes K_i$ are the restrictions of that of K and similarly for K', etc. Since the representations $\pi_i \otimes I_{K_i}$ and $\pi_j \otimes I_{K_j'}$ are disjoint if $i \neq j$, every morphism $s : \pi \to \pi'$ is given by a family (s_i), where $s_i \in \operatorname{Hom}(\pi_i \otimes I_{K_i}, \pi_i \otimes I_{K_i'})$. Here s_i is a morphism in the $*$-category $\operatorname{Rep}_f(A, \Delta)$ and therefore has an adjoint s_i^* defined by $(s_i u_i, v_i)_{H_i \otimes K_i'} = (u_i, s_i^* v_i)_{H_i \otimes K_i}$. Conversely, every such family constitutes a morphism in $\operatorname{Hom}(\pi, \pi')$. Thus we can define an element of $\operatorname{Hom}(\pi', \pi)$ by $s^* = (s_i^*)$. It is evident that this definition satisfies the properties of a $*$-operation and extends the $*$-operation of $\operatorname{Rep}_f(A, \Delta)$. Now $(su, v)_{K'} = (u, s^* v)_K$ is automatic since $(\cdot, \cdot)_K = \sum_i (\cdot, \cdot)_{H_i \otimes K_i}$, etc. The continuity of $\circ$ is also obvious. $\qquad\square$

Remark 2.4. (1) Note that $\operatorname{Rep}_*(A, \Delta)$ is not closed under tensor products, but it is stable under tensor products with finite-dimensional $*$-representations.

(2) For a general AQG the category, $\operatorname{Rep}_*(A, \Delta)$ may consist only of copies of ε. This does not happen in the discrete case to be discussed below.

Proposition 2.5. *Let (A, Δ) be a discrete AQG, so $A = \oplus_{i \in I} \operatorname{End} H_i$ with H_i finite-dimensional Hilbert spaces. Let I_i be the unit of $\operatorname{End} H_i$ and let $p_i \in \operatorname{Rep}(A, \Delta)$ denote the canonical projection from A to $\operatorname{End} H_i$. Then:*

(1) *For any $\pi \in \operatorname{Rep}(A, \Delta)$, we have $\pi \cong \oplus_i n_i p_i$ with $n_i = \dim \pi(I_i) K / \dim H_i$.*
(2) *$\pi_l \cong \oplus_{i \in I} \dim H_i \, p_i$, so $\pi_l \in \operatorname{Rep}_*(A, \Delta)$.*
(3) *$\operatorname{Rep}_f(A, \Delta)$ is equivalent to the tensor category of all finite-dimensional representations of (A, Δ) and $\operatorname{Rep}(A, \Delta)$ is equivalent to the tensor category of all representations.*
(4) *$R(A) \cong M(A)$ as unital algebras, whereas $\operatorname{Hom}(\pi_l, \pi_l)$ and $M(A)$ are antiisomorphic as unital $*$-algebras.*

Proof. (1) The subspaces $K_i = \pi(I_i) K$ are clearly linear independent and a short argument using $\pi(A) K = K$ shows that $K \cong \oplus_i K_i$. Define $*$-representations π_i of (A, Δ) on K_i by $\pi_i(a) = \pi(a) \restriction K_i$ for $a \in A$, and note that $\pi \cong \oplus_i \pi_i$ with $\pi_i \cong n_i p_i$.

(2) This follows from (1) by noting that $K_i = \operatorname{End} H_i$ so $n_i = \dim H_i$.

(3) This follows from the facts that the decomposition in (1) holds also for representations which are not $*$-representations and that the irreducible representations p_i are $*$-representations.

(4) By definition $\operatorname{Hom}(\pi_l, \pi_l)$ and $R(A)$ are anti-isomorphic as unital algebras. Let $\phi \in R(A)$. In view of the definition of right multipliers we have $\phi(a) = \phi(I_i a) = I_i \phi(a) \in \operatorname{End} H_i$ for every $i \in I$, $a \in \operatorname{End} H_i$. Thus we obtain restrictions

$\phi_i = \phi \restriction \operatorname{End} H_i \in R(\operatorname{End} H_i)$ such that $\phi = \oplus_i \phi_i$. Conversely, the latter formula defines an element of $R(A)$ for every element $(\phi_i, i \in I)$ of $\prod_i R(\operatorname{End} H_i)$. Since the $\operatorname{End} H_i$ are unital we have $R(\operatorname{End} H_i) = \operatorname{End} H_i$, and therefore $R(A) = \prod_i \operatorname{End} H_i = M(A)$ as unital algebras. It follows that $\operatorname{Hom}(\pi_l, \pi_l)$ and $M(A)$ are anti-isomorphic as unital $*$-algebras. $\qquad\square$

For a discrete AQG we normalize the left invariant positive functional by requiring $\varphi(I_0) = 1$.

We aim now at understanding the relation between $\operatorname{Rep}_f(A, \Delta)$ and $\operatorname{Rep}(A, \Delta)$ in more categorical terms, whenever (A, Δ) is a discrete quantum group.

In order to make sense of infinite direct sums of objects, we need some categorical devices. Let $\mathcal{J}$ be a small category, the index category, and let $F : \mathcal{J} \to \mathcal{C}$ be a functor. We denote the objects of $\mathcal{J}$ by i, j, k and write $X_i = F(i)$. A pair (X, f_i), where $X \in \mathcal{C}$ and the morphisms $f_i : X_i \to X$ for $i \in \mathcal{J}$ satisfy $f_j \circ F(s) = f_i$ for every $s : i \to j$, is called a cone. We say F has an *inductive limit* (or colimit) if there exists a cone (X, f_i) that is universal, i.e. for any other cone (Y, g_i), there exists a unique $t : X \to Y$ such that $t \circ f_i = g_i$ for all $i \in \mathcal{J}$. The category $\mathcal{J}$ is *filtered* if it satisfies the following conditions:

(1) For every $i, j \in \mathcal{J}$, there exists $k \in \mathcal{J}$ and morphisms $u : i \to k$ and $v : j \to k$.
(2) For every $i, j \in \mathcal{J}$ and $u, v : i \to j$, there exists $s : j \to k$ such that $s \circ u = s \circ v$.

An inductive limit $F : \mathcal{J} \to \mathcal{C}$ is called *filtered* if $\mathcal{J}$ is a filtered category. Every directed partially ordered set J gives rise to a filtered category $\mathcal{J}$, where $\operatorname{Obj} \mathcal{J} = J$ and $\operatorname{Hom}_{\mathcal{J}}(i, j)$ contains one element if $i \leq j$ and none otherwise. Given a set S, the power set 2^S is a directed partially ordered set.

In our applications $\mathcal{C}$ has finite direct sums, and we define an infinite direct sum $\oplus_{j \in S} Y_j$ as a filtered inductive limit over $F : 2^S \to \mathcal{C}$. Here 2^S is the filtered category corresponding to the power set 2^S, and the functor F is given by choosing a finite direct sum for every $s \in 2^S$. An example of a category for which all filtered inductive limits exist is the category $\operatorname{Rep}(A, \Delta)$, where (A, Δ) is a discrete AQG.

We will now consider a completion $\hat{\mathcal{C}}$ with respect to all filtered inductive limits of a given category $\mathcal{C}$. Given any category $\mathcal{C}$ there exists a category $\operatorname{Ind} \mathcal{C}$ of "filtered inductive limits of objects in $\mathcal{C}$". The standard reference is [1]. We collect some of its properties that we shall need, none of which is new.

Proposition 2.6. *Let $\mathcal{C}$ be a category and denote $\hat{\mathcal{C}} = \operatorname{Ind} \mathcal{C}$. Then*

(1) $\hat{\mathcal{C}}$ *contains $\mathcal{C}$ as a full subcategory.*
(2) $\hat{\mathcal{C}}$ *is complete with respect to filtered inductive limits. In particular, there exist infinite sums $Z \cong \bigoplus_{i \in I} Z_i$, where $Z_i \in \mathcal{C}$.*
(3) *If $\mathcal{C}$ is abelian, in particular semisimple, then $\hat{\mathcal{C}}$ is abelian.*
(4) *If $\mathcal{C}$ is semisimple, then every object of $\hat{\mathcal{C}}$ is a filtered inductive limit of objects in $\mathcal{C}$. In this case, $\hat{\mathcal{C}}$ is uniquely characterized up to equivalence by this property and (1)–(2).*

(5) *If $\mathcal{C}$ is monoidal, then the tensor product extends uniquely to $\hat{\mathcal{C}}$. Similarly if $\mathcal{C}$ is braided or symmetric, then so is $\hat{\mathcal{C}}$.*

(6) *If $\mathcal{C}$ has exact tensor product, in particular if $\mathcal{C}$ has duals, then the tensor product of $\hat{\mathcal{C}}$ is also exact.*

Proof. We limit ourselves giving references for the interested reader. Statements (1)–(2) are proven in [1], whereas (3) follows from [1, 12]. Claim (4) is proven in [4, Sec. 4], and for (5), (6), see [5, 3]. □

Remark 2.7. Concerning the construction of $\hat{\mathcal{C}}$, we only note that its objects are pairs $(\mathcal{G}, F)$, where $\mathcal{G}$ is a small filtered category and $F : \mathcal{G} \to \mathcal{C}$ is a functor. Denoting objects of $\hat{\mathcal{C}}$ by (X_i), where $i \in \mathrm{Obj}\,\mathcal{G}$ and $X_i = F(i)$, the hom-sets are defined by

$$\mathrm{Hom}_{\hat{\mathcal{C}}}((X_i), (Y_j)) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

Lemma 2.8. *Let $\mathcal{C}$ be a semisimple tensor $*$-category, and let $\mathcal{C}_*$ be the full subcategory of $\hat{\mathcal{C}}$ consisting of direct sums of irreducible objects of $\mathcal{C}$ with finite multiplicities. Then $\mathcal{C}_*$ has a $*$-operation extending that of $\mathcal{C}$.*

Proof. Exactly as for Proposition 2.3. □

The following is an immediate, though very useful generalization of [4, Example 4.3.2].

Proposition 2.9. *Let (A, Δ) be a discrete AQG and let $\mathcal{C} = \mathrm{Rep}_f(A, \Delta)$. Then there is a canonical equivalence $\hat{F} : \hat{\mathcal{C}} \to \mathrm{Rep}(A, \Delta)$ of tensor categories which restricts to the identity on the full subcategory $\mathcal{C}$ of $\hat{\mathcal{C}}$ and restricts to an equivalence of $\mathcal{C}_*$ and $\mathrm{Rep}_*(A, \Delta)$.*

Proof. Note that if $\mathcal{C}$ is a category of vector spaces or of representations, then the filtered inductive limits above are inductive limits in the ordinary sense. The category $\mathrm{Rep}_f(A, \Delta)$ is semisimple and every object of $\mathrm{Rep}(A, \Delta)$ is an inductive limit of objects in $\mathrm{Rep}_f(A, \Delta)$. Since $\mathrm{Rep}(A, \Delta)$ is closed with respect to inductive limits, the equivalence $\hat{\mathcal{C}} \simeq \mathrm{Rep}(A, \Delta)$ follows from assertion (4) in Proposition 2.6. The last statement is obvious since both $\mathcal{C}_*$ and $\mathrm{Rep}_*(A, \Delta)$ are defined as the respective full subcategories of objects that contain the simple objects with finite multiplicities. □

2.2. *Construction of the regular monoid*

Let (A, Δ) be an AQG and $(\hat{A}, \hat{\Delta})$ its Pontryagin dual with the conventions that $\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)$ and $\hat{m}(\omega \otimes \omega') = (\omega \otimes \omega')\Delta$, where $a, b \in A$ and $\omega, \omega' \in \hat{A}$. Consider the Fourier transform $\mathcal{F} : A \to \hat{A}$, which is given by $\mathcal{F}(a) = \hat{a} = a\varphi$, for

$a \in A$. Here and in the sequel $c\varphi$ and φc denote the linear functionals on A given by $c\varphi = \varphi(\cdot\, c)$ and $\varphi c = \varphi(c\,\cdot)$, for $c \in M(A)$. It is known that $\mathcal{F}$ is a bijective linear map satisfying Plancherel's formula $\hat{\psi}(\mathcal{F}(a)^*\mathcal{F}(b)) = \varphi(a^*b)$, for $a, b \in A$. Here $\hat{\psi}$ is the right invariant functional on $(\hat{A}, \hat{\Delta})$ determined by $\hat{\psi}\mathcal{F} = \varepsilon$. If (A, Δ) is discrete, then $(\hat{A}, \hat{\Delta})$ is a Hopf $*$-algebra and $\hat{\psi}$ is a bounded functional on $\hat{A}$ which is both left- and right invariant.

Lemma 2.10. *Let (A, Δ) be an AQG. Then*

$$(\varphi \otimes \varphi)(\Delta(c)(a \otimes b)) = \varphi(c(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(b))(a \otimes 1)])$$
$$= \varphi(c(\iota \otimes \varphi)[(1 \otimes S^{-1}(b))\Delta(a)]),$$

for all $a, b \in A$ and $c \in M(A)$.

Proof. The formula $\varphi((\omega S \otimes \iota)\Delta(c)b) = \varphi(c(\omega \otimes \iota)\Delta(b))$ holds for any $\omega \in \hat{A}$, $b \in A$ and $c \in M(A)$, and is known as the strong left invariance property [16]. Thus

$$(\varphi \otimes \varphi)(\Delta(c)(a \otimes b)) = \varphi(((\hat{a}S^{-1}S \otimes \iota)\Delta(c))b)$$
$$= \varphi(c(\hat{a}S^{-1} \otimes \iota)\Delta(b)) = \varphi(c(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(b))(a \otimes 1)])$$

for $a, b \in A$ and $c \in M(A)$. The computation

$$\varphi(c(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(b))(a \otimes 1)])$$
$$= \varphi(c(\hat{a}S^{-1} \otimes \iota)\Delta(b))$$
$$= \hat{a}S^{-1}((\iota \otimes \varphi c)\Delta(b)) = \varphi(S^{-1}[(\iota \otimes \varphi c)\Delta(b)]a)$$
$$= \varphi((\varphi cS \otimes \iota)\Delta S^{-1}(b)a) = \varphi(S^{-1}(b)(\varphi c \otimes \iota)\Delta(a))$$
$$= \varphi(c(\iota \otimes \varphi)[(1 \otimes S^{-1}(b))\Delta(a)])$$

proves the second identity. $\qquad\square$

Let $\hat{m} : \hat{A} \otimes \hat{A} \to \hat{A}$ be the linearized multiplication on $\hat{A}$, so $\hat{m}(\omega \otimes \eta) = \omega\eta$, for $\omega, \eta \in \hat{A}$, which means that

$$\hat{m}(\hat{a} \otimes \hat{b})(c) = (\hat{a}\hat{b})(c) = (\hat{a} \otimes \hat{b})\Delta(c) = (\varphi \otimes \varphi)(\Delta(c)(a \otimes b)),$$

for $a, b, c \in A$, and remains valid also for $c \in M(A)$.

Definition 2.11. A semigroup in a (strict) tensor category $\mathcal{C}$ is a pair (Q, m), where Q is an object and $m : Q \otimes Q \to Q$ satisfies $m \circ (m \otimes \mathrm{id}_Q) = m \circ (\mathrm{id}_Q \otimes m)$. A monoid is a triple (Q, m, η) where (Q, m) is a semigroup and $\eta : \mathbf{1} \to Q$ satisfies $m \circ (\eta \otimes \mathrm{id}_Q) = m \circ (\mathrm{id}_Q \otimes \eta) = \mathrm{id}_Q$. Two semigroups (monoids) (Q, m, η), (Q', m', η') are isomorphic if there exists an isomorphism $s : Q \to Q'$ such that $s \circ m = m' \circ (s \otimes s)$ (and $s \circ \eta = \eta'$).

Proposition 2.12. *Let notation be as above and consider the linear map* $\tilde{m} = \mathcal{F}^{-1}\hat{m}(\mathcal{F} \otimes \mathcal{F}) : A \otimes A \to A$. *Then:*

(1) $\varphi(c\tilde{m}(x)) = (\varphi \otimes \varphi)(\Delta(c)x)$ *for* $x \in A \otimes A$ *and* $c \in M(A)$.
(2) $\tilde{m}(a \otimes b) = (\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(b))(a \otimes 1)] = (\iota \otimes \varphi)[(1 \otimes S^{-1}(b))\Delta(a)]$ *for* $a, b \in A$.
(3) $\tilde{m}(\tilde{m} \otimes \iota) = \tilde{m}(\iota \otimes \tilde{m})$, *so* $\tilde{m}$ *is a multiplication on* A.
(4) $\tilde{m}(\Delta(a)x) = a\tilde{m}(x)$ *for* $a \in A$ *and* $x \in A \otimes A$.

Proof. The identity $\mathcal{F}\tilde{m} = \hat{m}(\mathcal{F} \otimes \mathcal{F})$ means that

$$\varphi(c\tilde{m}(a \otimes b)) = (\mathcal{F}\tilde{m}(a \otimes b))(c) = (\hat{m}(\mathcal{F} \otimes \mathcal{F})(a \otimes b))(c)$$
$$= \hat{m}(\hat{a} \otimes \hat{b})(c) = (\varphi \otimes \varphi)(\Delta(c)(a \otimes b)),$$

for $a, b, c \in A$, which proves statement (1). Statement (2) is now immediate from Lemma 2.10 and faithfulness of φ. To show (3), calculate

$$\tilde{m}(\tilde{m} \otimes \iota) = \mathcal{F}^{-1}\hat{m}(\mathcal{F} \otimes \mathcal{F})(\mathcal{F}^{-1}\hat{m}(\mathcal{F} \otimes \mathcal{F}) \otimes \iota) = \mathcal{F}^{-1}\hat{m}(\hat{m} \otimes \iota)(\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})$$
$$= \mathcal{F}^{-1}\hat{m}(\iota \otimes \hat{m})(\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}) = \tilde{m}(\iota \otimes \tilde{m}).$$

Claim (4) is checked by using (1) and computing

$$\varphi(c\tilde{m}(\Delta(a)x)) = (\varphi \otimes \varphi)(\Delta(c)\Delta(a)x) = (\varphi \otimes \varphi)(\Delta(ca)x) = \varphi(ca\tilde{m}(x)),$$

for $x \in A \otimes A$ and $a, c \in A$. Now (4) follows by faithfulness of φ. □

Corollary 2.13. *We have* $\tilde{m} \in \mathrm{Hom}(\pi_l \times \pi_l, \pi_l)$, *and* $(\pi_l, \tilde{m})$ *is a semigroup in* $\mathrm{Rep}(A, \Delta)$.

Proof. By the previous proposition, the linear map $\tilde{m} : A \otimes A \to A$ is associative and satisfies

$$\tilde{m}(\pi_l \times \pi_l)(a)x = \tilde{m}(\Delta(a)x) = \pi_l(a)\tilde{m}x,$$

for $a \in A$ and $x \in A \otimes A$. Thus $\tilde{m}$ is an intertwiner from $\pi_l \times \pi_l$ to π_l. □

Remark 2.14. If $s : \pi \to \pi'$ is bounded with respect to the scalar products on K, K', then s^* can be defined as the adjoint of the unique extension of s to the Hilbert space completions. Therefore, the reader might wonder why we do not work with the usual tensor $*$-category of non-degenerate $*$-representations of a discrete AQG on Hilbert spaces. Considering bounded morphisms is, however, not sufficient for our purposes, since the morphism $\tilde{m}$, which plays a fundamental role in our considerations, is not bounded with respect to the 2-norms on $H \otimes H$ and H. To see this it suffices to consider the simple case of $(\hat{A}, \hat{\Delta})$ with $\hat{A}_r = C(\mathbb{T})$, so $\hat{m}(f \otimes g)(s, t) = f(s)g(t)$, for $f, g \in C(\mathbb{T})$ and $s, t \in \mathbb{T}$. Here $(\hat{A}_r, \hat{\Delta}_r)$ is the analytic extension of $(\hat{A}, \hat{\Delta})$ in the sense of [16], so $\hat{A}_r$ is a unital C*-algebra and $(\hat{A}_r, \hat{\Delta}_r)$ is a compact quantum group in the sense of Woronowicz. By $C(\mathbb{T})$ we then mean

the unital C*-algebra of all continuous complex valued functions of the circle $\mathbb{T}$ with pointwise algebraic operations and uniform norm. Since $\mathcal{F}$ is an isometry by Plancherel's formula, we must then require $\hat{m}$ to be bounded with respect to the 2-norms on the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{T} \times \mathbb{T})$ of square integrable functions on $\mathbb{T}$ and $\mathbb{T} \times \mathbb{T}$ (obtained from the GNS-constructions of $\hat{A}$ and $\hat{A} \otimes \hat{A}$ described in [16]), and this is clearly false. Thus one cannot define $\tilde{m}^*$ by extension to the Hilbert space completion. Also Proposition 2.3 is not applicable, since in general $\pi_l \times \pi_l$ is not in $\mathrm{Rep}_*(A, \Delta)$.

Proposition 2.15. *Let (A, Δ) be an AQG. Then $\mathrm{Hom}(\varepsilon, \pi_l) \neq \{0\}$ if and only if (A, Δ) is discrete. In this case, the map $\tilde{\eta} : c \mapsto c\mathcal{F}^{-1}(1_{\hat{A}})$ belongs to $\mathrm{Hom}(\varepsilon, \pi_l)$ and $(\pi_l, \tilde{m}, \tilde{\eta})$ is a monoid, which we call the regular monoid. We have $\tilde{\eta}c = cI_0$. Since $\tilde{\eta} : \varepsilon \to \pi_l$ is a morphism in $\mathrm{Rep}_*(A, \Delta)$, the adjoint $\tilde{\eta}^*$ exists and $\tilde{\eta}^* = \varepsilon$.*

Proof. For every morphism $\eta \in \mathrm{Hom}(\varepsilon, \pi_l)$, we have

$$\varepsilon(a)\eta(1) = \eta(\varepsilon(a)1) = \pi_l(a)\eta(1) = a\eta(1)$$

for $a \in A$, saying that $\eta(1)$ is a left integral in A. Thus $\eta \to \eta(1)$ is a bijection from $\mathrm{Hom}(\varepsilon, \pi_l)$ to the space of left integrals in A. By definition an AQG (A, Δ) is discrete if and only if a non-zero left integral exists, and in this case it is unique up to a scalar.

If (A, Δ) is discrete then $(\hat{A}, \hat{\Delta})$ is compact, i.e. $\hat{A}$ has a unit $1_{\hat{A}}$. Now

$$\mathcal{F}(I_0)(a) = \varphi(aI_0) = \varepsilon(a)\varphi(I_0) = \varepsilon(a).$$

Thus $\mathcal{F}(I_0) = \varepsilon = 1_{\hat{A}}$ and $\tilde{\eta}(1) = I_0$, which is a left integral in A, so $\tilde{\eta} \in \mathrm{Hom}(\varepsilon, \pi_l)$.
    Finally, the equalities

$$(\tilde{\eta}(c), a)_A = \varphi(a^*cI_0) = c\varepsilon(a^*) = c\overline{\varepsilon(a)} = (c, \varepsilon(a))_{\mathbb{C}},$$

for $c \in \mathbb{C}$ and $a \in A$, show that $\tilde{\eta}^* = \varepsilon$. $\square$

Remark 2.16. (1) The above result shows in particular that a monoid structure on the regular representation exists only if (A, Δ) is discrete. It turns out that the multiplication $\tilde{m}$ is in general not unique, not even up to isomorphisms of π_l.

(2) If (A, Δ) is a discrete and quasitriangular AQG with R-matrix R, the categories $\mathrm{Rep}_f(A, \Delta)$ and $\mathrm{Rep}(A, \Delta)$ are braided. It is therefore natural to ask whether the monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ is commutative in the sense that $\tilde{m} \circ c_{\pi_l, \pi_l} = \tilde{m}$, where c denotes the braiding. One can easily show that this is the case if and only if $R = 1 \otimes 1$. In that case, (A, Δ) is cocommutative and the representation categories are symmetric.

Definition 2.17. A comonoid in a (strict) tensor category $\mathcal{C}$ is a triple (Q, Δ, ε), where Q is an object and $\Delta : Q \to Q \otimes Q$, $\varepsilon : Q \to \mathbf{1}$ satisfy $\Delta \otimes \mathrm{id}_Q \circ \Delta = \mathrm{id}_Q \otimes \Delta \circ \Delta$ and $\varepsilon \otimes \mathrm{id}_Q \circ \Delta = \mathrm{id}_Q \otimes \varepsilon \circ \Delta = \mathrm{id}_Q$.

For a compact AQG, we have the following easy result.

Proposition 2.18. *Let (A, Δ) be an AQG. The map $\varepsilon : A \to \mathbb{C}$ is in $\mathrm{Hom}(\pi_l, \varepsilon)$. Furthermore, (A, Δ) is compact if and only if $\Delta(A) \subset A \otimes A$ if and only if $\Delta \in \mathrm{Hom}(\pi_l, \pi_l \times \pi_l)$. In this case $(\pi_l, \Delta, \varepsilon)$ is a comonoid in $\mathrm{Rep}(A, \Delta)$, which we call the regular comonoid.*

Proof. For any AQG, we have the equation $m(S \otimes \iota)\Delta(a) = \varepsilon(a)I$ in $M(A)$. If $\Delta(A) \subset A \otimes A$, the left-hand side and therefore the unit I belongs to A. The remaining facts are obvious consequences of ε and Δ being algebra homomorphisms. $\qquad\square$

Remark 2.19. Again, one might try to work with the usual tensor $*$-category of unital $*$-representations of a compact AQG (A, Δ) on Hilbert spaces. There is no problem with Δ, as it is an isometry, but $\varepsilon : A \to \mathbb{C}$ has in the general case no continuous extension with respect to the 2-norm on A given by the GNS-construction.

Definition 2.20. A Frobenius algebra in a tensor category $\mathcal{C}$ is a quintuple $(Q, m, \eta, \Delta, \varepsilon)$ such that (Q, m, η) is a monoid in $\mathcal{C}$, (Q, Δ, ε) is a comonoid in $\mathcal{C}$, and the following compatibility condition holds

$$\mathrm{id}_Q \otimes m \circ \Delta \otimes \mathrm{id}_Q = \Delta \circ m = m \otimes \mathrm{id}_Q \circ \mathrm{id}_Q \otimes \Delta. \tag{2.1}$$

Proposition 2.21. *Let (A, Δ) be a finite-dimensional AQG. Then $\tilde{m}^* = \Delta$ so the regular monoid and comonoid are each others adjoints: $(\pi_l, \tilde{m}, \tilde{\eta})^* \equiv (\pi_l, \tilde{m}^*, \tilde{\eta}^*) = (\pi_l, \Delta, \varepsilon)$. Furthermore, the quintuple $(\pi_l, \tilde{m}, \tilde{\eta}, \Delta, \varepsilon)$ is a Frobenius algebra in $\mathrm{Rep}_f(A, \Delta)$, which we call the regular Frobenius algebra.*

Proof. That $\tilde{\eta}^* = \varepsilon$ is shown in Proposition 2.15. If (A, Δ) is finite-dimensional, $\tilde{m}^* = \Delta$ follows from statement (1) of Proposition 2.12. The Frobenius property (2.1) will be shown at the end of the next subsection. (Cf. also [20].) $\qquad\square$

Remark 2.22. (1) Conversely, the existence of both the regular monoid and the regular comonoid requires (A, Δ) to be discrete and compact, thus A is finite-dimensional.

(2) Given a Frobenius algebra $(Q, m, \eta, \Delta, \varepsilon)$, it is easy to show that the morphisms $\varepsilon \circ m : Q \otimes Q \to \mathbf{1}$ and $\Delta \circ \eta : \mathbf{1} \to Q \otimes Q$ satisfy the triangular equations [14], i.e. the object Q is its own two-sided dual. If $\mathcal{C}$ is a $*$-category and $m^* = \Delta$, $\eta^* = \varepsilon$, we have $(\Delta \circ \eta)^* = \varepsilon \circ m$ and we obtain a solution of the conjugate equations [18]. In the case considered above, this in particular implies that π_l is a finite-dimensional object in $\mathrm{Rep}(A, \Delta)$, thus again A is finite-dimensional.

We show now how one can recover the intrinsic group, cf. [22], from the regular monoid of a discrete AQG.

Definition 2.23. Let $\mathcal{C}$ be a tensor category and $\mathcal{C}_*$ a full $*$-subcategory. Let (Q, m, η) be a monoid in $\mathcal{C}$ with $Q \in \mathcal{C}_*$. Denote by G_Q the group in $\text{End}\,Q$ given by

$$G_Q = \{t \in \text{End}\,Q \mid t \circ t^* = t^* \circ t = \text{id}_Q,\ m \circ t \otimes t = t \circ m\}$$

with group multiplication being composition of arrows, so the unit of G_Q is id_Q and the inverse t^{-1} of $t \in G_Q$ is t^*. The group G_Q is called the intrinsic group of the monoid (Q, m, η).

Proposition 2.24. *Let* (A, Δ) *be a discrete AQG with intrinsic group G defined by*

$$G = \{g \in M(A) \mid \Delta g = g \otimes g,\ g^*g = gg^* = I\},$$

which is compact with respect to the product topology on $M(A)$. Let $G_{\pi_l} \subset \text{Hom}(\pi_l, \pi_l)$ be the intrinsic group of the regular monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ with topology defined in Proposition 2.3. *Then $G \cong G_{\pi_l}$ as topological groups.*

Proof. By Proposition 2.5, $M(A) \cong R(A)$ as unital algebras, and $R(A)$ and $\text{Hom}(\pi_l, \pi_l)$ are anti-isomorphic as unital $*$-algebras. Let $\rho : A \to A$ denote the linear map such that $\varphi(ab) = \varphi(b\rho(a))$ for $a, b \in A$. It suffices to show that for any $g \in R(A)$, we have $\tilde{m} \circ (g \otimes g) = g \circ \tilde{m}$ if and if only $\Delta g = g \otimes g$. But $\tilde{m} \circ (g \otimes g) = g \circ \tilde{m}$ means that $\tilde{m} \circ (g \otimes g)(a \otimes b) = g \circ \tilde{m}(a \otimes b)$, for $a, b \in A$, or $\tilde{m}(ag \otimes bg) = \tilde{m}(a \otimes b)g$, which by faithfulness of φ, can be expressed as

$$\varphi(c\tilde{m}(ag \otimes bg)) = \varphi(c\tilde{m}(a \otimes b)g) = \varphi(\rho^{-1}(g)c\tilde{m}(a \otimes b)),$$

for $a, b, c \in A$. Hence by Proposition 2.12, the formula $\tilde{m} \circ (g \otimes g) = g \circ \tilde{m}$ can be rewritten as

$$(\varphi \otimes \varphi)(\Delta c(ag \otimes bg)) = (\varphi \otimes \varphi)(\Delta(\rho^{-1}(g)c)(a \otimes b))$$
$$= (\varphi \otimes \varphi)(\Delta \rho^{-1}(g)\Delta c(a \otimes b)) = (\varphi \otimes \varphi)(\Delta(c)(a \otimes b)(\rho \otimes \rho)\Delta \rho^{-1}(g)),$$

for $a, b, c \in A$. Thus again by faithfulness of φ, we see that $\tilde{m} \circ (g \otimes g) = g \circ \tilde{m}$ if and only if $(\rho \otimes \rho)\Delta \rho^{-1}(g) = g \otimes g$.

We assert now that $(\rho \otimes \rho)\Delta \rho^{-1} = \Delta$ for any discrete AQG, which clearly completes the proof of the proposition. Since $(S^2 \otimes \rho)\Delta = \Delta \rho$ for any AQG, we see that $(\rho \otimes \rho)\Delta \rho^{-1} = \Delta$ if and only if $\rho = S^2$, which holds for discrete AQG. □

2.3. *The absorbing property*

The following obvious fact will be used without further reference.

Lemma 2.25. *Let* (A, Δ) *be a AQG. If θ is a $*$-representation of (A, Δ) on K then*

$$(\theta \times \pi_l)(a)(v \otimes x) = \textstyle\sum_i \theta(a_i)v \otimes b_i, \quad \text{where} \quad \textstyle\sum_i a_i \otimes b_i = \Delta(a)(1 \otimes x),$$
$$(\pi_l \times \theta)(a)(x \otimes v) = \textstyle\sum_i a_i \otimes \theta(b_i)v, \quad \text{where} \quad \textstyle\sum_i a_i \otimes b_i = \Delta(a)(x \otimes 1),$$

for $v \in K$ and $x \in A$.

Proposition 2.26. *Let (A, Δ) be an AQG. For every $\theta \in \mathrm{Rep}(A, \Delta)$, we have the absorption property*

$$\theta \times \pi_l \; \cong \; \pi_l \times \theta \; \cong \; I_\theta \times \pi_l \; \cong \; \dim K \, \pi_l$$

for π_l, where I_θ is the $*$-representation of A on K given by $I_\theta(a) = \varepsilon(a)\mathrm{id}_K$ for $a \in A$.

Proof. We start by showing $\theta \times \pi_l \cong I_\theta \times \pi_l$. Define a linear map $U_\theta : K \otimes A \to K \otimes A$ by

$$U_\theta(\theta(a)v \otimes x) = \sum_i \theta(a_i)v \otimes x_i,$$

where $\sum_i a_i \otimes x_i = \Delta(x)(a \otimes 1)$ for $a, x \in A$. To see that U_θ is well-defined, suppose $\sum_j \theta(a^j)v^j \otimes x^j = 0$, where $a^j, x^j \in A$ and $v^j \in K$, and write $\sum_j \Delta(x^j)(a^j \otimes 1) = \sum_{ij} a_i^j \otimes x_i^j$ with $a_i^j, x_i^j \in A$. We must show that $\sum_{ij} \theta(a_i^j)v^j \otimes x_i^j = 0$, and in doing so, we can assume that (x^j) are linearly independent, so $\theta(a^j)v^j = 0$ for all j. Pick a two sided local unit e for the collection $\{a_i^j, x_i^j\}$ and a local unit e' for e. Then

$$\sum_{ij} \theta(a_i^j)v^j \otimes x_i^j = \sum_{ij} (\theta(e) \otimes e)(\theta(a_i^j)v^j \otimes x_i^j e')(v^j \otimes e)$$

$$= (\theta(e) \otimes e) \sum_j (\theta \otimes \iota)(\Delta(x^j)(a^j \otimes 1))(v^j \otimes e'e)$$

$$= (\theta(e) \otimes e) \sum_j (\theta \otimes \iota)(\Delta(x^j)(1 \otimes e'))(\theta(a^j)v^j \otimes e) = 0$$

as $\theta(a^j)v^j = 0$ for all j. Thus U_θ is well-defined.

Furthermore, for $a, b, x \in A$ and $v \in K$, we have

$$(\theta \times \pi_l)(b)U_\theta(\theta(a)v \otimes x) = \sum_i (\theta \times \pi_l)(b)(\theta(a_i)v \otimes x_i) = \sum_{ik} \theta(b_k^i)\theta(a_i)v \otimes y_k^i$$

$$= \sum_{ik} \theta(b_k^i a_i)v \otimes y_k^i,$$

where $\sum_i a_i \otimes x_i = \Delta(x)(a \otimes 1)$ and $\sum_k b_k^i \otimes y_k^i = \Delta(b)(1 \otimes x_i)$. But

$$\sum_{ik} b_k^i a_i \otimes y_k^i = \sum_i \Delta(b)(1 \otimes x_i)(a_i \otimes 1) = \Delta(b)\Delta(x)(a \otimes 1) = \Delta(bx)(a \otimes 1),$$

so $(\theta \times \pi_l)(b)U_\theta(\theta(a)v \otimes x) = U_\theta(\theta(a)v \otimes bx)$ for $a, b, x \in A$ and $v \in K$. On the other hand, if we write $\Delta(b)(1 \otimes x) = \sum_i c_i \otimes z_i$, for $b, x \in A$, and calculate

$$(I_\theta \times \pi_l)(b)(\theta(a)v \otimes x) = \sum_i I_\theta(c_i)\theta(a)v \otimes z_i = \sum_i \varepsilon(c_i)\theta(a)v \otimes z_i$$

$$= \theta(a)v \otimes \left(\sum_i \varepsilon(c_i)z_i \right) = \theta(a)v \otimes (\varepsilon \otimes \iota)[\Delta(b)(1 \otimes x)] = \theta(a)v \otimes bx,$$

for $a \in A$ and $v \in K$, we see that

$$(\theta \times \pi_l)(b)U_\theta(\theta(a)v \otimes x) = U_\theta(\theta(a)v \otimes bx) = U_\theta(I_\theta \times \pi_l)(b)(\theta(a)v \otimes x),$$

for $a, b, x \in A$ and $v \in K$, so

$$(\theta \times \pi_l)(b)U_\theta = U_\theta(I_\theta \times \pi_l)(b),$$

for $b \in A$ and therefore $U_\theta \in \mathrm{Hom}(I_\theta \times \pi_l, \theta \times \pi_l)$ in $\mathrm{Rep}(A, \Delta)$.

By the cancellation laws for (A, Δ), we see that U_θ is invertible, and thus $\theta \times \pi_l \cong I_\theta \times \pi_l$. In fact, the inverse of U_θ in $\mathrm{Rep}(A, \Delta)$ is given by the following formula $U_\theta^{-1}(\theta(a)v \otimes x) = \sum_i \theta(a_i)v \otimes x_i$, where

$$\sum_i a_i \otimes x_i = ((S^{-1} \otimes \iota)\Delta(x))(a \otimes 1)$$

for $a, x \in A$ and $v \in K$.

Similarly, one shows that the linear map $V_\theta : A \otimes K \to A \otimes K$ given by

$$V_\theta(x \otimes \theta(a)v) = \sum_i x_i \otimes \theta(a_i)v,$$

for $a, x \in A$ and $v \in K$, and where $\sum_i x_i \otimes a_i = \Delta(x)(1 \otimes a)$, is well-defined and is an isomorphism in $\mathrm{Hom}(\pi_l \times I_\theta, \pi_l \times \theta)$. Thus $\pi_l \times I_\theta \cong \pi_l \times \theta$ in $\mathrm{Rep}(A, \Delta)$.

From the previously derived expression

$$(I_\theta \times \pi_l)(b)(\theta(a)v \otimes x) = \theta(a)v \otimes bx,$$

valid for $a, b, x \in A$ and $v \in K$, we see that $(I_\theta \times \pi_l)(b) = I_{B(K)} \otimes \pi_l(b)$, so $I_\theta \times \pi_l \cong (\dim K)\pi_l$ and similarly $\pi_l \times I_\theta \cong (\dim K)\pi_l$. □

Remark 2.27. In the case of a discrete AQG, where $\mathrm{Rep}(A, \Delta)$ is semisimple, Proposition 2.26 can also be proven using (1) in Proposition 2.5 together with Proposition 3.15 below.

Proposition 2.28. *Let (A, Δ) be a discrete AQG and $\theta \in \mathrm{Rep}_f(A, \Delta)$. Then $\pi_l \times \theta, \pi_l \times I_\theta, \theta \times \pi_l, I_\theta \times \pi_l \in \mathrm{Rep}_*(A, \Delta)$, and the morphisms U_θ, V_θ considered in the preceding proposition are unitary.*

Proof. In the discrete case $\pi_l \in \mathrm{Rep}_*(A, \Delta)$ by Proposition 2.5 and the same is true for $\pi_l \times \theta$ and $\theta \times \pi_l$. Since we know that U_θ is invertible, we need only show that it is an isometry. (As always, the regular representation π_l is understood to come with its scalar product defined using φ.) Recall that U_θ is defined by

$$U_\theta(\theta(a)v \otimes x) = \sum_i \theta(a_i)v \otimes x_i,$$

where $\sum_i a_i \otimes x_i = \Delta(x)(a \otimes 1)$ for $a, x \in A$. We compute

$$(U_\theta(\theta(a)v \otimes x), U_\theta(\theta(a)v \otimes x)) = \sum_{ij}(\theta(a_i)v \otimes x_i, \theta(a_j)v \otimes x_j)$$

$$= \sum_{ij}(\theta(a_i)v, \theta(a_j)v)\varphi(x_j^* x_j) = \left(\theta\left(\sum_{ij} a_j^* a_i \varphi(x_j^* x_j)\right)v, v\right)$$

$$= \left(\theta \left(\sum_{ij} (\iota \otimes \varphi)(a_j^* a_i \otimes x_j^* x_j) \right) v, v \right)$$

$$= (\theta((\iota \otimes \varphi)([\Delta(x)(a \otimes 1)]^*[\Delta(x)(a \otimes 1)]))v, v)$$

$$= (\theta((\iota \otimes \varphi)((a^* \otimes 1)\Delta(x^* x)(a \otimes 1)))v, v) = (\theta(a^* a\varphi(x^* x))v, v)$$

$$= (\theta(a)v, \theta(a)v)(x, x) = (\theta(a)v \otimes x, \theta(a)v \otimes x),$$

for $a, x \in A$ and $v \in K$. Thus U_θ is an isometry. The same is true for V_θ by a similar computation. $\qquad\square$

Remark 2.29. (1) Let θ be a $*$-representation of (A, Δ) with U_θ as above. Clearly, there is a $*$-representation $\tilde{\theta}$ of $(A, \Delta_{\mathrm{op}})$ which coincides with θ as a map from A to $\mathrm{End}\, K$. It is then easy to see that $V_{\tilde{\theta}} = \Sigma U_\theta \Sigma^{-1}$, where $\Sigma : K \otimes A \to A \otimes K$ is the flip map. This observation obviates separate proofs for V_θ.

(2) Note that the assumptions on (A, Δ) and θ were only made in order for U_θ^* to be definable in $\mathrm{Rep}_*(A, \Delta)$. The computation showing that U_θ is isometric holds in general and provides an alternative proof for the well-definedness of U_θ.

Proposition 2.30. *The morphisms $V_\theta : \pi_l \times I_\theta \to \pi_l \times \theta$ are natural with respect to θ, i.e. the diagrams*

$$
\begin{array}{ccc}
\pi_l \times I_\theta & \xrightarrow{\ V_\theta\ } & \pi_l \times \theta \\[2pt]
\Big\downarrow{\scriptstyle \mathrm{id}_{\pi_l} \otimes s} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\pi_l} \otimes s} \\[2pt]
\pi_l \times I_{\theta'} & \xrightarrow{\ V_{\theta'}\ } & \pi_l \times \theta'
\end{array}
$$

commute for all $s : \theta \to \theta'$, and similarly for U_θ.

Proof. This is obvious by definition of V_θ. $\qquad\square$

Having defined monoids in tensor categories, we will also need the notion of a module over a monoid.

Definition 2.31. Let $\mathcal{C}$ be a tensor category and (Q, m) a semigroup in $\mathcal{C}$. Then a (left) Q-module is a pair (X, μ), where $X \in \mathcal{C}$ and $\mu : Q \otimes X \to X$ satisfies

$$\mu \circ m \otimes \mathrm{id}_X = \mu \circ \mathrm{id}_Q \otimes \mu.$$

For a monoid (Q, m, η) we require in addition that $\mu \circ \eta \otimes \mathrm{id}_X = \mathrm{id}_X$. With

$$\mathrm{Hom}_{Q\text{-mod}}((X, \mu), (X', \mu')) = \{ s \in \mathrm{Hom}(X, X') \mid s \circ \mu = \mu' \circ \mathrm{id}_Q \otimes s \}$$

as morphisms, the Q-modules form a category which we denote by Q-mod.

Proposition 2.32. *The diagram*

$$\begin{array}{ccc}
\pi_l \times \pi_l \times I_\theta & \xrightarrow{\tilde{m} \otimes \mathrm{id}_\theta} & \pi_l \times I_\theta \\
{\scriptstyle \mathrm{id}_{\pi_l} \otimes V_\theta} \downarrow & & \downarrow {\scriptstyle V_\theta} \\
\pi_l \times \pi_l \times \theta & \xrightarrow{\tilde{m} \otimes \mathrm{id}_\theta} & \pi_l \times \theta
\end{array}$$

commutes. We have similar commutative diagrams for the morphisms $V_\theta^* : \pi_l \times \theta \to \pi_l \times I_\theta$.

Proof. We must show that $(\tilde{m} \otimes \iota)(\iota \otimes V_\theta) = V_\theta(\tilde{m} \otimes \iota)$ as maps from $A \otimes A \otimes K$ to $A \otimes K$. Let $a, c, x, y \in A$ and $v \in K$ and write $\Delta(x)(1 \otimes a) = \sum_i x_i \otimes a_i$, where $a_i, x_i \in A$. Then

$$(\varphi c \otimes \iota)(\tilde{m} \otimes \iota)(\iota \otimes V_\theta)(y \otimes x \otimes \theta(a)v)$$

$$= \sum_i \varphi(c\tilde{m}(y \otimes x_i))\theta(a_i)v$$

$$= \theta\left(\sum_i (\varphi \otimes \varphi)(\Delta(c)(y \otimes x_i))a_i \right) v$$

$$= \theta((\varphi \otimes \varphi \otimes \iota)((\Delta(c) \otimes 1)(1 \otimes \Delta(x))(y \otimes 1 \otimes a)))v,$$

whereas if we write

$$\Delta(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(x))(y \otimes 1)](1 \otimes a) = \sum_j y_j \otimes b_j$$

for $b_j, y_j \in A$ and use (2) in Proposition 2.12, we get

$$(\varphi c \otimes \iota)V_\theta(\tilde{m} \otimes \iota)(y \otimes x \otimes \theta(a)v)$$

$$= (\varphi c \otimes \iota)V_\theta((\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(x))(y \otimes 1)] \otimes \theta(a)v) = \theta\left(\sum_j \varphi(cy_j)b_j \right) v.$$

Hence $(\tilde{m} \otimes \iota)(\iota \otimes V_\theta) = V_\theta(\tilde{m} \otimes \iota)$ follows if

$$(\varphi \otimes \varphi \otimes \iota)((\Delta(c) \otimes 1)(1 \otimes \Delta(x))(y \otimes 1 \otimes a)) = \sum_j \varphi(cy_j)b_j.$$

Now,

$$\sum_j \varphi(cy_j)b_j = (\varphi c \otimes \iota)\left(\sum_j y_j \otimes b_j \right) = (\varphi c \otimes \iota)[\Delta(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(x))(y \otimes 1)](1 \otimes a)],$$

so $(\tilde{m} \otimes \iota)(\iota \otimes V_\theta) = V_\theta(\tilde{m} \otimes \iota)$ if

$$(\varphi c \otimes \iota)\Delta(\varphi \otimes \iota)[((S^{-1} \otimes \iota)\Delta(x))(y \otimes 1)] = (\varphi \otimes \varphi \otimes \iota)((\Delta(c) \otimes 1)(1 \otimes \Delta(x))(y \otimes 1 \otimes 1)).$$

But

$$
\begin{aligned}
\text{L.H.S.} &= (\varphi c \otimes \iota)\Delta(\hat{y} \otimes \iota)(S^{-1} \otimes \iota)\Delta(x) = (\varphi c \otimes \iota)\Delta(\hat{y}S^{-1} \otimes \iota)\Delta(x) \\
&= (\hat{y}S^{-1} \otimes \varphi c \otimes \iota)(\iota \otimes \Delta)\Delta(x) = (\hat{y}S^{-1} \otimes \varphi c \otimes \iota)(\Delta \otimes \iota)\Delta(x),
\end{aligned}
$$

whereas by strong left-invariance of φ, we get

$$
\begin{aligned}
\text{R.H.S.} &= (\hat{y} \otimes \varphi \otimes \iota)((\Delta(c) \otimes 1)(1 \otimes \Delta(x))) \\
&= (\hat{y}S^{-1}S \otimes \varphi \otimes \iota)((\Delta(c) \otimes 1)(1 \otimes \Delta(x))) \\
&= (\varphi \otimes \iota)(c \otimes 1((\hat{y}S^{-1} \otimes \iota)\Delta \otimes \iota)\Delta(x)) = (\hat{y}S^{-1} \otimes \varphi c \otimes \iota)(\Delta \otimes \iota)\Delta(x),
\end{aligned}
$$

as desired. Now, replacing V_θ by its inverse V_θ^*, the direction of the vertical arrows in the diagram is reversed, and we see that also V_θ^* is a π_l-module morphism. $\square$

Corollary 2.33. *The morphisms $V_\theta : \pi_l \times I_\theta \to \pi_l \times \theta$ and $V_\theta^* : \pi_l \times \theta \to \pi_l \times I_\theta$ are morphisms of π_l-modules.*

Proposition 2.34. *Let (A, Δ) be a discrete AQG. Then*

$$(\tilde{m} \otimes \iota)(y \otimes (\Delta(x)(1 \otimes a))) = \Delta(\tilde{m}(y \otimes x))(1 \otimes a) \tag{2.2}$$

holds for $x, y, a \in A$.

Proof. First note that for V_θ with $\theta = \pi_l$, we have $V_\theta(x \otimes a) = \Delta(x)(1 \otimes a)$ for $a, x \in A$. To see this write $\Delta(x)(1 \otimes a) = \sum_i x_i \otimes a_i$ with $a_i, x_i \in A$ and pick a right-sided local unit $e \in A$ for $\{a, a_i\}$. Then

$$V_\theta(x \otimes a) = V_\theta(x \otimes \pi_l(a)e) = \sum_i x_i \otimes \pi_l(a_i)e = \sum_i x_i \otimes a_i e = \sum_i x_i \otimes a_i = \Delta(x)(1 \otimes a).$$

By Proposition 2.32, we have $(\tilde{m} \otimes \iota)(\iota \otimes V_\theta) = V_\theta(\tilde{m} \otimes \iota)$. Thus

$$
\begin{aligned}
(\tilde{m} \otimes \iota)(y \otimes (\Delta(x)(1 \otimes a))) &= (\tilde{m} \otimes \iota)(\iota \otimes V_\theta)(y \otimes x \otimes a) \\
&= V_\theta(\tilde{m} \otimes \iota)(y \otimes x \otimes a) = \Delta(\tilde{m}(y \otimes x))(1 \otimes a)
\end{aligned}
$$

for $x, y, a \in A$. $\square$

Remark 2.35. Clearly, it follows from the proof of this proposition that $\tilde{m}$ has the property stated in the proposition if and only if V_{π_l} is a π_l-module map. In the discrete case, where every representation is a direct sum of representations contained in π_l, Proposition 2.32 can therefore also be deduced using the naturality property of Proposition 2.30.

End of Proof of Proposition 2.21. If (A, Δ) is finite-dimensional, we can put $a = 1$ in (2.2) and obtain

$$(\tilde{m} \otimes \iota)(\iota \otimes \Delta)(y \otimes x) = \Delta(\tilde{m}(y \otimes x))$$

for $x, y \in A$. In categorical terms, this is the equality $\tilde{m} \otimes \mathrm{id}_{\pi_l} \circ \mathrm{id}_{\pi_l} \otimes \Delta = \Delta \circ \tilde{m}$ in $\mathrm{End}(\pi_l \otimes \pi_l)$. Using the $*$-operation and $\Delta^* = \tilde{m}$, we also find $\mathrm{id}_{\pi_l} \otimes \tilde{m} \circ \Delta \otimes \mathrm{id}_{\pi_l} = \Delta \circ \tilde{m}$. This completes the proof of the Frobenius property in Proposition 2.21. $\square$

We close this section by summarizing the results on the regular representation.

Theorem 2.36. *Let (A, Δ) be an AQG with left regular representation π_l. Then there exists a morphism $\tilde{m} : \pi_l \times \pi_l \to \pi_l$ such that $(\pi_l, \tilde{m})$ is a semigroup in the tensor category $\mathrm{Rep}(A, \Delta)$. The representation π_l has the absorbing property $\pi_l \times \theta \cong \pi_l \times I_\theta \cong \dim K \, \pi_l$ with respect to a natural family of equivalences $V_\theta : \pi_l \times I_\theta \to \pi_l \times \theta$ of (left) π_l-modules. Similarly, there are natural equivalences $U_\theta : I_\theta \times \pi_l \to \theta \times \pi_l$ of right π_l-modules. These equivalences are unitary whenever (A, Δ) is discrete and $\theta \in \mathrm{Rep}_f(A, \Delta)$.*

There exists a morphism $\tilde{\eta} : \varepsilon \to \pi_l$ such that $(\pi_l, \tilde{m}, \tilde{\eta})$ is a monoid if and only if (A, Δ) is discrete. In the discrete case, there exists a non-monoidal $*$-subcategory $\mathrm{Rep}_*(A, \Delta) \subset \mathrm{Rep}(A, \Delta)$ containing π_l.

3. On Monoids, Embedding Functors and AQG

3.1. *From monoids to embedding functors*

Some of the results in this section will be formulated over any ground field $\mathbb{F}$. Let $\mathrm{Vect}_{\mathbb{F}}$ denote the tensor category of finite-dimensional vector spaces over $\mathbb{F}$.

Lemma 3.1. *Let $\mathcal{C}$ be an $\mathbb{F}$-linear semisimple category. Then an $\mathbb{F}$-linear functor $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ is faithful (i.e. $F(s) = 0$ for $s : X \to Y$ implies $s = 0$) if $F(X)$ is non-zero for every irreducible $X \in \mathcal{C}$.*

Proof. Suppose $F(X)$ is non-zero for every irreducible X and consider $s : X \to Y$ such that $F(s) = 0$. Let $I_\mathcal{C}$ be the set of isomorphism classes of irreducible objects with chosen representatives $X_i, i \in I_\mathcal{C}$. Let $(v_{i\alpha})$ be bases in $\mathrm{Hom}(X_i, X)$ with dual bases $(v'_{i\alpha})$ satisfying $v'_{i\alpha} \circ v_{j\beta} = \delta_{ij} \delta_{\alpha, \beta} \mathrm{id}_{X_i}$ and $\sum_\alpha v_{i\alpha} \circ v'_{i\alpha} = \mathrm{id}_X$. Pick $w_{j\beta} \in \mathrm{Hom}(X_i, Y)$ and $w'_{j\beta}$ similarly. Since $\mathrm{Hom}(X_i, X_j) = \delta_{ij} \mathrm{id}_{X_i} \mathbb{F}$, which implies $w'_{j\beta} \circ s \circ v_{i\alpha} = \delta_{i,j} \, c_{i\alpha\beta} \mathrm{id}_{X_i}$, we can write

$$s = \sum_{i\alpha, j\beta} w_{j\beta} \circ w'_{j\beta} \circ s \circ v_{i\alpha} \circ v'_{i\alpha} = \sum_{i\alpha\beta} c_{i\alpha\beta} \, w_{i\beta} \circ v'_{i\alpha}.$$

Thus

$$0 = F(w'_{k\eta}) \circ F(s) \circ F(v_{k\xi}) = \sum_{i\alpha\beta} c_{i\alpha\beta} \, F(w'_{k\eta} \circ w_{i\beta} \circ v'_{i\alpha} \circ v_{k\xi}) = c_{k\xi\eta} \, F(\mathrm{id}_{X_k}),$$

for k, ξ and η. By assumption $F(\mathrm{id}_{X_k}) \neq 0$ for $k \in I_{\mathcal{C}}$, thus all $c_{i\alpha\beta}$ vanish and $s = 0$. $\qquad\square$

Proposition 3.2. *Let $\mathcal{C}$ be a semisimple $\mathbb{F}$-linear tensor category with* $\mathrm{End}\mathbf{1} \cong \mathbb{F}$, *and let (Q, m, η) be a monoid in $\hat{\mathcal{C}}$ such that:*

(1) $\dim \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathbf{1}, Q) = 1$.
(2) *For every $X \in \mathcal{C}$, there is an isomorphism $Q \otimes X \cong n(X)Q$ of Q-modules with $n(X) \in \mathbb{N}$.*

Then the functor $E : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ defined by $X \mapsto \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathbf{1}, Q \otimes X)$ and

$$E(s)\phi = \mathrm{id}_Q \otimes s \circ \phi, \tag{3.1}$$

where $s : X \to Y$ and $\phi \in \mathrm{Hom}(\mathbf{1}, Q \otimes X)$, is a faithful (strong) tensor functor with $\dim E(X) = n(X)$.

Proof. We have $E(X) = \mathrm{Hom}(\mathbf{1}, Q \otimes X) \cong \mathrm{Hom}(\mathbf{1}, n(X)Q) \cong d(X)\mathrm{Hom}(\mathbf{1}, Q) \cong \mathbb{F}^{n(X)}$, thus $E(X)$ is a vector space of dimension $n(X)$. Since $E(X) \neq 0$ for every $X \in \mathcal{C}$, Lemma 3.1 tells us that E is faithful.

To see that E is monoidal, first observe that $E(\mathbf{1}) = \mathrm{Hom}(\mathbf{1}, Q) = \mathbb{F}\eta$ by (2). Thus there is a canonical isomorphism $e : \mathbb{F} = \mathbf{1}_{\mathrm{Vect}_{\mathbb{F}}} \to E(\mathbf{1}) = \mathrm{Hom}(\mathbf{1}, Q)$ defined by $c \mapsto c\eta$. Next we define morphisms

$$d^E_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y), \quad \phi \otimes \psi \mapsto m \otimes \mathrm{id}_{X\otimes Y} \circ \mathrm{id}_Q \otimes \phi \otimes \mathrm{id}_Y \circ \psi.$$
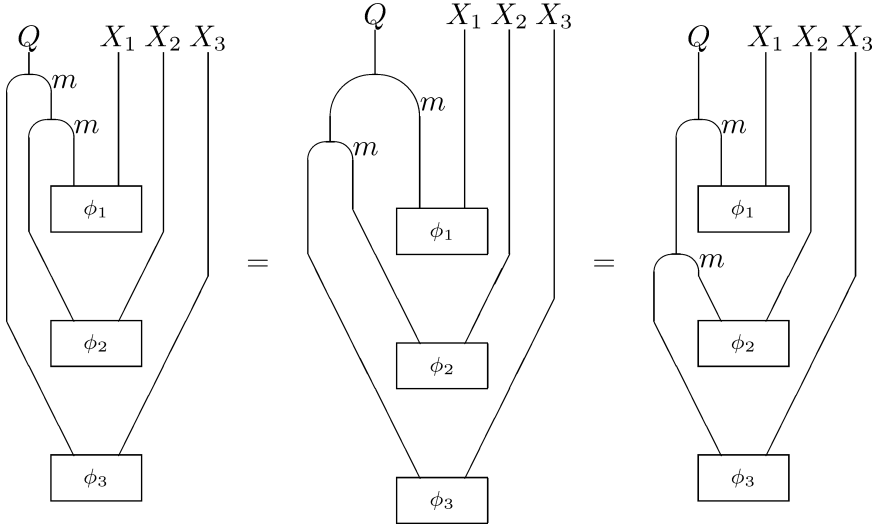
In terms of a diagram, this means

$$d^E_{X,Y}(\phi \otimes \psi) = $$
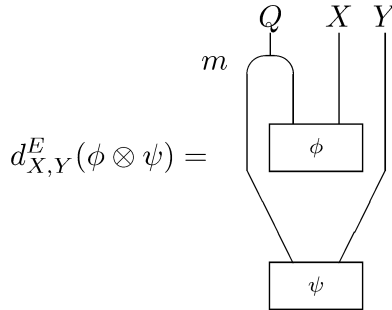
By definition (3.1) of the map $E(s) : E(X) \to E(Y)$, it is obvious that the family $(d^E_{X,Y})$ is natural with respect to both arguments. The equation

$$d^E_{X_1 \otimes X_2, X_3} \circ d^E_{X_1, X_2} \otimes \mathrm{id}_{E(X_3)} = d^E_{X_1, X_2 \otimes X_3} \circ \mathrm{id}_{E(X_1)} \otimes d^E_{X_2, X_3} \quad \forall X_1, X_2, X_3 \in \mathcal{C}$$

required from a tensor functor is immediate by associativity of m:

That $(E, (d_{X,Y}), e)$ satisfies the unit axioms is almost obvious. The first condition follows by

$$d_{X,\mathbf{1}}(\mathrm{id}_{E(X)} \otimes e)\phi = d_{X,\mathbf{1}}(\phi \otimes \eta) = m \otimes \mathrm{id}_X \circ \mathrm{id}_Q \otimes \phi \circ \eta = \phi,$$

and the second is shown analogously.

So far we have shown that E is a weak tensor functor for which $e : \mathbf{1}_{\mathrm{Vect}_{\mathbb{F}}} \to E(\mathbf{1})$ is an isomorphism. In order to conclude that E is a (strong) tensor functor it remains to show that the morphisms $d_{X,Y}^E$ are isomorphisms. Let $X, Y \in \mathcal{C}$. We consider the bilinear map

$$\gamma_{X,Y} : \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X) \boxtimes \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes Y) \to \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X \otimes Y)$$

$$s \boxtimes t \mapsto s \otimes \mathrm{id}_Y \circ t,$$

and we write $\boxtimes$ rather than $\otimes_{\mathbb{F}}$ for the tensor product of $\mathrm{Vect}_{\mathbb{F}}$ in order to avoid confusion with the tensor product in Q-mod. By (2), we have Q-module morphisms $s_i : Q \to Q \otimes X, s_i' : Q \otimes X \to Q$ for $i = 1, \dots, n(X)$ satisfying $s_i' \circ s_j = \delta_{ij}\mathrm{id}_Q$, and $\sum_i s_i \circ s_i' = \mathrm{id}_{Q \otimes X}$, and similar morphisms t_i, t_i', $i = 1, \dots, n(Y)$ with X replaced by Y. Then the $\gamma_{ij} = \gamma_{X,Y}(s_i \otimes t_j)$ are linearly independent because $\gamma_{i'j'}' \circ \gamma_{ij} = \delta_{i'i}\delta_{j'j}\mathrm{id}_Q$, where $\gamma_{i'j'}' = t_j' \circ s_i' \otimes \mathrm{id}_Y$. Bijectivity of $\gamma_{X,Y}$ follows now from the fact that both the domain and codomain of $\gamma_{X,Y}$ have dimension $n(X)n(Y)$.

For any $X \in \mathcal{C}$, we have a Q-module $(Q \otimes X, m \otimes \mathrm{id}_X)$. If (Q, m, η) is a monoid in the tensor category, then it is straightforward to check that the following maps are inverses of each other:

$$\delta_X : \ \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X) \to \mathrm{Hom}(\mathbf{1}, Q \otimes X), \qquad s \mapsto s \circ \eta,$$
$$\delta_X^{-1} : \ \mathrm{Hom}(\mathbf{1}, Q \otimes X) \to \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X), \qquad \tilde{s} \mapsto m \otimes \mathrm{id}_X \circ \mathrm{id}_Q \otimes \tilde{s}.$$

But

$$d^E_{X,Y} = \delta_{X \otimes Y} \circ \gamma_{X,Y} \circ \delta_X^{-1} \boxtimes \delta_Y^{-1},$$

which shows that $d^E_{X,Y}$ is an isomorphism for every $X, Y \in \mathcal{C}$.   □

Remark 3.3. From the assumptions, it follows that $Q \cong \oplus_i n(\overline{X_i}) X_i$. Such an object Q cannot exist in $\mathcal{C}$ if $\mathcal{C}$ has infinitely many isomorphism classes of irreducible objects. This is the reason why we consider monoids living in a larger category $\hat{\mathcal{C}}$.

The previous considerations being valid over any field $\mathbb{F}$, we now turn to $*$-categories where $\mathbb{F} = \mathbb{C}$.

Proposition 3.4. *Let $\mathcal{C}$ be a semisimple tensor $*$-category and let (Q, m, η) be a monoid in $\hat{\mathcal{C}}$ satisfying the conditions of Proposition 3.2 and in addition:*

(3) $Q \in \mathcal{C}_*$.
(4) *For every $s \in \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X)$ we have $s^* \in \mathrm{Hom}_{Q\text{-mod}}(Q \otimes X, Q)$.*

Then the functor E defined in Proposition 3.2 is $*$-preserving with respect to the scalar products on $E(X)$ given by $(\phi, \psi)\mathrm{id}_{\mathbf{1}} = \psi^* \circ \phi$, and the isomorphisms $d_{X,Y}$ are unitary for all $X, Y \in \mathcal{C}$.

Proof. Clearly the inner products are positive definite, thus the $E(X)$ are Hilbert spaces. Let $s : X \to Y$, $\phi \in \mathrm{Hom}(\mathbf{1}, Q \otimes X)$ and $\psi \in \mathrm{Hom}(\mathbf{1}, Q \otimes Y)$. Then

$$(E(s)\phi, \psi) = \psi^* \circ \mathrm{id}_Q \otimes s \circ \phi = (\phi^* \circ \mathrm{id}_Q \otimes s^* \circ \psi)^* = \overline{(E(s^*)\psi, \phi)} = (\phi, E(s^*)\psi).$$

Thus $E(s^*) = E(s)^*$, so E is a $*$-preserving functor.

By assumption (2), we have the isomorphism $Q \otimes X \cong n(X)Q$ in the category Q-mod, to wit there exist $s_i \in \mathrm{Hom}_{Q\text{-mod}}(Q, Q \otimes X)$, $t_i \in \mathrm{Hom}_{Q\text{-mod}}(Q \otimes X, Q)$, $i = 1, \ldots, n(X)$ satisfying $t_i \circ s_j = \delta_{ij}\mathrm{id}_Q$ and $\sum_i s_i \circ t_i = \mathrm{id}_{Q \otimes X}$. Now (4) implies that we can choose the s_i, t_i such that $t_i = s_i^*$. We must show that $d^E_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y)$ is unitary for every $X, Y \in \mathcal{C}$. Since we already know that it is an isomorphism, it suffices to show that it is an isometry. Since δ_X and δ_Y are isomorphisms, we need only show that

$$(d^E_{X,Y}(\delta_X(s_i) \otimes \delta_Y(s_j)), d^E_{X,Y}(\delta_X(s_{i'}) \otimes \delta_Y(s_{j'})))_{E(X \otimes Y)}$$
$$= (\delta_X(s_i), \delta_X(s_{i'}))_{E(X)}(\delta_Y(s_j), \delta_Y(s_{j'}))_{E(Y)}$$

for all i, i', j, j'. But definition of the inner products, the R.H.S. equals

$$(\eta^* \circ s_{i'}^* \circ s_i \circ \eta)(\eta^* \circ s_{j'}^* \circ s_j \circ \eta) = \delta_{ii'}\delta_{jj'}(\eta^* \circ \eta)^2 = \delta_{ii'}\delta_{jj'},$$

whereas the L.H.S. equals

$$(\delta_{X\otimes Y} \circ \gamma_{X,Y}(s_i \boxtimes s_j), \delta_{X\otimes Y} \circ \gamma_{X,Y}(s_{i'} \boxtimes s_{j'}))_{E(X\otimes Y)}$$

$$= (\delta_{X\otimes Y} \circ (s_i \otimes \mathrm{id}_Y) \circ s_j, \delta_{X\otimes Y} \circ (s_{i'} \otimes \mathrm{id}_Y) \circ s_{j'})_{E(X\otimes Y)}$$

$$= ((s_i \otimes \mathrm{id}_Y) \circ s_j \circ \eta, (s_{i'} \otimes \mathrm{id}_Y) \circ s_{j'} \circ \eta)_{E(X\otimes Y)}$$

$$= \eta^* \circ s_{j'}^* \circ (s_{i'}^* \otimes \mathrm{id}_Y) \circ (s_i \otimes \mathrm{id}_Y) \circ s_j \circ \eta = \delta_{ii'}\delta_{jj'}\eta^* \circ \eta = \delta_{ii'}\delta_{jj'},$$

as desired. $\square$

Remark 3.5. In the situation where $\mathcal{C} = \mathrm{Rep}_f(A, \Delta)$ for a discrete AQG, we have seen that $\hat{\mathcal{C}} \simeq \mathrm{Rep}(A, \Delta)$ and $\mathcal{C}_* \simeq \mathrm{Rep}_*(A, \Delta)$. The regular monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ satisfies all assumptions of Proposition 3.4: as to assumption (3), recall from Proposition 2.5 that $\pi_l \in \mathrm{Rep}_*(A, \Delta)$. Assumption 4 follows from unitarity of the isomorphism $V_\theta : \pi_l \times I_\theta \to \pi_l \times \theta$ and the fact that V_θ and V_θ^* are morphisms of π_l-modules.

Lemma 3.6. *Let $\mathcal{C}$ be as in Proposition 3.2. Let $Q \in \hat{\mathcal{C}}$ be a direct sum of irreducible objects in $\mathcal{C}$ with finite multiplicities, where $\mathbf{1}$ appears with multiplicity one. Consider the functor $\mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ defined by $E(X) = \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathbf{1}, Q \otimes X)$. Then the map $a : \mathrm{End}\, Q \to \mathrm{Nat}\, E$, $s \mapsto (a_X(s))$ with $a_X(s) = s \otimes \mathrm{id}_X \in \mathrm{End} E(X)$, is an isomorphism. It restricts to an isomorphism $\mathrm{Aut}\, Q \to \mathrm{Aut}\, E$.*

Proof. That $(a_X(s))$ is a natural transformation from E to itself is obvious. Injectivity follows from $a_{\mathbf{1}}(s) = s \otimes \mathrm{id}_{\mathbf{1}} = s$. The fact $Q \cong \oplus_i n_i X_i$, where i runs through I and $n_i \in \mathbb{Z}_+$, implies $\mathrm{End}\, Q \cong \prod_i M_{n_i}(\mathbb{F})$. On the other hand, by semisimplicity of $\mathcal{C}$ we have $\mathrm{Nat}\, E \cong \prod_i \mathrm{End}\, E(X_i)$, cf. e.g. [22]. Now it is easy to see that the composition of the latter two isomorphisms with the map $a : \mathrm{End}\, Q \to \mathrm{Nat}\, E$ preserves the factors in the respective direct products. Then surjectivity follows from $\dim E(X_i) = n_i$. $\square$

Lemma 3.7. *Let $\mathcal{C}$ and the monoids (Q, m, η) and (Q', m', η') be as in Proposition 3.2. Assume in addition that $\mathcal{C}$ has duals and that Q, Q' are direct sums of irreducibles in $\mathcal{C}$ with finite multiplicities. Let $E, E' : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ be the ensuing embedding functors. Then there is a bijection between monoidal natural isomorphisms $b : E \to E'$ and isomorphisms $s : Q \to Q'$ of monoids.*

Proof. One direction is easy: if $s : Q \to Q'$ is an isomorphism such that $s \circ m = m' \circ s \otimes s$ and $\eta = \eta' \circ s$, then we define $a_X(s) : E(X) \to E'(X)$ by $a_X(s)\phi = s \otimes \mathrm{id}_X \circ \phi \in E'(X)$ for $\phi \in E(X)$. The family (a_X) obviously is a natural isomorphism of E and E', and that it is monoidal, i.e. satisfies $d_{X,Y}^{E'} \circ a_X \otimes a_Y = a_{X\otimes Y} \circ d_{X,Y}^E$ for all X, Y, is obvious by the definition of $d^E, d^{E'}$ and the fact that s is an isomorphism of monoids.

As to the converse, the existence of a monoidal natural isomorphism $b : E \to E'$ implies $\dim \mathrm{Hom}(\mathbf{1}, Q \otimes X) = \dim \mathrm{Hom}(\mathbf{1}, Q' \otimes X)$ for $X \in \mathcal{C}$. By duality we have

$\dim \operatorname{Hom}(X_i, Q) = \dim \operatorname{Hom}(X_i, Q')$ for all irreducible $X_i \in \mathcal{C}$, which implies that Q and Q' are isomorphic. Fix an arbitrary isomorphism $s : Q' \to Q$ and consider the monoid (Q, m'', η'') where $m'' = s \circ m' \circ s^{-1} \otimes s^{-1}$ and $\eta'' = s \circ \eta'$. Let E'' be the embedding functor corresponding to (Q, m'', η''). By construction, $(Q, m'', \eta'') \cong (Q', m', \eta')$, and by the preceding considerations we have the monoidal natural isomorphism $a(s) = (a_X(s)) : E' \to E''$. If $b : E \to E'$ is a monoidal natural isomorphism, then the composition $c = a(s) \circ b : E \to E''$ is monoidal, and there exists $t \in \operatorname{Aut} Q$ such that $c = c(t)$. Since E and E'' coincide as functors, the condition $c_{X \otimes Y} \circ d^E_{X,Y} = d^{E''}_{X,Y} \circ c_X \otimes c_Y$ is equivalent to

$$(t \circ m) \otimes \operatorname{id}_{X \otimes Y} \circ \operatorname{id}_Q \otimes \phi \otimes \operatorname{id}_Y \circ \psi = (m'' \circ t \otimes t) \otimes \operatorname{id}_{X \otimes Y} \circ \operatorname{id}_Q \otimes \phi \otimes \operatorname{id}_Y \circ \psi$$

for $X, Y \in \mathcal{C}$ and $\phi \in E(X)$, $\psi \in E(Y)$. Since $\mathcal{C}$ has duals, this means that

$$m'' \circ t \otimes t \circ u \otimes v = t \circ m \circ u \otimes v$$

for $X, Y \in \mathcal{C}$ and $u : X \to Q$, $v : Y \to Q$. But Q is a direct sum of simple objects in $\mathcal{C}$, so we can cancel $u \otimes v$ and conclude $m'' \circ t \otimes t = t \circ m$. The equality $\eta'' \circ t = \eta$ is proven in a similar fashion using the morphisms $e^E : \mathbb{F} \to E(\mathbf{1})$ and $e^{E''} : \mathbb{F} \to E''(\mathbf{1})$. Thus we have an isomorphism $t : (Q, m, \eta) \to (Q, m'', \eta'')$ of monoids and composing with the isomorphism $s^{-1} : (Q, m'', \eta'') \to (Q', m', \eta')$ implies the claim. Clearly this gives us a bijection between isomorphisms of embedding functors and of monoids, respectively. □

Remark 3.8. Having assumed throughout that the tensor category $\mathcal{C}$ is strict, we now comment briefly on the non-strict case. If a tensor category $\mathcal{C}$ has a non-trivial associativity constraint

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

the definition of a monoid in $\mathcal{C}$ is changed in an obvious way: the associativity condition becomes

$$m \circ (m \otimes \operatorname{id}_Q) = m \circ (\operatorname{id}_Q \otimes m) \circ \alpha_{Q,Q,Q},$$

and the first equation in Definition 2.31 relating elements in $\operatorname{Hom}((Q \otimes Q) \otimes X, X)$ becomes

$$\mu \circ m \otimes \operatorname{id}_X = \mu \circ \operatorname{id}_Q \otimes \mu \circ \alpha_{Q,Q,X}.$$

It then remains true that an absorbing monoid gives rise to an embedding functor, but we omit the proofs.

3.2. *Main result*

Given a discrete AQG, it is occasionally convenient to consider an abstract tensor $*$-category $\operatorname{Rep}^{\operatorname{abs}}_f(A, \Delta)$ together with an embedding functor E, rather than the concrete category $\operatorname{Rep}_f(A, \Delta)$ and the forgetful functor K.

Lemma 3.9. *Let (A, Δ) be a discrete AQG and write $\mathcal{C} = \operatorname{Rep}_f^{\mathrm{abs}}(A, \Delta)$. Let $E : \mathcal{C} \to \mathcal{H}$ be the obvious embedding functor. Let $(\pi_l, \tilde{m}, \tilde{\eta})$ be the regular monoid in $\hat{\mathcal{C}} \simeq \operatorname{Rep}(A, \Delta)$ and $E' : \mathcal{C} \to \mathcal{H}$ the embedding functor that it gives rise to by Proposition 3.4. Then there exists a unitary equivalence $u : E \to E'$ of tensor functors.*

Proof. For $X \in \mathcal{C}$ we have $(E(X), \pi_X) \in \operatorname{Rep}_f(A, \Delta)$, and let us write V_X instead of V_{π_X}. For $\phi \in E(X)$ define $u_X \phi \in A \otimes E(X)$ by $u_X \phi = V_X(I_0 \otimes \phi)$. Then

$$(\pi_l \times \pi_X)(a)u_X\phi = (\pi_l \times \pi_X)(a)V_X(I_0 \otimes \phi) = V_X(\pi_l \times I_{\pi_X})(a)(I_0 \otimes \phi)$$
$$= V_X(\pi_l \otimes \varepsilon)\Delta(a)(I_0 \otimes \phi) = V_X(\pi_l(a)I_0 \otimes \phi)$$
$$= V_X(\varepsilon(a)I_0 \otimes \phi) = \varepsilon(a)V_X(I_0 \otimes \phi) = \varepsilon(a)u_X\phi,$$

thus $u_X\phi \in \operatorname{Hom}(\varepsilon, \pi_l \times \pi_X)$. In order to show that (u_X) is a natural transformation, we consider $s : X \to X'$ and compute

$$u_{X'}E(s)\phi = V_{X'}(I_0 \otimes s\phi) = V_{X'}(1 \otimes s)(I_0 \otimes \phi) = (1 \otimes s)V_X(I_0 \otimes \phi)$$
$$= (1 \otimes s)u_X\phi = E'(s)u_X\phi,$$

where we have used Proposition 2.30. Since V_X is invertible, the map $\phi \mapsto u_X\phi$ is injective and therefore bijective by equality of the dimensions.

Thus (u_X) is a natural isomorphism. It remains to show that it is monoidal, i.e.

$$d_{X,X'}^{E'} \circ u_X \otimes u_{X'} = u_{X \otimes X'}$$

for $X, X' \in \mathcal{C}$. Here we have as usual identified the vector spaces $E(X) \otimes E(X')$ and $E(X \otimes X')$. Let $\phi \in E(X)$ and $\phi' \in E(X')$. Then

$$u_{X \otimes X'}(\phi \otimes \phi') = V_{X \otimes X'}(I_0 \otimes \phi \otimes \phi'),$$

whereas

$$d_{X,X'}^{E'} \circ (u_X \otimes u_{X'})(\phi \otimes \phi') = (\tilde{m} \otimes \operatorname{id}_X \otimes \operatorname{id}_{X'}) \circ (\operatorname{id}_A \otimes u_X\phi \otimes \operatorname{id}_{X'}) \circ u_{X'}\phi'$$
$$= (\tilde{m} \otimes \iota \otimes \iota)((V_{X'})_{14}(V_X)_{23}(1 \otimes 1 \otimes \phi \otimes \phi')).$$

Thus we must show that

$$V_{X \otimes X'}(I_0 \otimes \phi \otimes \phi') = (\tilde{m} \otimes \iota \otimes \iota)((V_{X'})_{14}(V_X)_{23}(1 \otimes 1 \otimes \phi \otimes \phi')).$$

By non-degeneracy of π_X and $\pi_{X'}$, we may assume $\phi = \pi_X(a)v$ and $\phi' = \pi_{X'}(b)v'$, for $a, b \in A$ and $v \in E(X)$ and $v' \in E(X')$. By the definition of V_X, $V_{X'}$ and $V_{X \otimes X'}$, it thus suffices to show that

$$(\tilde{m} \otimes \iota \otimes \iota)(\Delta(I_0)_{14}\Delta(I_0)_{23}(1 \otimes 1 \otimes a \otimes b)) = (\Delta \otimes \iota)\Delta(I_0)(1 \otimes a \otimes b)$$

for $a, b \in A$. Write $\Delta(I_0)(1 \otimes b) = \sum_i a_i \otimes b_i$ for $a_i, b_i \in A$. Then by Proposition 2.34 and $\tilde{m}(\mathrm{id} \otimes \tilde{\eta}) = \mathrm{id}_{\pi_l}$, we get

$$
\begin{aligned}
(\tilde{m} \otimes \iota \otimes \iota)&(\Delta(I_0)_{14}\Delta(I_0)_{23}(1 \otimes 1 \otimes a \otimes b)) \\
&= \sum_i (\tilde{m} \otimes \iota \otimes \iota)(a_i \otimes \Delta(I_0)(1 \otimes a) \otimes b_i) \\
&= \sum_i (\tilde{m} \otimes \iota)(a_i \otimes \Delta(I_0)(1 \otimes a)) \otimes b_i = \sum_i \Delta \tilde{m}(a_i \otimes I_0)(1 \otimes a) \otimes b_i \\
&= \sum_i \Delta(a_i)(1 \otimes a) \otimes b_i = \sum_i (\Delta \otimes \iota)(a_i \otimes b_i)(1 \otimes a \otimes 1) \\
&= (\Delta \otimes \iota)(\Delta(I_0)(1 \otimes b))(1 \otimes a \otimes 1) = (\Delta \otimes \iota)\Delta(I_0)(1 \otimes a \otimes b),
\end{aligned}
$$

as desired. $\square$

At this stage we need to recall the generalized Tannaka theorem for discrete AQG, as proven in [22].

Theorem 3.10. *Let $\mathcal{C}$ be a semisimple tensor $*$-category and let E be an embedding functor. Then there exists a discrete AQG (A, Δ) and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f(A, \Delta)$ of tensor $*$-categories, such that $K \circ F = E$, where $K : \mathrm{Rep}_f(A, \Delta) \to \mathcal{H}$ is the forgetful functor.*

We are now in a position to state our main result which describes the precise relationship between embedding functors, absorbing monoids and discrete AQG.

Theorem 3.11. (1) *Let $\mathcal{C}$ be a tensor $*$-category with conjugates and $\mathrm{End}\,\mathbf{1} \cong \mathbb{C}$ and let $E : \mathcal{C} \to \mathcal{H}$ be an embedding functor. Let (A, Δ) be the discrete AQG and $F : \mathcal{C} \to \mathrm{Rep}_f(A, \Delta)$ the monoidal equivalence provided by the generalized Tannaka theorem. Let $(\pi_l, \tilde{m}, \tilde{\eta})$ be the regular monoid in $\mathrm{Rep}(A, \Delta)$ and $E' : \mathcal{C} \to \mathcal{H}$ the embedding functor that it gives rise to. Then E and E' are naturally unitarily equivalent as tensor functors.*

(2) *Let (A, Δ) be a discrete AQG and $(\pi_l, \tilde{m}, \tilde{\eta})$ the regular monoid in $\mathrm{Rep}(A, \Delta)$. Let $E : \mathrm{Rep}_f^{\mathrm{abs}}(A, \Delta) \to \mathcal{H}$ be the embedding functor obtained from the latter via Proposition 3.2 and (A', Δ') the discrete AQG given by the generalized Tannaka theorem. Then (A, Δ) and (A', Δ') are isomorphic.*

(3) *Let $\mathcal{C}$ be a tensor $*$-category with conjugates and $\mathrm{End}\,\mathbf{1} \cong \mathbb{C}$ and let (Q, m, η) be a monoid in $\hat{\mathcal{C}}$ satisfying the assumptions in Proposition 3.4. Let E be the resulting embedding functor and (A, Δ) and F as in (2). Then the image (Q', m', η') of the regular monoid $(\pi_l, \tilde{m}, \tilde{\eta})$ under the equivalence $\mathrm{Rep}(A, \Delta) \to \hat{\mathcal{C}}$ is isomorphic to (Q, m, η).*

Proof. (1) Consider the equivalence $F : \mathcal{C} \to \mathrm{Rep}_f(A, \Delta)$ satisfying $K \circ F = E$ provided by the generalized Tannaka theorem. Then the claim is just a reformulation of Lemma 3.9.

(2) Let $\mathcal{C} = \mathrm{Rep}_f^{\mathrm{abs}}(A, \Delta)$ with the canonical embedding functor $E : \mathcal{C} \to \mathcal{H}$. Obviously, (A, Δ) is isomorphic to the AQG given by the generalized Tannaka theorem from the pair $(\mathcal{C}, E)$. Now the claim follows from Lemma 3.9 and the fact (cf. [21, Proposition 5.28]) that isomorphic embedding functors give rise to isomorphic discrete AQG.

(3) Given $\mathcal{C}$ and the monoid (Q, m, η) in $\hat{\mathcal{C}}$, we obtain an embedding functor $E : \mathcal{C} \to \mathcal{H}$ by Proposition 3.4. On the other hand, going from $(\mathcal{C}, E)$ to an AQG, then to the regular monoid in $\mathrm{Rep}(A, \Delta) \simeq \hat{\mathcal{C}}$ and, finally, from the latter to the embedding functor $E' : \mathcal{C} \to \mathcal{H}$, Lemma 3.9 again implies $E \overset{\otimes}{\cong} E'$. Thus the monoids (Q, m, η) and (Q', m', η') in $\hat{\mathcal{C}}$ give rise to equivalent embedding functors and are therefore isomorphic by Lemma 3.7. □

Remark 3.12. (1) The preceding result can be formalized more conceptually as follows. Let Disc be the category of discrete AQG with isomorphisms as arrows. Let Emb be the category of pairs $(\mathcal{C}, E)$ where $\mathcal{C}$ is a semisimple $\mathbb{F}$-linear tensor category with duals and $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$ and $E : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ is a faithful $\mathbb{F}$-linear tensor functor. The arrows in Emb are equivalences $F : \mathcal{C} \to \mathcal{C}'$ such that $E' \circ F = E$. Finally, let Mon be the category of pairs $(\mathcal{C}, (Q, m, \eta))$, where $\mathcal{C}$ is a semisimple $\mathbb{F}$-linear tensor category with duals and $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$ and (Q, m, η) is a monoid in $\hat{\mathcal{C}}$ satisfying the assumptions of Proposition 3.2. Here the arrows are equivalences $F : \mathcal{C} \to \mathcal{C}'$ such that $F((Q, m, \eta))$ is isomorphic to (Q', m', η') in $\mathcal{C}'$. Then the various constructions considered so far give rise to the equivalences $\mathrm{Mon} \simeq \mathrm{Emb} \simeq \mathrm{Disc}^{\mathrm{op}}$, where $\mathrm{Disc}^{\mathrm{op}}$ is the opposite category of Disc. More precisely, every oriented loop in the triangle with corners $\mathrm{Mon}, \mathrm{Emb}, \mathrm{Disc}^{\mathrm{op}}$ obtained as composition of these functors is naturally isomorphic to the identity functor.

(2) The preceding theorem remains valid if one replaces tensor $*$-categories with conjugates by semisimple $\mathbb{F}$-linear tensor categories with duals, and discrete AQG by regular multiplier Hopf algebras with left invariant functionals. The arguments are essentially unchanged, provided one appeals to the version of the generalized Tannaka theorem stated in [22, Sec. 5.4].

3.3. *Dimension functions vs. absorbing objects*

Definition 3.13. A dimension function on a C^*-tensor category $\mathcal{C}$ with conjugates is a map $n : \mathrm{Obj}\,\mathcal{C} \to \mathbb{R}_+$ such that $n(X \oplus Y) = n(X) + n(Y)$ and $n(X \otimes Y) = n(X)n(Y)$ and $n(X) = n(\overline{X})$.

Remark 3.14. Note that a dimension function automatically satisfies $n(\mathbf{1}) = 1$. Every C^*-tensor category $\mathcal{C}$ with conjugates comes with a distinguished dimension function, the intrinsic dimension, cf. [18]. The representation categories associated with q-deformations of simple Lie groups show that the intrinsic dimension need not be integer valued, cf. [23]. On the other hand, an embedding functor $E : \mathcal{C} \to \mathcal{H}$

gives rise to an integer valued dimension function by $n(X) = \dim E(X)$. This also shows that one and the same category can have a dimension function which is integer valued and one which is not. We remark further that C^*-tensor categories having only finitely many irreducible objects admit only one dimension function, namely the intrinsic one, as can be shown using Perron–Frobenius theory. Furthermore, every embedding functor must preserve dimensions whenever $\mathcal{C}$ is amenable, which in particular holds when $\mathcal{C}$ admits a unitary braiding, cf. [18]. Thus if the intrinsic dimension of $\mathcal{C}$ is not integer valued and $\mathcal{C}$ is finite or has a unitary braiding, an embedding functor cannot exist.

Assuming the existence of an integer valued dimension function we arrive at the following partial converse of Proposition 3.2.

Proposition 3.15. *Let $\mathcal{C}$ be a semisimple $\mathbb{F}$-linear tensor category with two-sided duals and integer valued dimension function n. Let $n_i = n(X_i)$ for $i \in I_\mathcal{C}$ and consider the direct sum*

$$Q = \bigoplus_{i \in I_\mathcal{C}} n_{\overline{i}} X_i$$

in $\hat{\mathcal{C}}$. Then $Q \otimes X \cong X \otimes Q \cong n(X)Q$ for all $X \in \mathcal{C}$.

Conversely, assume $Q \in \hat{\mathcal{C}}$ is a direct sum of irreducible objects of $\mathcal{C}$ and that $Q \otimes X \cong n(X)Q$ with $n(X) \in \mathbb{N}$ for $X \in \mathcal{C}$. Then

$$Q \cong N \bigoplus_{i \in I_\mathcal{C}} n_{\overline{i}} X_i,$$

where $N = \dim \mathrm{Hom}(\mathbf{1}, Q)$. If $N < \infty$ then $n : \mathrm{Obj}\,\mathcal{C} \to \mathbb{N}$ is additive and multiplicative. If $I_\mathcal{C}$ is a finite set then $n(X) = d(X)$ for all $X \in \mathcal{C}$; thus in this case an absorbing object exists if and only if all intrinsic dimensions are integers.

Proof. By $\bigoplus_{i \in I_\mathcal{C}} n_{\overline{i}} X_i$ we mean the filtered inductive limit over partial finite direct sums, which defines an object of $\hat{\mathcal{C}}$ unique up to isomorphism. Let $j \in I_\mathcal{C}$. We compute

$$Q \otimes X_j \cong \bigoplus_{i \in I_\mathcal{C}} n_{\overline{i}} X_i \otimes X_j \cong \bigoplus_{i \in I_\mathcal{C}} n_{\overline{i}} \bigoplus_{k \in I_\mathcal{C}} N_{ij}^k X_k \cong \bigoplus_{k \in I_\mathcal{C}} \left(\sum_{i \in I_\mathcal{C}} N_{ij}^k n_{\overline{i}} \right) X_k.$$

Using standard properties of the coefficients, cf. e.g. [22], we calculate

$$\sum_{i \in I_\mathcal{C}} N_{ij}^k n_{\overline{i}} = \sum_{i \in I_\mathcal{C}} N_{j\overline{k}}^{\overline{i}} n_{\overline{i}} = n_j n_{\overline{k}},$$

and therefore $Q \otimes X_j \cong n_j \bigoplus_{k \in I_\mathcal{C}} n_{\overline{k}} X_k \cong n_j Q$. For a reducible object X the claim now follows by semisimplicity. The argument for $X \otimes Q$ is similar.

As to the converse, for irreducible $X \in \mathcal{C}$ we compute

$$\dim \mathrm{Hom}(X, Q) = \dim \mathrm{Hom}(\mathbf{1}, Q \otimes \overline{X}) = \dim \mathrm{Hom}(\mathbf{1}, Q \otimes \mathbf{1}^{\oplus n(\overline{X})})$$

$$= n(\overline{X}) \dim \mathrm{Hom}(\mathbf{1}, Q) = n(\overline{X})N.$$

Since Q is a direct sum of irreducibles in $\mathcal{C}$, we thus have

$$Q \cong N \bigoplus_{i \in I_{\mathcal{C}}} n_{\overline{\imath}} X_i,$$

and the claim follows. Assume now that $N < \infty$. Then we find

$$n(X \otimes Y)N = \dim \operatorname{Hom}(\overline{X \otimes Y}, Q) = \dim \operatorname{Hom}(\overline{Y} \otimes \overline{X}, Q)$$
$$= \dim \operatorname{Hom}(\overline{Y}, Q \otimes X) = n(X) \dim \operatorname{Hom}(\overline{Y}, Q) = n(Y)n(X)N$$

and thus $n(X)n(Y) = n(X \otimes Y)$ for $X, Y \in \mathcal{C}$.

If $\mathcal{C}$ is finite, it is well-known that the intrinsic dimension function is the only additive and multiplicative function on $\operatorname{Obj}\mathcal{C}$. $\qquad\square$

Remark 3.16. (1) Note that an additive and multiplicative function on $\operatorname{Obj}\mathcal{C}$ determines and is determined by a function $n' : I_{\mathcal{C}} \to \mathbb{N}$ which satisfies $\sum_{k \in I_{\mathcal{C}}} N_{ij}^k n_k' = n_i' n_j'$ for all $i, j \in I_{\mathcal{C}}$.

(2) It is important to note that the existence of an integer valued dimension function does not obviously imply the existence of a monoid structure on the absorbing object Q. By our earlier constructions, an embedding functor gives rise to a quantum group, and therefore to the regular monoid in $\hat{\mathcal{C}}$. (One can also construct the latter directly from the embedding functor, but we refrain from giving the details.) Since any dimension function n satisfies $n(\mathbf{1}) = 1$, we have $\dim \operatorname{Hom}(\mathbf{1}, Q) = 1$, thus there exists a morphism $\eta : \mathbf{1} \to Q$ that is unique up to a scalar. But the main issue clearly is constructing an associative morphism $m : Q \otimes Q \to Q$ such that (Q, m, η) is a monoid. This is a difficult cohomological problem.

(3) Another approach for constructing an absorbing monoid might be to generalize Deligne's proof to the braided case. However, as our earlier mentioned counter examples show, assuming just the existence of a braiding does not suffice.

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