Rickart's proof of $\sigma(a) \neq \emptyset$ and of the Beurling-Gelfand formula

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October 19, 2020

1.1 Lemma *Let $\mathcal{A}$ be a normed unital algebra and $\mathrm{Inv}\,\mathcal{A} \subseteq \mathcal{A}$ the set of invertible elements. Then $\mathrm{Inv}(\mathcal{A})$ is a topological group (w.r.t. the norm topology).*

Proof. (i) It is clear that $\mathrm{Inv}(\mathcal{A})$ is a group and that multiplication is continuous, since multiplication $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is jointly continuous. It remains to show that the inverse map $\sigma : \mathrm{Inv}(\mathcal{A}) \to \mathrm{Inv}(\mathcal{A}), a \mapsto a^{-1}$ is continuous. To this purpose, let $r, r+h \in \mathrm{Inv}(\mathcal{A})$ and put $(r+h)^{-1} = r^{-1} + k$. We must show that $\|h\| \to 0$ implies $\|k\| \to 0$. From $\mathbf{1} = (r^{-1}+k)(r+h) = \mathbf{1} + r^{-1}h + kr + kh$ we obtain $r^{-1}h + kr + kh = 0$. Multiplying this on the right by r^{-1} we have $r^{-1}hr^{-1} + k + khr^{-1} = 0$, thus $k = -r^{-1}hr^{-1} - khr^{-1}$. Therefore $\|k\| \leq \|r^{-1}\|^2\|h\| + \|k\|\|h\|\|r^{-1}\|$, which is equivalent to $\|k\|(1 - \|h\|\|r^{-1}\|) \leq \|r^{-1}\|^2\|h\|$ and, for $\|h\| < \|r^{-1}\|^{-1}$, to

$$\|k\| \leq \frac{\|r^{-1}\|^2}{1 - \|h\|\|r^{-1}\|}\|h\|.$$

From this it is clear that $\|h\| \to 0$ implies $\|k\| \to 0$. ■

1.2 Definition *If $\mathcal{A}$ is a unital algebra and $a \in \mathcal{A}$, the spectrum of a is defined as*

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda\mathbf{1} \not\in \mathrm{Inv}\,\mathcal{A}\}.$$

The spectral radius of a is $r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$.

1.3 Theorem *Let $\mathcal{A}$ be a unital normed algebra and $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$, and*

$$r(a) \geq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}. \tag{1}$$

Proof. We try to make the argument digestible by breaking it up in pieces.

Claim: With $\nu = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$ $(\leq \|A\|)$ we have $\lim_{m\to\infty} \|a^m\|^{1/m} = \nu$.

Proof. For every $a \in \mathcal{A}$, with $\|a^n\| \leq \|a\|^n$ we trivially have

$$0 \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n} \leq \|a\| < \infty. \tag{2}$$

Abbreviating $\nu = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$, for every $\varepsilon > 0$ there is a k such that $\|a^k\|^{1/k} < \nu + \varepsilon$. Now every $m \in \mathbb{N}$ is of the form $m = sk + r$ with unique $k \in \mathbb{N}_0$ and $0 \leq r < k$. Then

$$\|a^m\| = \|a^{sk+r}\| \leq \|a^k\|^s\|a\|^r < (\nu+\varepsilon)^{sk}\|a\|^r,$$

$$\|a^m\|^{1/m} \leq (\nu+\varepsilon)^{\frac{sk}{sk+r}}\|a\|^{\frac{r}{sk+r}}.$$

Now $m \to \infty$ means $\frac{sk}{sk+r} \to 1$ and $\frac{r}{sk+r} \to 0$, so that $\limsup_{m\to\infty} \|a^m\|^{1/m} \leq \nu + \varepsilon$. Since this holds for every $\varepsilon > 0$, we have $\limsup_{m\to\infty} \|a^m\|^{1/m} \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$. Together with (2) this implies that $\lim_{m\to\infty} \|a^m\|^{1/m}$ exists and equals $\inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$. □

Claim: (i) holds if $\nu = 0$. Assume $a \in \mathrm{Inv}(\mathcal{A})$. Then there is $b \in \mathcal{A}$ such that $ab = ba = \mathbf{1}$. Then $\mathbf{1} = a^nb^n$, thus with $1 \leq \|\mathbf{1}\|$ we have $1 \leq \|\mathbf{1}\| = \|a^nb^n\| \leq \|a^n\|\|b^n\| \leq \|a^n\|\|b\|^n$. Taking n-th roots, we have $1 \leq \|a^n\|^{1/n}\|b\|$, and taking the limit gives the contradiction $1 \leq \nu\|b\| = 0$. Thus if $\nu = 0$ then a is not invertible, so that $0 \in \sigma(a)$, thus $\sigma(a) \neq \emptyset$. Now (1) is obviously true. $\square$

Claim C: For all $\mu > \nu$ we have $\left(\frac{a}{\mu}\right)^n \to 0$ as $n \to \infty$, but $\left(\frac{a}{\nu}\right)^n \not\to 0$ provided $\nu > 0$. (This is of course trivial if $\mu > \|a\|$, but our hypothesis is weaker when $\nu < \|a\|$.)
Proof. Let $\mu > \nu$, and choose μ' such that $\nu < \mu' < \mu$. Since $\|a^n\|^{1/n} \to \nu$ by the first step, there is a n_0 such that $n \geq n_0 \Rightarrow \|a^n\|^{1/n} < \mu'$. For such n we have

$$\left\|\left(\frac{a}{\mu}\right)^n\right\| = \frac{\|a^n\|}{\mu^n} \leq \left(\frac{\mu'}{\mu}\right)^n \stackrel{n\to\infty}{\longrightarrow} 0.$$

Thus for every $\mu > \nu$ we have that $(a/\mu)^n \to 0$ as $n \to \infty$. On the other hand, for all $n \in \mathbb{N}$ we have $\|a^n\|^{1/n} \geq \nu$. With $\nu > 0$ this implies $\|(a/\nu)^n\| \geq 1\ \forall n$, and therefore $(a/\nu)^n \not\to 0$. $\square$

From now on assume $\nu > 0$. Assume that there is no $\lambda \in \sigma(a)$ with $|\lambda| \geq \nu$. This implies that $(a - \lambda\mathbf{1})^{-1}$ exists for all $|\lambda| \geq \nu$ and depends continuously on λ by Lemma 1.1. The same holds (since $|\lambda| \geq \nu > 0$) for the slightly more convenient function

$$\phi : \{\lambda \in \mathbb{C} \mid |\lambda| \geq \nu\} \to \mathcal{A}, \quad \lambda \mapsto \left(\frac{a}{\lambda} - \mathbf{1}\right)^{-1}.$$

Claim A: For all $|\lambda| \geq \nu,\ n \in \mathbb{N}$ we have $(\frac{a}{\lambda})^n - \mathbf{1} \in \mathrm{Inv}\mathcal{A}$ and

$$((\frac{a}{\lambda})^n - \mathbf{1})^{-1} = \frac{1}{n}\sum_{k=1}^{n} \phi(\lambda_k), \quad \text{where} \quad \lambda_k = e^{\frac{2\pi i}{n}k}\lambda. \tag{3}$$

Before we give the proof, which is elementary algebra, we show how Claims C and A imply the theorem.

Pick any $\eta > \nu$. Since the annulus $\Lambda = \{\lambda \in \mathbb{C} \mid \nu \leq |\lambda| \leq \eta\}$ is compact, the continuous map $\phi : \Lambda \to \mathcal{A}$ is uniformly continuous. I.e., for every $\varepsilon > 0$ we can find $\delta > 0$ such that $\lambda, \lambda' \in \Lambda,\ |\lambda - \lambda'| < \delta \Rightarrow \|\phi(\lambda) - \phi(\lambda')\| < \varepsilon$. If $\nu < \mu < \nu + \delta$, we have $|\nu_k - \mu_k| = |\nu - \mu| < \delta$ and therefore $\|\phi(\nu_k) - \phi(\mu_k)\| < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1, \ldots, n$. Combining this with (3) we have $\|((\frac{a}{\nu})^n - \mathbf{1})^{-1} - ((\frac{a}{\mu})^n - \mathbf{1})^{-1}\| \leq \frac{1}{n}\sum_{k=1}^n \|\phi(\nu_k) - \phi(\mu_k)\| < \varepsilon\quad \forall n \in \mathbb{N}$. Thus:

$$\forall \varepsilon > 0\ \ \exists \mu > \nu\ \ \forall n \in \mathbb{N} : \quad \|((\frac{a}{\nu})^n - \mathbf{1})^{-1} - ((\frac{a}{\mu})^n - \mathbf{1})^{-1}\| < \varepsilon. \tag{4}$$

By Claim C, $\mu > \nu$ implies $(a/\mu)^n \to 0$ as $n \to \infty$. With continuity of the inverse map, $((a/\mu)^n - \mathbf{1})^{-1} \to -\mathbf{1}$. Thus for n large enough we have $\|((a/\mu)^n - \mathbf{1})^{-1} + \mathbf{1}\| < \varepsilon$, and combining this with (4) we have $\|((a/\nu)^n - \mathbf{1})^{-1} + \mathbf{1}\| < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $((a/\nu)^n - \mathbf{1})^{-1} \to -\mathbf{1}$ as $n \to \infty$ and therefore $(a/\nu)^n \to 0$. This contradicts Claim C, so that our assumption that there is no $\lambda \in \sigma(a)$ with $|\lambda| \geq \nu$ is false. Existence of such a λ obviously gives $\sigma(a) \neq \emptyset$ and $r(a) \geq \nu$, completing the proof.

It remains to prove (3): For $0 \neq \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, put $\lambda_k = \lambda e^{\frac{2\pi i}{n}k}$, where $k = 1, \ldots, n$. (One should really write $\lambda_{n,k}$, but we suppress the n.) Then $\lambda_1, \ldots, \lambda_n$ are the solutions of $z^n = \lambda^n$, and we have $z^n - \lambda^n = \prod_k (z - \lambda_k)$. Let $|\lambda| \geq \nu$ and $n \in \mathbb{N}$. Then our assumption ($|\lambda| \geq \nu \implies \lambda \notin \sigma(a)$) implies $\lambda_k \notin \sigma(a)$ for all $k = 1, \ldots, n$. Thus all $\frac{a}{\lambda_k} - \mathbf{1}$ are invertible, and so is $(\frac{a}{\lambda})^n - \mathbf{1} = \prod_k(\frac{a}{\lambda_k} - \mathbf{1})$. A telescoping computation proves the well-known formula

$$z^n - 1 = (z - 1)(1 + z + z^2 + \cdots + z^{n-1}) \tag{5}$$

for finite geometric sums. Putting $z = a/\lambda_k$ and observing $\lambda_k^n = \lambda^n$, we have

$$(\frac{a}{\lambda})^n - \mathbf{1} = (\frac{a}{\lambda_k})^n - \mathbf{1} = (\frac{a}{\lambda_k} - \mathbf{1})(\mathbf{1} + \frac{a}{\lambda_k} + \cdots + (\frac{a}{\lambda_k})^{n-1})$$

and therefore with the invertibilities proven above,

$$\phi(\lambda_k) = (\frac{a}{\lambda_k} - \mathbf{1})^{-1} = ((\frac{a}{\lambda})^n - \mathbf{1})^{-1}(\mathbf{1} + \frac{a}{\lambda_k} + \cdots + (\frac{a}{\lambda_k})^{n-1}) = ((\frac{a}{\lambda})^n - \mathbf{1})^{-1}\sum_{l=0}^{n-1}\left(\frac{a}{\lambda}\right)^l e^{-\frac{2\pi i k l}{n}}.$$

Summing over $k \in \{1, \ldots, n\}$, we have

$$\sum_{k=1}^{n} \phi(\lambda_k) = ((\frac{a}{\lambda})^n - \mathbf{1})^{-1} \sum_{l=0}^{n-1} \left(\frac{a}{\lambda}\right)^l \sum_{k=1}^{n} e^{-\frac{2\pi i k l}{n}}. \tag{6}$$

If $l \in \{1, \ldots, n-1\}$ then $z = e^{\frac{2\pi i}{n} l}$ satisfies $z \neq 1$ and $z^n = 1$. Thus (5) gives

$$\sum_{k=1}^{n} e^{\frac{2\pi i}{n} kl} = e^{\frac{2\pi i}{n} l} \sum_{k=0}^{n-1} z^k = e^{\frac{2\pi i}{n} l} \frac{z^n - 1}{z - 1} = 0,$$

so that only $l = 0$ in (6) survives, and the r.h.s. equals $n((\frac{a}{\lambda})^n - \mathbf{1})^{-1}$, yielding (3). ■

The above proof is from [3] (which is written in terms of 'quasi-inverses', for which one does not need a unit). Versions of it can be found in [4, 2]. Compare also [5, 1].

1.4 Corollary *If $\mathcal{A}$ is a normed division algebra (thus unital) over $\mathbb{C}$ then $\mathcal{A} = \mathbb{C}\mathbf{1}$.*

Proof. Let $a \in \mathcal{A}$. By the above Theorem, $\sigma(a) \neq \emptyset$. Picking $\lambda \in \sigma(a)$, we have $a - \lambda\mathbf{1} \notin \mathrm{Inv}(\mathcal{A})$. Since $\mathcal{A}$ is a division algebra, we have $a - \lambda\mathbf{1} = 0$, thus $a \in \mathbb{C}\mathbf{1}$. Since a was arbitrary, we have $\mathcal{A} = \mathbb{C}\mathbf{1}$. ■

Note that completeness was neither assumed nor used so far! The following material is standard and is included for the sake of completeness.

1.5 Theorem *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then*

(i) $\mathbf{1} - b \in \mathrm{Inv}\,\mathcal{A}$ *whenever* $b \in \mathcal{A}$, $\|b\| < 1$, *and* $\mathrm{Inv}\,\mathcal{A} \subseteq \mathcal{A}$ *is open.*

(ii) $\sigma(a)$ *is closed.*

(iii) $r(a) \leq \|a\|$.

(iv) $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. *(Beurling-Gelfand formula)*

Proof. (i) If $\|b\| < 1$ then $\sum_{n=0}^{\infty} \|b^n\| \leq \sum_{n=0}^{\infty} \|b\|^n < \infty$, so that the series $\sum_{n=0}^{\infty} b^n$ converges to some $c \in \mathcal{A}$ by completeness. Now clearly $c = \mathbf{1} + bc = \mathbf{1} + cb$, which is equivalent to $c(\mathbf{1} - b) = \mathbf{1} = (\mathbf{1} - b)c$ to that $\mathbf{1} - b \in \mathrm{Inv}\,\mathcal{A}$. If now $a \in \mathrm{Inv}\,\mathcal{A}$ and $\|a - a'\| < \|a^{-1}\|^{-1}$ then $\|\mathbf{1} - a^{-1}a'\| = \|a^{-1}(a - a')\| \leq \|a^{-1}\|\|a - a'\| < 1$ so that $a^{-1}a' = \mathbf{1} - (\mathbf{1} - a^{-1}a') \in \mathrm{Inv}\,\mathcal{A}$, thus $a' = a(a^{-1}a') \in \mathrm{Inv}\,\mathcal{A}$. This proves that $\mathrm{Inv}\,\mathcal{A}$ is open.

(ii) If $a \in \mathcal{A}$ then $f_a : \mathbb{C} \to \mathcal{A}$, $\lambda \mapsto a - \lambda\mathbf{1}$ is continuous, thus $f_a^{-1}(\mathrm{Inv}\,\mathcal{A}) \subseteq \mathbb{C}$ is open by (i). Now $\sigma(a) = \mathbb{C} \backslash f_a^{-1}(\mathrm{Inv}\,\mathcal{A})$ is closed.

(iii) If $\lambda \in \mathbb{C}, |\lambda| > \|a\|$ then $\|a/\lambda\| < 1$ so that $\mathbf{1} - a/\lambda \in \mathrm{Inv}\,\mathcal{A}$ by (i). Thus $\lambda\mathbf{1} - a \in \mathrm{Inv}\,\mathcal{A}$, so that $\lambda \notin \sigma(a)$.

(iv) Replacing z in (5) by $a \in \mathcal{A}$, both factors on the r.h.s. commute. If $\lambda \in \sigma(a)$ then $a - \lambda$ is not invertible, thus $a^n - \lambda^n$ is not invertible, so that $\lambda^n \in \sigma(a^n)$. Thus $r(a) \leq \inf_{n\in\mathbb{N}} r(a^n)^{1/n}$. Using (iii), we have $r(a) \leq \inf_n r(a^n)^{1/n} \leq \inf_n \|a^n\|^{1/n} = \nu$. Combining with Theorem 1.3 gives the claim. ■

References

[1] S. Kametani: An elementary proof of the fundamental theorem of normed fields. J. Math. Soc. Japan **4**, 96-99 (1952).

[2] E. Kaniuth: *A course in commutative Banach algebras.* Springer, 2009.

[3] C. E. Rickart: An elementary proof of a fundamental theorem in the theory of Banach algebras. Michigan Math. J. **5**, 75-78 (1958).

[4] C. E. Rickart: *General theory of Banach algebras.* Van Nostrand, 1960.

[5] L. Tornheim: Normed fields over the real and complex fields. Michigan Math. J. **1**, 61-68 (1952).