Superselection theory in low dimensions, modular invariants and categorical ramifications

Michael Müger

Radboud Universiteit Nijmegen

(Partially based on joint work with
A. Kitaev; A. Davydov, D. Nikshych, V. Ostrik)

LMS Regional Meeting and workshop on operator algebras
Cardiff, June 21-25, 2010

Outline

Superselection theory for $d \geq 2+1$

Algebraic QFT (Local Quantum Physics):

- Spacetime M $(\mathbb{R}, S^1, M^{s+1}, (M,g), \ldots)$
- Nice regions: $I \subset \mathbb{R}$, $\mathcal{O} \subset M$.
- Vacuum Hilbert space H_0
- Local (von Neumann) algebras: $\mathcal{O} \mapsto A(\mathcal{O}) \subset \mathcal{B}(H_0)$.

Axioms:

- Isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; A(\mathcal{O}_1) \subset A(\mathcal{O}_2)$.
- Locality: $\mathcal{O}_1 \perp \mathcal{O}_2 \;\Rightarrow\; [A(\mathcal{O}_1), A(\mathcal{O}_2)] = \{0\}$.
- Irreducibility: $\bigvee A(\mathcal{O}) = \mathcal{B}(H_0)$.
- Vacuum $\Omega \in H_0$.
- Covariance, Positive energy, ...

Superselection theory (DHR)

Let $\mathcal{A}$ be the global C^*-algebra associated to $\{A(\mathcal{O})\}$.
A DHR representation (Doplicher-Haag-Roberts representation) is a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ such that

$$\pi \restriction A(\mathcal{O}') \;\cong\; \pi_0 \restriction A(\mathcal{O}') \quad \forall \mathcal{O}.$$

Assuming Haag duality (strong version of locality)

$$A(\mathcal{O})' = A(\mathcal{O}') \quad \forall \mathcal{O}$$

and $d \geq 2+1$, DHR (1971) prove that the $*$-category DHR$(\mathcal{A})$ of DHR representations is a symmetric tensor $*$-category with irreducible tensor unit $\mathbf{1}$.
The full subcategory DHR$_f(\mathcal{A})$ of finite representations is rigid (has duals). From now on we only consider DHR$_f(\mathcal{A})$ and drop the subscript.

Galois theory for local fields

- Doplicher/Roberts 1989/90:
  - Unique compact (super)group G s.th. Rep $G \simeq \mathrm{DHR}(\mathcal{A})$.
  - Net $\mathcal{O} \mapsto F(\mathcal{O}) \subset \mathcal{B}(H)$ satisfying (graded) locality.
  - G acts unitarily on $\mathcal{F}$ and $\mathcal{F}^G \cong \mathcal{A}$.
  - $F(\mathcal{O}) \cap A(\mathcal{O})' = \mathbb{C}\mathbf{1} \ \forall \mathcal{O}$.
  - Restricted to $\mathcal{A}$, the vacuum representation of $\mathcal{F}$ contains all DHR representations of $\mathcal{A}$.
  - DHR($\mathcal{F}$) is trivial. (Conti/Doplicher/Roberts)
  - For every local extension $\mathcal{B} \supset \mathcal{A}$, there is closed subgroup $H \subset G$ s.th. $\mathcal{B} \cong \mathcal{F}^H$.
- Mathematical interpretation: Galois theory for local fields. $G(\mathrm{DHR}(\mathcal{A}))$ = absolute Galois group, $\mathcal{F}$ = Galois closure of $\mathcal{A}$, etc.
- Proof is involved and somewhat monolithic.

Alternative approach

based on work by Roberts (unpublished), Deligne, Bichon + ε. Cf. Halvorson/M. (2006).

1. $\mathcal{C}$ symmetric semisimple rigid tensor category with simple $\mathbf{1}$ $\rightsquigarrow$ symm. fiber functor $E : \mathcal{C} \to \text{Vect}_{\mathbb{C}}$. (Deligne 1990, Bichon 1998)
2. $\mathcal{C}$ $*$-category $\rightsquigarrow$ fiber functor can be chosen $*$-preserving. (M.)
3. $\rightsquigarrow$ $G := \mathrm{Aut}_{\otimes} E$ is compact and $\mathcal{C} \simeq \mathrm{Rep}\, G$. (Tannaka 1939)
4. $\rightsquigarrow$ field net $\mathcal{F} = \mathcal{A} \rtimes_E \mathcal{C}$ acted upon by G. (Roberts 1970s). Idea: $\mathcal{F} = \{(A, \rho, \psi) \mid A \in \mathcal{A}, \rho \in \mathrm{DHR}(\mathcal{A}), \psi \in E(\rho)\}/\sim$, where $(AT, \rho, \psi) \sim (A, \rho', E(T)\psi)$ when $T \in \mathrm{Hom}(\rho, \rho')$.

Advantages:

1. Proof is modular and quite transparent.
2. Simplifies matters in applications.
3. Partially applicable in low dimensions.

Low dimensional spacetimes

Low dimensions: $\mathbb{R}, S^1, M^{1+1}$. What changes?
Fredenhagen/Rehren/Schroer (1989): DHR($\mathcal{A}$) has same properties as before, except: DHR($\mathcal{A}$) is only braided.
In fact (Kawahigashi/Longo/M. 2001): If $\mathcal{A}$ is completely rational theory on $\mathbb{R}$, i.e.

1. Strong additivity.
2. Split property.
3. $I \subset\subset J \;\Rightarrow \mu := [A(J) \cap A(I)' : A(J \cap I')] < \infty$.

then

- All irreducible DHR reps are finite, thus have duals.
- Finitely many irreps, $\dim \mathrm{DHR}(\mathcal{A}) \equiv \sum_i d(\rho_i)^2 = \mu$.
- DHR($\mathcal{A}$) is modular category, i.e. maximally non-symmetric: ρ simple, $\not\cong \mathbf{1} \;\Rightarrow \exists\sigma$ s.th. $c(\rho,\sigma) \circ c(\sigma,\rho) \neq \mathrm{id}$.

Clear: DHR($\mathcal{A}$) $\not\simeq \mathrm{Rep}\, G$ for G compact group.

What to do?

- DHR($\mathcal{A}$) not symmetric $\Rightarrow$ no proof of existence of fiber functor.
- At least: Given a $*$-preserving fiber functor E, Roberts' construction $\rightsquigarrow$ field net $\mathcal{F}$, Woronowicz's Tannaka theorem $\rightsquigarrow$ discrete quantum group Q acting on $\mathcal{F}$ with $\mathcal{F}^Q \cong \mathcal{A}$ and R-matrix describing space-like commutation relations.
- In general: fiber functor $E : \mathrm{DHR}(\mathcal{A}) \to \mathrm{Hilb}$ does not exist.
- Hayashi (1980s)/Ostrik (2003): $\mathcal{C}$ finite semisimple fusion category $\rightsquigarrow$ weak Hopf algebra H s.th. $\mathcal{C} \simeq H$-Mod. Related to 'reduced field bundle' (FRS 1990).
- Problems:
  - H non-unique $\rightsquigarrow$ No good physical interpretation.
  - $F(\mathcal{O}) \cap A(\mathcal{O})' \neq \mathbb{C}\mathbf{1}$ in reduced field bundle.
- Solution:
  - Consider the category DHR($\mathcal{A}$) as fundamental.
  - Categorical approach to local extensions etc.

Local extensions

- Local extension of QFT $\mathcal{A}$: Inclusion of local nets $A(\mathcal{O}) \hookrightarrow B(\mathcal{O}) \subset \mathcal{B}(\widehat{H})$ (where $\widehat{H} \supsetneq H$).
- Index $[\mathcal{B} : \mathcal{A}] = [\mathcal{B}(\mathcal{O}) : \mathcal{A}(\mathcal{O})] \in [1, \infty]$ (indep. of $\mathcal{O}$)
- $\mathcal{A} \subset \mathcal{B}$ with $[\mathcal{B} : \mathcal{A}] < \infty \;\Rightarrow\; \mathcal{A}$ cpl. rtl. $\Leftrightarrow$ $\mathcal{B}$ cpl. rtl. and

$$\dim \mathrm{DHR}(\mathcal{A}) = [\mathcal{B} : \mathcal{A}]^2 \cdot \dim \mathrm{DHR}(\mathcal{B}).$$

- Obvious consequence: There are maximal local extensions; every local extension is contained in a maximal one.
- Maximal local extensions usually not unique!
- But all maximal local extensions have equivalent representation categories, denoted $\mathrm{DHR}(\mathcal{B}_{\max})$.
- But: In general, $\mathrm{DHR}(\mathcal{B}_{\max})$ is not trivial!
- Question: Under which condition on $\mathcal{A}$ is $\mathrm{DHR}(\mathcal{B}_{\max})$ trivial?

Longo-Rehren: Classification of local extensions

- Longo/Rehren (1995): Bijection local extensions $B \supset A \leftrightarrow$ commutative Q-systems (Γ, m, η) in DHR($\mathcal{A}$).
- Under this correspondence, $[\mathcal{B} : \mathcal{A}] = d(\Gamma)$.
- Modulo technicalities, a Q-system in a $\otimes$-category $\mathcal{C}$ is just an algebra in $\mathcal{C}$, i.e. $\Gamma \in \mathrm{Obj}(\mathcal{C})$, $m : \Gamma \otimes \Gamma \to \Gamma$, $\eta : \mathbf{1} \to \Gamma$ s.th. $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$, $m \circ \eta \otimes \mathrm{id} = m = m \circ \mathrm{id} \otimes \eta$. A Q-system (Γ, m, η) is commutative if $m \circ c(\Gamma, \Gamma) = m$.
- Question: Determine braided category DHR($\mathcal{B}$) for local extension $\mathcal{B} \supset \mathcal{A}$ corresponding to $(\Gamma, m, \eta) \in \mathrm{DHR}(\mathcal{A})$.
- Other issues: Composition of extensions, intermediate extensions, . . .

Representation Categories of Local extensions

- Candidate: Category $\Gamma - \mathrm{Mod}_{\mathcal{C}}$ of 'Γ-modules in $\mathcal{C}$', i.e. pairs (X, μ) with $\mu : \Gamma \otimes X \to X$ satisfying obvious axioms.
- $\mathcal{C}$ symmetric $\rightsquigarrow \Gamma - \mathrm{Mod}_{\mathcal{C}}$ symmetric.
- $\mathcal{C}$ braided $\rightsquigarrow \Gamma - \mathrm{Mod}_{\mathcal{C}}$ monoidal, but not necessarily braided!!
- Pareigis (1995): Γ-module (X, μ) is local ('dyslexic') if $\mu \circ c(X, \Gamma) \circ c(\Gamma, X) = \mu$. The full subcategory $\Gamma - \mathrm{Mod}^0_{\mathcal{C}} \subset \Gamma - \mathrm{Mod}_{\mathcal{C}}$ of local modules is monoidal and braided!
- Kirillov Jr./Ostrik (2003): $\mathcal{C}$ modular, Γ commut. algebra in $\mathcal{C}$ $\Rightarrow \; \Gamma - \mathrm{Mod}^0_{\mathcal{C}}$ modular and $\dim \Gamma - \mathrm{Mod}^0_{\mathcal{C}} = \dim \mathcal{C}/d(\Gamma)^2$.
- Theorem (M.): $\mathcal{A}$ completely rational QFT, Γ commutative Q-system in $\mathrm{DHR}(\mathcal{A})$ with $d(\Gamma) < \infty$. If $\mathcal{B} \supset \mathcal{A}$ is the local extension corresponding to Γ then $\mathrm{DHR}(\mathcal{B}) \;\simeq\; \Gamma - \mathrm{Mod}^0_{\mathrm{DHR}}(\mathcal{A})$ as braided tensor $*$-categories.

Representation Categories of Local extensions

- Remarks:
  - Proof uses α-induction (Böckenhauer/Evans), quite simple.
  - Analogous result for VOAs stated by Kirillov/Ostrik, but no complete proof given.
- Corollary: A local extension $\mathcal{B} \supset \mathcal{A}$ with $\mathrm{DHR}(\mathcal{B})$ trivial exists iff $\exists$ commutative algebra $\Gamma \in \mathrm{DHR}(\mathcal{A})$ with $d(\Gamma)^2 = \dim(\mathcal{C})$ (which is the maximal possible dimension.)
- Question: Which modular categories satisfy this condition?
- Theorem (M. 2006/7): A modular category $\mathcal{C}$ contains a commutative algebra Γ with $d(\Gamma)^2 = \dim \mathcal{C}$ iff there exists a fusion category $\mathcal{D}$ such that $\mathcal{C} \simeq Z(\mathcal{D})$. (Drinfeld centre)
- Conjectured by A. Kitaev (2006). $\Rightarrow$ also proven by Drinfeld/Gelaki/Nikshych/Ostrik (2007), Work related to $\Leftarrow$ by Bruguières/Virelizier (2008).

Reminder: The Drinfeld centre

- Let $\mathcal{C}$ be tensor category, $X \in \mathcal{C}$. A half braiding for X is a family $\{e_X(Y) : X \otimes Y \stackrel{\cong}{\longrightarrow} Y \otimes X\}_{Y \in \mathcal{C}}$, natural w.r.t. Y and satisfying

$$e_X(Y \otimes Z) = \mathrm{id}_Y \otimes e_X(Z) \circ e_X(Y) \otimes \mathrm{id}_Z \quad \forall Y, Z.$$

- $Z(\mathcal{C})$ = category with objects (X, e_X),

$$\mathrm{Hom}_{Z(\mathcal{C})}((X, e_X), (Y, e_Y)) = \{s : X \to Y \mid e_Y(Z) \circ s \otimes \mathrm{id}_Z = \mathrm{id}_Z \otimes s \circ e_X(Z) \ \forall Z\}.$$

With $(X, e_X) \otimes (Y, e_Y) = (X \otimes Y, e_{X \otimes Y})$, where

$$e_{X \otimes Y}(Z) = e_X(Z) \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes e_Y(Z) \ \ \forall Z$$

and $c((X, e_X), (Y, e_Y)) = e_X(Y)$, one proves that $Z(\mathcal{C})$ is braided tensor category. (Drinfeld, Majid, Joyal/Street ~ 1990).
- Theorem (M. 2003): Let $\mathcal{C}$ be a fusion category ($\mathbb{C}$-linear spherical tensor category, semisimple, finitely many simple objects, simple unit) satisfying $\dim \mathcal{C} \neq 0$ then $Z(\mathcal{C})$ is semisimple, finite with $\dim Z(\mathcal{C}) = (\dim \mathcal{C})^2$ and modular. If $\mathcal{C}$ is modular then $Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \widetilde{\mathcal{C}}$. ($\widetilde{\mathcal{C}} = \mathcal{C}$ as $\otimes$-category, opposite braiding.)
- Remark: Proof inspired by work of Ocneanu, Longo/Rehren, Izumi in subfactor theory.
- June 18, 2010: Turaev/Virelizier: For every $\mathcal{C}$ as above and every 3-manifold: $RT(M, Z(\mathcal{C})) = BW(M, C)\ \forall \mathcal{C}$. (BW: Barrett-Westbury state-sum invariant, no braiding on $\mathcal{C}$ needed.)

Applications

- Thus: Characterization of those modular categories that arise from Drinfeld's centre.
- NB: $\exists$ many modular categories $\mathcal{C} \not\simeq Z(\mathcal{D})$, e.g. most of those obtained from quantum groups at roots of unity and all arising from lattices.
- Corollary: A completely rational QFT $\mathcal{A}$ admits local extensions $\mathcal{B} \supset \mathcal{A}$ with trivial $\mathrm{DHR}(\mathcal{B})$ $\Leftrightarrow$ $\mathrm{DHR}(\mathcal{A})$ is Drinfeld centre of some fusion category.
- Corollary: Let $\mathcal{B}$ be completely rational with trivial $\mathrm{DHR}(\mathcal{B})$ ('holomorphic'), G finite group of unitarily implemented automorphisms and $\mathcal{A} := \mathcal{B}^G$. Then $\exists![\omega] \in H^3(G, \mathbb{T})$ s.th.

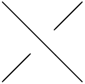
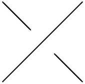
$$\mathrm{DHR}(\mathcal{A}) \simeq D^\omega(G) - \mathrm{Mod},$$

where $D^\omega(G)$ is the twisted quantum double (Dijkgraaf/Pasquier/Roche 1990).

Preliminaries

- Turaev 1992: A modular category $\mathcal{C}$ gives rise to a fin.dim. (unitary) representation π of the modular group $SL(2,\mathbb{Z})$.
- Long-standing problem: Construction of $d=2$ conformal field theories out of two chiral ("$d=1$") CFTs A_L, A_R.
- (Actually: What is a $d=2$ CFT?? By now various proposals.)
- Old fashioned approach to $d=2$ CFT: Study modular invariants: $\mathbb{Z}_{\geq 0}$-valued matrices (Z_{ij}) indexed by the simple objects of the representation categories of A_L, A_R, resp., satisfying $Z_{00}=1$ and $Z\pi_R(\cdot)=\pi_L(\cdot)Z$.
- More sophisticated approach needed.
- First proposal: Böckenhauer/Evans/Kawahigashi (1998-), based on subfactor theory.
- Categorical reformulation: Ostrik (2003), Fuchs/Runkel/Schweigert (2006). (Appl. to $d=2$ CFT.)

The Böckenhauer-Evans-Kawahigashi approach to modular invariants according to Ostrik-Fuchs-Runkel-Schweigert

- Non-commutative k-algebra Γ $\rightsquigarrow$ centre $Z(\Gamma)$.
- Same works in any semisimple symmetric tensor category $\mathcal{C}$, giving rise to commutative algebra $Z(\Gamma) \in \mathcal{C}$.
- $\mathcal{C}$ only braided $\rightsquigarrow$ two centres $Z_L(\Gamma), Z_R(\Gamma)$. (The definitions of the two centres differ only by  $\rightsquigarrow$  .)
- $\mathcal{C}$ modular $\Rightarrow$ One also obtains an equivalence

$$E : \Gamma_L^0 - \mathrm{Mod} \xrightarrow{\simeq} \Gamma_R^0 - \mathrm{Mod}$$

  of braided categories, where $\Gamma_{L/R} := Z_{L/R}(\Gamma)$.
- The triple (Γ_L, Γ_R, E) depends only on the Morita-class of Γ.
- F/R/S construct 'topological $d = 2$ CFT' starting from modular category $\mathcal{C}$ and (non-comm.) algebra $\Gamma \in \mathcal{C}$.

Rehren's QFT Approach

- Drawbacks of BEK/O/FRS approach: 1) Works only in left-right symmetric situation: $\mathcal{C}_L = \mathcal{C}_R$. 2) Involves non-commutative algebras in $\otimes$-categories, not corresponding to local extensions.
- Rehren (2000): Local QFT approach to modular invariants: Let A_L, A_R be completely rational CFTs on $\mathbb{R}$. Define local net of VNAs on M^{1+1} by

$$\mathcal{O} = I_L \times I_R \mapsto \mathcal{A}(\mathcal{O}) = A_L(I_L) \otimes \overline{A_R}(I_R)$$

  and study finite local extensions $\mathcal{B} \supset \mathcal{A}$.
- $\rightsquigarrow$ $\mathbb{Z}_{\geq 0}$-valued matrix (Z_{ij}) satisfying $Z_{00} = 1$ and $ZT_R = T_L Z$ where $T_{L/R} = \pi_{L/R} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Rehren's QFT Approach

- $ZS_R = S_L Z$ for $S_{L/R} = \pi_{L/R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (and thus $Z\pi_R(\cdot) = \pi_L(\cdot)Z$) holds iff $\mathrm{DHR}(\mathcal{B})$ is trivial. (Conjectured by Rehren, proven by M. and by Kawahigashi/Longo.)
- Recall: Local extensions $\mathcal{B} \supset \mathcal{A}$ with trivial $\mathrm{DHR}(\mathcal{B})$ correspond to commutative algebras Γ with $d(\Gamma)^2 = \dim \mathrm{DHR}(\mathcal{A})$. Thus every such Γ gives a modular invariant matrix Z.
- In fact, one has more: Given local extension $\mathcal{B} \supset \mathcal{A} \equiv A_L \otimes A_R$, Rehren (2000) proves existence of maximal local chiral extensions $\widehat{A}_L \supset A_L$, $\widehat{A}_R \supset A_R$ satisfying $A_L \otimes \overline{A_R} \subset \widehat{A}_L \otimes \overline{\widehat{A}_R} \subset \mathcal{B}$. Also: The categories $\widehat{\mathcal{C}}_{L/R} = \mathrm{DHR}(\widehat{A}_{L/R})$ have isomorphic fusion rules.
- Even better: Braided monoidal equivalence $E : \widehat{\mathcal{C}}_L \stackrel{\simeq}{\rightarrow} \widehat{\mathcal{C}}_R$. (Kawahigashi/Longo 2003)

Categorical reformulation

- Recall: $\widehat{A}_{L/R} \supset A_{L/R}$ $\leftrightarrow$ comm. algebras $\Gamma_{L/R} \in \mathcal{C}_{L/R}$ and $\widehat{\mathcal{C}}_{L/R} := \mathrm{DHR}(\widehat{A}_{L/R}) \simeq \Gamma_{L/R} - \mathrm{Mod}^0_{\mathcal{C}_{L/R}}$.
- Thus: A commutative algebra of maximal dimension in $\mathcal{C}_L \boxtimes \widetilde{\mathcal{C}}_R$ gives rise to a triple (Γ_L, Γ_R, E), where $\Gamma_{L/R} \in \mathcal{C}_{L/R}$ are commutative algebras and $E : \Gamma_L - \mathrm{Mod}^0_{\mathcal{C}_L} \stackrel{\simeq}{\to} \Gamma_R - \mathrm{Mod}^0_{\mathcal{C}_R}$ is a braided monoidal equivalence. Converse is also true. (Follows from LR 1995).
- Def.: A modular invariant for a pair $(\mathcal{C}_L, \mathcal{C}_R)$ of modular categories is a triple (Γ_L, Γ_R, E) as above.
- Results are independent of whether or not $\mathcal{C}_L, \mathcal{C}_R$ arise from local nets:
- Theorem (DMNO): For modular categories $\mathcal{C}_L, \mathcal{C}_R$ there is a bijection, modulo suitable equivalences, between commutative algebras $\Gamma \in \mathcal{C}_L \boxtimes \widetilde{\mathcal{C}}_R$ with $d(\Gamma)^2 = \dim \mathcal{C}_L \cdot \dim \mathcal{C}_R$ and modular invariants for $(\mathcal{C}_L, \mathcal{C}_R)$.

Witt Group of Modular Categories

- Def.: Modular categories $\mathcal{C}_1, \mathcal{C}_2$ are called Witt equivalent ($\mathcal{C}_1 \sim \mathcal{C}_2$) if there exists a modular invariant for $(\mathcal{C}_1, \mathcal{C}_2)$.
- By the characterization of Drinfeld centres: $\mathcal{C}_1 \sim \mathcal{C}_2 \ \Leftrightarrow \mathcal{C}_1 \boxtimes \widetilde{\mathcal{C}_2} \simeq Z(\mathcal{D})$ for $\mathcal{D}$ fusion (Drinfeld centre).
- Theorem: Witt equivalence is an equivalence relation (including braided equivalence).
- Rem.: Symmetry is easy, reflexivity follows from $\mathcal{C} \boxtimes \widetilde{\mathcal{C}} \simeq Z(\mathcal{C})$. Transitivity requires more work.
- Def.: $W_M = \{\mathcal{C} \text{ modular}\}/\sim$.
- With $[\mathcal{C}_1] \cdot [\mathcal{C}_2] := [\mathcal{C}_1 \boxtimes \mathcal{C}_2]$ and $\mathbf{1} = \mathrm{Vect}_{\mathbb{C}}$, W_M is a commutative monoid.
- In view of $\mathcal{C} \boxtimes \widetilde{\mathcal{C}} \simeq Z(\mathcal{C}) \sim \mathbf{1}$, defining $[\mathcal{C}]^{-1} = [\widetilde{\mathcal{C}}]$ turns W_M into an abelian group, the Witt Group of modular categories. (Due to A. Kitaev (+M.), V. Drinfeld et al., A. Davydov.)

Comments

- For every fusion category $\mathcal{D}$, we have $Z(\mathcal{D}) \sim \mathrm{Vect}_{\mathbb{C}}$, i.e. $[Z(\mathcal{D})] = \mathbf{1}$. Thus passing to the Witt group kills all Drinfeld centres. This is good since there is no hope of 'classifying' all fusion categories. But W_M should be computable.
- Γ comm. algebra in $\mathcal{C}$ $\Rightarrow$ $[\Gamma - \mathrm{Mod}^0_{\mathcal{C}}] = [\mathcal{C}]$. Thus: $\mathcal{A} \subset \mathcal{B}$ cpl. rtl. with $[\mathcal{B} : \mathcal{A}] < \infty$ $\Rightarrow$ $[\mathrm{DHR}(\mathcal{A})] = [\mathrm{DHR}(\mathcal{B})]$.
- Def: A modular category $\mathcal{C}$ is completely anisotropic if $\mathbf{1}$ is the only commutative algebra in $\mathcal{C}$.
- Theorem: Every Witt class contains a unique completely anisotropic category (up to braided equivalence).
- $\rightsquigarrow$ Uniqueness of $\mathrm{DHR}(\mathcal{B}_{\max})$.
- W_M contains the classical Witt group W (related to quadratic forms) as a subgroup. The latter is known explicitly.
- Conjecture: W_M is generated by W and $[U_q(\mathfrak{g}) - \mathrm{Mod}]$ $(q^N = 1)$ with relations given by conformal extensions, cosets and low-dim. exceptions. (=Rigorous version of QFT folklore.)

Selected Open Problems

- Determine Witt group W_M of modular categories.
- Inverse problem: Which modular categories arise from completely rational QFTs? [Solved in symmetric case $d \geq 2+1$. (Doplicher/Piacitelli)]
- Classify completely rational QFTs $\mathcal{A}$ with trivial $\mathrm{DHR}(\mathcal{A})$. (Contains the classification of self-dual lattices, thus probably hopeless.)
- The requirement that $\mathrm{DHR}(\mathcal{B})$ be trivial is central in Rehren's approach to modular invariants. Use this as starting point for an analytic approach to defining full $d=2$ CFT.

Thank you!