Some examples of Fourier series

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Abstract

We explain in fairly complete detail the construction of two continuous 2π-periodic functions. The first has Fourier series that is uniformly but not absolutely convergent. The second has its Fourier series diverge at a point. All this well known and very old. The only point is to give an accessible exposition with minimal overhead, complementing the results proven in my course on Functional Analysis.

1 Introduction

We write $C(S^1, \mathbb{C})$ for the set of continuous 2π-periodic functions.

If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and integrable over bounded intervals we put

$$\widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

and

$$S_N(f, x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx}. \tag{1.1}$$

If $\sum_{k \in \mathbb{Z}} |\widehat{f}(k)| < \infty$ then (1.1) converges absolutely uniformly to some $g \in C(S^1, \mathbb{C})$. By the uniform convergence one has $\widehat{g}(k) = \widehat{f}(k)\ \forall k \in \mathbb{Z}$. This implies $g = f$, cf. Proposition 2.3(ii).

The goal of these notes is to define fairly simple $f_1, f_2 \in C(S^1, \mathbb{C})$ such that

- $S_N(f_1, x)$ converges to $f_1(x)$ uniformly in x as $N \to \infty$, but $\sum_{k \in \mathbb{Z}} |\widehat{f_1}(k)| = \infty$.
- $S_N(f_2, 0)$ diverges as $N \to \infty$.

2 Reminders

For 2π-periodic f, g we define the convolution product $f \star g$ by

$$(f \star g) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(x - t) dt.$$

Now $f \star g$ is 2π-periodic. Convolution is commutative and associative. Defining $\|f\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)| dx$ we have

$$|(f \star g)(x)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(t) g(x - t)\, dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)||g(x - t)|\, dt \leq \|f\|_\infty \|g\|_1$$

thus $\|f \star g\|_\infty \leq \|f\|_\infty \|g\|_1$.

It is straightforward to prove $S_N(f,x) = (D_N \star f)(x)$, where

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

is the Dirichlet kernel. (The quickest way to check the last identity is the telescoping calculation

$$(e^{ix/2} - e^{-ix/2})D_N(x) = \sum_{k=-N}^{N} e^{ix(k+1/2)} - \sum_{k=-N}^{N} e^{ix(k-1/2)} = e^{ix(N+1/2)} - e^{-ix(N+1/2)},$$

together with $e^{ix} - e^{-ix} = 2i\sin x$.) Now we define the Fejér kernel

$$F_N(x) = \frac{1}{N+1}(D_0(x) + \cdots + D_N(x)) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{ikx},$$

(some authors would call this F_{N+1}) which has the property that

$$\sigma_N(f,x) = (F_N \star f)(x) = \frac{1}{N+1}\big(S_0(f)(x) + S_1(f)(x) + \cdots + S_N(f)(x)\big)$$

is the N-th Cesàro mean of the sequence $\{S_N(f)\}$ of partial Fourier sums of f.

2.1 Exercise Prove

$$F_N(x) = \frac{1}{N+1} \frac{\sin^2 \frac{N+1}{2}x}{\sin^2 \frac{x}{2}} \quad \forall N \in \mathbb{N}. \tag{2.1}$$

2.2 Lemma $\{F_N\}_{N\in\mathbb{N}}$ *is an approximate unit for* $\star$, *i.e.*

(i) $F_N(x) \geq 0$ *for all* N, x.

(ii) $\|F_N\|_1 = 1$ $\forall N$.

(iii) $\lim_{N\to\infty} \int_{\varepsilon \leq |x| \leq \pi} |F_N(x)| dx \to 0$.

Proof. (i) is obvious from the explicit formula (2.1). (ii) follows from (i) since it gives $\|F_N\|_1 = \frac{1}{2\pi}\int_0^{2\pi} F_N(x) = 1$, the second equality being obvious from the definition of F_N as finite sum. (iii) It is easy to see that $\frac{\sin^2 \frac{N+1}{2}x}{\sin^2 \frac{x}{2}}$ is bounded on $\{x \mid \varepsilon \leq |x| \leq \pi\}$ for each $\varepsilon > 0$. Now the result is obvious. ■

We note for later that (ii) implies $\|\sigma_N(f)\|_\infty = \|f \star F_N\|_\infty \leq \|f\|_\infty$ $\forall N$.

2.3 Proposition *Let* $f, g \in C(S^1, \mathbb{C})$. *Then*

(i) $\sigma_N(f,\cdot) = F_N \star f$ *converges uniformly to* f *as* $N \to \infty$.

(ii) If $\widehat{f}(k) = \widehat{g}(k)$ $\forall k \in \mathbb{N}$ *then* $f = g$.

Proof. (i) Omitted. See e.g. [2, 4]. The proof uses uniform continuity of f and holds for every approximate unit.

(ii) If $\widehat{f}(k) = \widehat{g}(k)$ $\forall k \in \mathbb{Z}$ then clearly $S_N(f,x) = S_N(g,x)$ for all N, x, thus also $\sigma_N(f,x) = \sigma_N(g,x)$ for all N, x. Now the result follows from (i). ■

3 Preparations

3.1 Proposition *If $f \in L^\infty(\mathbb{R}, \mathbb{C})$ is 2π-periodic and $\widehat{f}(k) = O(1/k)$ then $\{S_N(f,x)\}$ is bounded uniformly in N, x.*

Proof. We compute

$$\begin{aligned} S_N(f,x) - \sigma_N(f,x) &= \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} - \sum_{k=-N}^{N} \widehat{f}(k) \left(1 - \frac{|k|}{N+1}\right) e^{ikx} \\ &= \sum_{k=-N}^{N} \widehat{f}(k) \frac{|k|}{N+1} e^{ikx}, \end{aligned}$$

which gives

$$|S_N(f,x) - \sigma_N(f,x)| \le \sum_{k=-N}^{N} \left| \widehat{f}(k) \frac{|k|}{N+1} e^{ikx} \right| = \frac{1}{N+1} \sum_{k=-N}^{N} |k \widehat{f}(k)|.$$

In view of $\widehat{f}(k) = O(1/k)$, equivalent to $k\widehat{f}(k) = O(1)$, this gives that $|S_N(f,x) - \sigma_N(f,x)|$ is bounded uniformly in N, x. We have seen before that $\|\sigma_N(f)\|_\infty \le \|f\|_\infty < \infty$ for all N, thus $\sigma_N(f,x)$ is bounded uniformly in N, x. With $|S_N(f,x)| \le |S_N(f,x) - \sigma_N(f,x)| + |\sigma_N(f,x)|$ the same holds for $S_N(f,x)$. ■

3.2 Exercise Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = i(\pi - x)$ for $x \in (0, \pi)$. For $x \in (-\pi, 0)$ define f by $f(x) = i(\pi + x)$, so as to be odd on $(-\pi, \pi)$. Now extend f to be $\mathbb{R}$ by 2π-periodicity. (i) Prove

$$\widehat{f}(k) = \frac{1}{k} \quad \forall k \neq 0, \qquad \widehat{f}(0) = 0.$$

(ii) Conclude that the partial sums $S_N(f,x)$ are bounded uniformly in N, x.

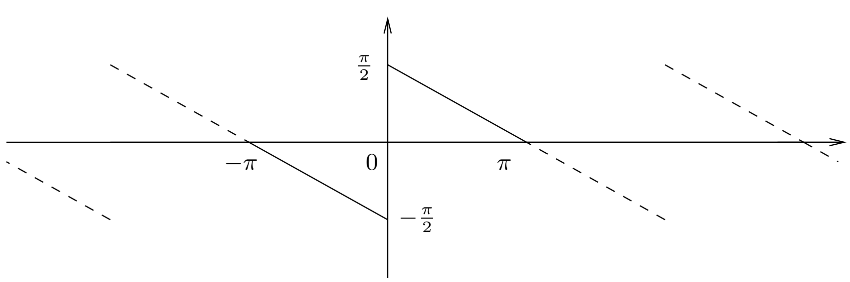

Figure 1: The sawtooth function $\frac{f}{2i}$ (picture from [4])

Using standard results from Fourier analysis and the fact that f has left and right derivatives at all points, it is easy to show that the Fourier series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ converges for all x, namely to zero if $x \in 2\pi\mathbb{Z}$ and to the sawtooth function $f/2i$ elsewhere. In the next section we will show that one has uniform convergence, thus a continuous limit function, if the denominator grows faster than k.

4 A function with uniformly but not absolutely convergent Fourier series

4.1 Proposition *Let $\{a_k\}_{k\in\mathbb{N}} \subseteq [0,\infty)$ such that $ka_k \searrow 0$ as $k\to\infty$. Then $\sum_{k=1}^\infty a_k \sin kx$ converges uniformly in x.*

Proof. We put $T_0(x)=0$ and $T_N(x)=\sum_{k=1}^N \frac{\sin kx}{k}$. In the preceding section we have proven boundedness, uniform in N,x, of $\sum_{1\le|k|\le N}\frac{e^{ikx}}{k}=\sum_{k=1}^N\frac{e^{ikx}-e^{-ikx}}{k}=2iT_N(x)$. Thus $|T_N(x)|\le C<\infty$ for all N,x and some C. Now a standard partial summation gives

$$\begin{aligned}
\sum_{k=M}^N a_k\sin kx &= \sum_{k=1}^N ka_k\,\frac{\sin kx}{k}=\sum_{k=M}^N ka_k(T_k(x)-T_{k-1}(x))\\
&= \sum_{k=M}^N ka_kT_k(x)-\sum_{k=M}^N ka_kT_{k-1}(x)\\
&= \sum_{k=M}^N ka_kT_k(x)-\sum_{k=M-1}^{N-1}(k+1)a_{k+1}T_k(x)\\
&= Na_NT_N(x)-Ma_MT_{M-1}(x)+\sum_{k=M}^{N-1}(ka_k-(k+1)a_{k+1})T_k(x).
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\sum_{k=M}^N a_k\sin kx\right| &\le |Na_NT_N(x)|+|Ma_MT_{M-1}(x)|+\sum_{k=M}^{N-1}|(ka_k-(k+1)a_{k+1})T_k(x)|\\
&\le (Na_N+Ma_M)C+\sum_{k=M}^{N-1}|(ka_k-(k+1)a_{k+1})|C\\
&= (Na_N+Ma_M)C+\sum_{k=M}^{N-1}((ka_k-(k+1)a_{k+1}))C\\
&= (Na_N+Ma_M)C+(Ma_M-Na_N)C=2Ma_M,
\end{aligned}$$

where we used $ka_k-(k+1)a_{k+1}\ge 0$ to remove the absolute value sign. Now the fact that $Ma_M\to 0$ implies that the series is Cauchy uniformly in x, thus uniformly convergent. ■

4.2 Corollary *(i) If $ka_k\searrow 0$ but $\sum_{k=1}^\infty a_k=\infty$ then $f(x)=\sum_{k=1}^\infty a_k\sin kx$ is a continuous function whose Fourier series converges uniformly, but not absolutely: $\sum_{k\in\mathbb{Z}}|\widehat{f}(k)|=\infty$.*

(ii) One such choice is $a_1=0,\ a_k=\frac{1}{k\log k}$ for $k\ge 2$, leading to $f_1(x)=\sum_{k\ge 2}\frac{\sin kx}{k\log k}$.

Proof. (i) Continuity of f follows from the uniform convergence. Now $\widehat{f}(0)=0$, and for $n\ne 0$ we have $\widehat{f}(k)=\frac{\mathrm{sgn}(k)a_{|k|}}{2i}$. The Fourier series just is the series $\sum_k a_k\sin kx$, and $\sum_{k\in\mathbb{Z}}|\widehat{f}(k)|=\infty$.

(ii) We have $(\log k)^{-1}\searrow 0$ and $\sum_{k=2}^\infty\frac{1}{k\log k}=\infty$, cf. e.g. [3, Theorem 3.29]. ■

4.3 Remark 1. The uniform convergence of $\sum_{k=1}^\infty a_k\sin kx$ remains valid if we only assume that $a_k\searrow 0$ and $ka_k\to 0$, which is weaker than $ka_k\searrow 0$. But the proof becomes more complicated, involving the conjugate Dirichlet kernel. Cf. [5, Section V.1] or [1, Section 7.2].

2. The above f_1 probably is the simplest function in $C(S^1, \mathbb{C})$ with non-absolutely summable Fourier coefficients. Of course others can be found using (i), but there are also examples not in this class. E.g. $\sum_{k=1}^{\infty} \frac{e^{i\alpha k \log k}}{k^\beta} e^{ikx}$ for all $\alpha > 0$ and $\beta \in (1/2, 1]$. But for these functions the proof of uniform convergence is more involved. Cf. [5]. $\square$

5 A function with Fourier series diverging at a point

We now discuss Fejér's example of a continuous function whose Fourier series diverges at a point. We will closely follow the expositions in [5, 1] quite closely. Note that the proof 'recycles' the fundamental ingredient already used in the preceding section.

For $p, q \in \mathbb{N}$ we define $t_{p,q}(x) = 2 \sin px \sum_{k=1}^{q} \frac{\sin kx}{k}$. By Exercise 3.2 the sums $\sum_{k=1}^{q} \frac{\sin kx}{k}$ are bounded uniformly in x, q, thus also the family $\{t_{p,q}\}$ is bounded uniformly in p, q, x. Thus if $\{\alpha_k\}_{k \in \mathbb{N}}$ satisfies $\sum_k |\alpha_k| < \infty$ then the series

$$f(x) = \sum_{l=1}^{\infty} \alpha_k \, t_{p_k, q_k}$$

converges uniformly to a continuous 2π-periodic function for any choice of the $p_k, q_k \in \mathbb{N}$. Since all the $t_{p,q}$ are even functions, so is f. Thus $f(x) = \sum_{k=0}^{\infty} c_k \cos kx$ and $S_N(f)(0) = \sum_{k=0}^{N} c_k$.

The goal now is to choose the p_k, q_k in such way as to have $S_N(f, 0) = \sum_{k=-N}^{N} \widehat{f}(k)$ diverge as $N \to \infty$. (This may seem to contradict the fact that $t_{p,q}(0) = 0$ for all p, q, but this is not the case.)

Not wanting to look up trigonometric identities, we compute

$$\begin{aligned} \sin x \sin y &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{iy} - e^{-iy}}{2i}\right) = \frac{-1}{4}\left(e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{-i(x-y)}\right) \\ &= \frac{\cos(x-y) - \cos(x+y)}{2}, \end{aligned}$$

so that

$$t_{p,q}(x) = \frac{\cos(p-q)x}{q} + \frac{\cos(p-q+1)x}{q-1} + \cdots + \frac{\cos(p-1)x}{1} - \frac{\cos(p+1)x}{1} + \cdots - \frac{\cos(p+q)x}{q}. \tag{5.1}$$

If we impose the restrictions

$$1 \le q_k \le p_k, \qquad p_k + q_k < p_{k+1} - q_{k+1}, \tag{5.2}$$

the summands t_{p_k,q_k} for different k have 'frequencies' in non-overlapping intervals. Thus

$$|S_{p_k+q_k}(f, 0) - S_{p_k}(f, 0)| = |\alpha_k| \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{q_k}\right) > |\alpha_k| \log q_k, \tag{5.3}$$

where we used $\sum_{k=1}^{n} 1/k \ge \int_1^{n+1} dx/x = \log(n+1) > \log n$. If we choose the α_k so as to have $\liminf_{k \to \infty} |\alpha_k| \log q_k > 0$ then (5.3) implies that $\{S_N(f, 0)\}$ is divergent (since it clearly is not Cauchy).

A possible choice of parameters satisfying the above restrictions is:

$$p_k = 2^{k^3+1}, \quad q_k = 2^{k^3}, \quad \alpha_k = \frac{1}{k^2}.$$

E.g., $|\alpha_k| \log q_k = k^{-2} k^3 \log 2$. The verification of (5.2) is left to the reader. $\blacksquare$

5.1 Exercise Give a more precise description of $S_N(f,0)$ as a function of N, for the above choice of parameters. In particular give an upper bound for the growth of $S_N(f,0)$.

5.2 Remark For $f \in C(S^1,\mathbb{C})$ then as an immediate consequence of $\|D_n\|_1 = O(\log n)$ and $|S_N(f,0)| \le \|D_N \star f\|_\infty \le \|D_N\|_1 \|f\|_\infty$ one has $S_N(f,0) = O(\log N)$. This can be improved to $o(\log N)$, cf. [5, Theorem II.11.9]. And for every sequence $\{\lambda_N\}$ with $\lambda_N = o(\log N)$ one can find a continuous 2π-periodic f with $|S_N(f,0)| \ge \lambda_N$ for infinitely many N. Cf. [5, Theorem VIII.1.2]. □

References

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