On Ikehara type Tauberian theorems with $O(x^\gamma)$ remainders

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Abstract

Motivated by analytic number theory, we explore remainder versions of Ikehara's Tauberian theorem yielding power law remainder terms. More precisely, for $f : [1,\infty) \to \mathbb{R}$ non-negative and non-decreasing we prove $f(x) - x = O(x^\gamma)$ with $\gamma < 1$ under certain assumptions on f. We state a conjecture concerning the weakest natural assumptions and show that we cannot hope for more.

1 Motivation and results

The following was proven in 1931 by Wiener's student Ikehara [4]. (For a much better proof see [1] or [2, Section 3.5].)

1.1 Theorem (Ikehara, 1931) *Let $f : [1,\infty) \to \mathbb{R}$ be non-negative and non-decreasing. Assume that*

$$F(s) = \int_1^\infty f(x) x^{-s} \frac{dx}{x}$$

converges for $s > 1$ (thus F is holomorphic on $\{\mathrm{Re}\, s > 1\}$). Assume that $F(s) - \frac{A}{s-1}$ has a continuous extension to the closed half-plane $\{\mathrm{Re}\, s \geq 1\}$. Then $f(x) = Ax + o(x)$.

This result gives rise to what still is the simplest proof of the prime number theorem $\pi(x) \sim \frac{x}{\log x}$. In most approaches to giving more precise estimates of $\pi(x) - \mathrm{Li}(x)$, or rather of $\psi(x) - x$, Tauberian theorems have not played a major rôle. One exception is provided by [5, 3], where remainder terms of the form $\frac{x}{\log^k x}$ are proven under somewhat stronger assumptions than in Theorem 1.1, which are then used to give the simplest known proofs of $\psi(x) - x = O(\frac{x}{\log^k x})\ \forall k \in \mathbb{N}$, invoking only properties of $\zeta(s)$ for $\mathrm{Re}\, s \geq 1$. More general results on remainder estimates in Ikehara's theorem are found in [7, §7.5], [6], but the Tauberian conditions considered here are different:

1.2 Question *Assume that $f : [1,\infty) \to \mathbb{R}$ is non-negative and non-decreasing, the integral $F(s) = \int_1^\infty f(x) x^{-s-1} dx$ converges for $s > 1$ (thus for $\mathrm{Re}\, s > 1$), and $F(s) - \frac{A}{s-1}$ has a holomorphic extension to the half-plane $\{\mathrm{Re}\, s > \alpha\}$, where $\alpha \in (0,1)$. Does this imply $f(x) = Ax + O(x^{\lambda+\varepsilon})$ for some $\lambda < 1$?*

1.3 Remark 1. Ikehara's theorem shows that there is a unique A such that $f(x) - Ax = o(x)$.

2. It is trivial that if $g : [1,\infty) \to \mathbb{R}$ is measurable and $g(x) = O(x^\gamma)$, then $G(s) = \int_1^\infty g(x) x^{-s-1} dx$ is convergent for $\mathrm{Re}\, s > \gamma$ and defines a holomorphic function on this domain. The above question is equivalent to asking to which extent this can be inverted under the additional

assumption that $x \mapsto g(x) + Ax$ is non-decreasing for some A (keeping in mind that the domain of holomorphicity of G can be larger then the domain of convergence of the integral).

3. If the answer to the question was positive with $\lambda = \alpha$ in case $A > 0$, it would provide a very simple deduction of $\psi(x) - x = O(x^{\alpha+\varepsilon})$ from $\mathrm{Re}\, s > \alpha \Rightarrow \zeta(s) \neq 0$. However, we will prove see that this is not the case. □

But there is a weaker positive answer to the question. To wit, we will prove the following:

1.4 Theorem *Let $f : [1, \infty) \to \mathbb{R}$ be non-negative and non-decreasing. Assume that $F(s) = \int_1^\infty f(x)x^{-s-1}dx$ converges for $s > 1$ and that $F(s) - \frac{A}{s-1}$ has a holomorphic continuation to $\{\mathrm{Re}\, s > \alpha\}$, where $\alpha \in (0, 1)$. Then $f(x) = Ax + O(x^{\gamma+\varepsilon})$, where*

(I) $\gamma = \alpha$ if $A = 0$,

(II) $\gamma = \frac{\alpha+1}{2}$ if $A > 0$ and $f(x) - Ax$ is of fixed sign for $x \geq x_0$ for some x_0.

These exponents are optimal under the given assumptions.

These results will be proven in Section 2. In Section 3 we show that the statement in Case II is false without the sign condition. We also conjecture that $\gamma = \frac{\alpha+2}{3}$ always works and provide some evidence.

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2 Proofs

Case (I) in Theorem 1.4 is fairly trivial and surely well-known. We only include the proof as a preparation for the following one.

2.1 Lemma *Let $\alpha > 0$ and $f : [1, \infty) \to \mathbb{R}$ be non-negative and non-decreasing. If $\int_1^\infty f(x)x^{-s-1}dx$ converges for all $s > \alpha$ then $f(x) = O(x^{\alpha+\varepsilon})$.*

Proof. Assume $f(x) = \Omega(x^\gamma)$ with $\gamma > \alpha$. Then there are $C > 0$ and arbitrarily large z such that $f(z) \geq Cz^\gamma$. For such a z, we have $f(x) \geq f(z) \geq Cz^\gamma$ whenever $x \geq z$. Taking $s = \gamma$, we have $\int_z^{2z} f(x)x^{-s-1}dx \geq Cz^\gamma \int_z^{2z} x^{-s-1}dx = Cs^{-1}(1 - 2^{-s}) > 0$. This contradicts the assumed convergence of $\int_1^\infty f(x)x^{-s-1}dx$ (since the latter means that for every $\varepsilon > 0$ there is a T such that $T \leq x_1 \leq x_2$ implies $|\int_{x_1}^{x_2} f(x)x^{-s-1}dx| < \varepsilon$). This contradiction proves that $f(x) = o(x^\gamma)$ for all $\gamma > \alpha$, which is equivalent to the assertion. ■

2.2 Remark With $f(x) = x^\alpha \log x$, we have convergence of $\int_1^\infty f(x)x^{-s-1}dx$ for all $s > \alpha$, but $f(x) = O(x^\gamma)$ holds if and only if $\gamma > \alpha$. This proves optimality of the result of the lemma. □

2.3 Proposition *Let $f : [1, \infty) \to \mathbb{R}$ be non-negative and non-decreasing. Then*

(i) If $A > 0$ and $\gamma \in (0, 1)$ are such that $f(x) - Ax = \Omega(x^\gamma)$ then $\int_1^\infty (f(x) - Ax)x^{-s-1}dx$ diverges whenever $s \leq 2\gamma - 1$.

(ii) If the integral $\int_1^\infty (f(x) - Ax)x^{-s-1}dx$ converges in the half-plane $\{\mathrm{Re}\, s > \alpha\}$ then $f(x) - Ax = O(x^{\gamma+\varepsilon})$, where $\gamma = \frac{\alpha+1}{2}$.

Proof. (i) The assumption $f(x) - Ax = \Omega(x^\gamma)$ means that there are $C > 0$ and arbitrarily large x such that $|f(x) - Ax| \geq Cx^\gamma$. Assume x_1 is such that $f(x_1) - Ax_1 \geq Cx_1^\gamma$. Since f is non-decreasing, we have $f(x) \geq f(x_1) \geq Ax_1 + Cx_1^\gamma$ for all $x \geq x_1$. Put $x_2 = x_1 + \frac{Cx_1^\gamma}{2A}$. Then for all $x \in [x_1, x_2]$ we have

$$f(x) - Ax \geq Ax_1 + Cx_1^\gamma - Ax_2 = Cx_1^\gamma - A(x_2 - x_1) = Cx_1^\gamma - A\frac{Cx_1^\gamma}{2A} = \frac{Cx_1^\gamma}{2}.$$

Thus for $s \geq -1$ we have

$$\int_{x_1}^{x_2} \frac{f(x) - Ax}{x^{s+1}} dx \geq \frac{Cx_1^\gamma}{2A} \frac{Cx_1^\gamma}{2} \frac{1}{x_2^{s+1}} = \frac{C^2 x_1^{2\gamma}}{4A(x_1 + \frac{Cx_1^\gamma}{2A})^{s+1}} = \frac{C^2}{4A} \frac{x_1^{2\gamma - s - 1}}{(1 + \frac{C}{2Ax_1^{1-\gamma}})^{s+1}}.$$

Now assume x_2 is such that $f(x_2) - Ax_2 \leq -Cx_2^\gamma$. Since f is non-decreasing, this implies that $f(x) \leq Ax_2 - Cx_2^\gamma$ for all $x \leq x_2$. Define $x_1 = x_2 - \frac{Cx_2^\gamma}{2A}$. By a reasoning similar to the one above we have $f(x) - Ax \leq -\frac{Cx_2^\gamma}{2}$ for all $x \in [x_1, x_2]$, implying

$$\int_{x_1}^{x_2} \frac{f(x) - Ax}{x^{s+1}} dx \leq -\frac{C^2}{4A} \frac{x_2^{2\gamma - s - 1}}{(1 - \frac{C}{2Ax_2^{1-\gamma}})^{s+1}}.$$

If $s \leq 2\gamma - 1$ then $2\gamma - s - 1 \geq 0$. Then the above computations and $f(x) - Ax = \Omega(x^\gamma)$ imply that for every $T \geq 1$ we can find an interval $[x_1, x_2] \subseteq [T, \infty)$ such that

$$\left| \int_{x_1}^{x_2} \frac{f(x) - Ax}{x^{s+1}} dx \right| \geq \frac{C^2}{4A \cdot 2} x_1^{2\gamma - s - 1} \geq \frac{C^2}{4A \cdot 2}.$$

But this clearly implies that $\int_1^\infty (f(x) - Ax)x^{-s-1}dx$ diverges.

(ii) This is essentially the contraposition of (i). ∎

Note that the above does not assume $f(x) - Ax$ to be ultimately of constant sign.

The following is a version of the Phragmén-Landau theorem on Dirichlet series with positive coefficients, cf. e.g. [7, Sec. II.1, Theorem 1.9].

2.4 Proposition *Assume that $g : [1, \infty) \to \mathbb{R}$ is non-negative and measurable, that*

$$G(s) = \int_1^\infty g(x)x^{-s} \frac{dx}{x} \tag{1}$$

converges for $\operatorname{Re} s > 1$ and that the function G has a holomorphic extension to the half-plane $\{\operatorname{Re} s > \alpha\}$, where $\alpha \in (0, 1)$. Then the integral in (1) converges to G whenever $\operatorname{Re} s > \alpha$.

Proof. Let $\alpha < t < 1 < s$. Since G is holomorphic at s, it has a power series expansion

$$G(z) = \sum_{n=0}^\infty \frac{(z - s)^n \, G^{(n)}(s)}{n!}. \tag{2}$$

Since $\int_1^\infty g(x)x^{-s}\frac{dx}{x} < \infty$ for all $s > 1$ and $(\log x)^k = O(x^\varepsilon)$ for any $k, \varepsilon > 0$, we also have $\int_1^\infty |g(x)|(\log x)^n x^{-s} \frac{dx}{x} < \infty$ for all $s > 1, n \in \mathbb{N}$. With $\frac{d^n}{ds^n} x^{-s} = (-\log x)^n x^{-s}$ and Lebesgue's dominated convergence theorem we can differentiate $G(s)$ under the integral sign and obtain

$$G^{(n)}(s) = \int_1^\infty g(x)(-\log x)^n x^{-s} \frac{dx}{x}.$$

Since G is holomorphic on the half-plane $\{\operatorname{Re} z > \alpha\}$, the domain of convergence of (2) includes t. Thus

$$G(t) = \sum_{n=0}^{\infty} \frac{(t-s)^n}{n!} \int_1^{\infty} g(x)(-\log x)^n x^{-s} \frac{dx}{x} = \sum_{n=0}^{\infty} \int_1^{\infty} \frac{(s-t)^n}{n!} g(x)(\log x)^n x^{-s} \frac{dx}{x}.$$

Since the integrand is non-negative and the double integral converges by our assumptions, by Fubini-Tonelli we may reverse the order of summation and integration:

$$\begin{aligned} G(t) &= \int_1^{\infty} \left(\sum_{n=0}^{\infty} \frac{(s-t)^n}{n!} (\log x)^n \right) g(x) x^{-s} \frac{dx}{x} = \int_1^{\infty} e^{(s-t)\log x} g(x) x^{-s} \frac{dx}{x} \\ &= \int_1^{\infty} x^{s-t} g(x) x^{-s} \frac{dx}{x} = \int_1^{\infty} g(x) x^{-t} \frac{dx}{x}, \end{aligned}$$

where the rightmost integral converges. ■

Proof of Theorem 1.4, Case (II). Assume that $g(x) := f(x) - Ax$ has constant sign for $x \geq x_0$. Since $\int_1^{x_0} (f(x) - Ax) x^{-s-1} dx$ converges for all $s \in \mathbb{C}$ and defines an entire function, we may replace the lower integration bound 1 by x_0 in the argument that follows, so that g has constant sign on the domain of integration. It is clear that $G(s) = \int_{x_0}^{\infty} g(x) x^{-s-1} dx$ converges for all $s > 1$, and the function G by assumption continues holomorphically to $\{\operatorname{Re} s > \alpha\}$. Thus Proposition 2.4 (which of course also holds for non-positive functions) implies that the integral converges to G for $\operatorname{Re} s > \alpha$. Now the claim follows from Proposition 2.3. ■

That the statements of Proposition 2.3 and Case II of the theorem are optimal follows from the following example:

2.5 Proposition *For every $\gamma \in (0,1)$ there exists a function $f : [1, \infty) \to \mathbb{R}$ such that*

- *f is non-decreasing and $f(x) \geq x \ \forall x \geq 1$,*
- *$f(x) - x$ is $O(x^{\gamma})$ and $\Omega(x^{\gamma})$ (thus not $O(x^{\gamma'})$ for any $\gamma' < \gamma$).*
- *$F(s) = \int_1^{\infty} f(x) x^{-s-1} dx$ converges for $s > 1$,*
- *$G(s) = \int_1^{\infty} (f(x) - x) x^{-s-1} dx$ converges if and only if $s > 2\gamma - 1$,*
- *$G(s) = F(s) - \frac{1}{s-1}$ is holomorphic on $\{\operatorname{Re} s > 2\gamma - 1\}$ and has a singularity at $2\gamma - 1$.*

Proof. Let $\{x_n\}, \{h_n\}$ be sequences satisfying $x_1 \geq 1$ and $x_{n+1} \geq x_n + h_n \ \forall n$. Define $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} x_i + h_i & \text{if } x \in [x_i, x_i + h_i] \text{ for some } i \\ x & \text{otherwise} \end{cases}$$

It is obvious that f is non-negative, non-decreasing and satisfies $f(x) - x \geq 0 \ \forall x$. With $h_n = x_n^{\gamma}$ it is immediate that $f(x) - x$ is $O(x^{\gamma})$ and $\Omega(x^{\gamma})$ (since $f(x_i) = x_i + x_i^{\gamma}$ for all i and $x_i \to \infty$). In view of $0 \leq f(x) - x \leq h_i = x_i^{\gamma}$ for $x \in [x_i, x_i + h_i]$, we have

$$G(s) = \int_1^{\infty} (f(x) - x) x^{-s-1} dx \leq \sum_{i=1}^{\infty} x_i^{2\gamma_i} x_i^{-s-1}.$$

Taking $x_i = 2^i$, the r.h.s. becomes $\sum_{i=1}^{\infty} 2^{(2\gamma - s - 1)i}$, which converges whenever $2\gamma - s - 1 < 0$, or $s > 2\gamma - 1$. Thus for the integral defining G we have $\sigma_c = \sigma_a \leq 2\gamma - 1$, and G is holomorphic on $\{\operatorname{Re} s > 2\gamma - 1\}$. Proposition 2.3 gives $\sigma_c \geq 2\gamma - 1$ and Proposition 2.4 implies that G has a singularity at σ_c (which a more careful computation shows to be a pole of order one). ■

3 Another example and a conjecture

Let $\gamma \in (0,1)$. Given a sequence $\{x_n\}$ satisfying $x_n + h_n \le x_{n+1} - h_{n+1}$, where $h_n = x_n^\gamma$, put

$$f(x) = \begin{cases} x_i - h_i & \text{if } x \in [x_i - h_i, x_i] \\ x_i + h_i & \text{if } x \in (x_i, x_i + h_i] \\ x & \text{otherwise} \end{cases}$$

Again, f is non-negative, non-decreasing and both $O(x^\gamma)$ and $\Omega(x^\gamma)$. The abscissas σ_c, σ_a of convergence of $\int_1^\infty (f(x) - x)x^{-s-1}dx$ satisfy $\sigma_c \ge 2\gamma - 1$ by Proposition 2.3, while comparison of f with the function considered in the proof of Proposition 2.5 gives $\sigma_a \le 2\gamma - 1$. This implies $\sigma_c = \sigma_a = 2\gamma - 1$.

With $g(x) = f(x) - x$, we have

$$G(s) = \int_1^\infty g(x)x^{-s-1}dx = \sum_{i=1}^\infty \left(\int_{x_i - h_i}^{x_i} \frac{x_i - h_i - t}{t^{s+1}}dt + \int_{x_i}^{x_i+h_i} \frac{x_i + h_i - t}{t^{s+1}}dt \right).$$

Since g assumes positive and negative values, we must argue more carefully than above. Focusing on a summand for fixed i, we have

$$\begin{aligned}
&\int_{x_i - h_i}^{x_i} \frac{x_i - h_i - t}{t^{s+1}}dt + \int_{x_i}^{x_i+h_i} \frac{x_i + h_i - t}{t^{s+1}}dt \\
&= \frac{x_i - h_i}{s}\left(\frac{1}{(x_i-h_i)^s} - \frac{1}{x_i^s}\right) + \frac{x_i + h_i}{s}\left(\frac{1}{x_i^s} - \frac{1}{(x_i+h_i)^s}\right) - \frac{1}{1-s}\left((x_i+h_i)^{1-s} - (x_i-h_i)^{1-s}\right) \\
&= \frac{2h_i}{sx_i^s} + \frac{1}{s(1-s)}\left((x_i-h_i)^{1-s} - (x_i+h_i)^{1-s}\right) \\
&= \frac{2h_i}{sx_i^s} + \frac{x_i^{1-s}}{s(1-s)}\left((1-\frac{h_i}{x_i})^{1-s} - (1+\frac{h_i}{x_i})^{1-s}\right). \qquad (3)
\end{aligned}$$

In view of $h_i << x_i$, we expand the term in the large brackets using the binomial series:

$$\begin{aligned}
&(1-\frac{h_i}{x_i})^{1-s} - (1+\frac{h_i}{x_i})^{1-s} \\
&\quad = \left(1 - (1-s)\frac{h_i}{x_i} + \frac{(1-s)(-s)}{2}(\frac{h_i}{x_i})^2 - \frac{(1-s)(-s)(-s-1)}{3!}(\frac{h_i}{x_i})^3\right) \\
&\quad - \left(1 + (1-s)\frac{h_i}{x_i} + \frac{(1-s)(-s)}{2}(\frac{h_i}{x_i})^2 + \frac{(1-s)(-s)(-s-1)}{3!}(\frac{h_i}{x_i})^3\right) + O((\frac{h_i}{x_i})^5) \\
&\quad = -2(1-s)\frac{h_i}{x_i} - \frac{1}{3}(1-s)s(s+1)(\frac{h_i}{x_i})^3 + O((\frac{h_i}{x_i})^5).
\end{aligned}$$

Plugging this into (3), the first order terms cancel and we get

$$\int_{x_i-h_i}^{x_i+h_i} g(x)x^{-s-1}dx = -2(s+1)x^{1-s}\left(\frac{1}{3!}(\frac{h_i}{x_i})^3 + \frac{(s+2)(s+3)}{5!}(\frac{h_i}{x_i})^5 + \cdots\right).$$

With $h_i = x_i^\gamma$, where $\gamma \in (0,1)$, we have

$$\begin{aligned}
\int_1^\infty g(x)x^{-s-1}dx &= -\frac{1}{3}(s+1)\sum_{i=1}^\infty x_i^{1-s}\left((\frac{x_i^\gamma}{x_i})^3 + O((\frac{h_i}{x_i})^5\right) \\
&= -\frac{1}{3}(s+1)\sum_{i=1}^\infty x_i^{3\gamma-s-2} + O\left(\sum_i x_i^{5\gamma-s-4}\right).
\end{aligned}$$

With $x_i = 2^i$, the leading term equals $-\frac{1}{3}(s+1)\sum_{i=1}^\infty 2^{(3\gamma-s-2)i}$. From this it follows that the series converges if $\mathrm{Re}(3\gamma - s - 2) < 0$, or $\mathrm{Re}\, s > 3\gamma - 2$, while the sum over the higher order terms converges for $\mathrm{Re}\, s > 5\gamma - 4$. In view of $\sum_{i=1}^\infty 2^{(3\gamma-s-2)i} = \frac{1}{1-2^{3\gamma-s-2}} - 1$, G is meromorphic on $\{\mathrm{Re}\, s > 5\gamma - 4\}$ with first order poles at $3\gamma - 2 + i\frac{2\pi}{\log 2}\mathbb{Z}$.

For $\mathrm{Re}\, s > 2\gamma - 1 = \sigma_a$, it is clear that the above series converges to $\int_1^\infty (f(x) - x)x^{-s-1}dx$, so that the series gives an analytic continuation of the integral to $\{\mathrm{Re}\, s > 3\gamma - 2\}$. We thus have:

3.1 Proposition *For every $\gamma \in (0,1)$ there exists a function $f : [1,\infty) \to \mathbb{R}$ such that*

- *f is non-negative and non-decreasing,*
- *$f(x) - x$ is $O(x^\gamma)$ and $\Omega(x^\gamma)$,*
- *$F(s) = \int_1^\infty f(x)x^{-s-1}dx$ converges for $s > 1$,*
- *$G(s) = \int_1^\infty (f(x) - x)x^{-s-1}dx$ converges if and only if $s > 2\gamma - 1$,*
- *$G(s) = F(s) - \frac{1}{s-1}$ analytically continues to $\{\mathrm{Re}\, s > 3\gamma - 2\}$ and has a singularity at $3\gamma - 2$.*

3.2 Remark 1. In view of $3\gamma - 2 < 2\gamma - 1$, the maximal half-plane of holomorphicity of G is larger than the half-plane of convergence of the integral defining it, reflecting the fact that the Phragmén-Landau theorem does not apply to the function g, which is not ultimately of one sign.

2. This shows that holomorphicity of G on $\{\mathrm{Re}\, s > \alpha\}$ is compatible with $f(x) - Ax = O(x^\gamma)$ being true only for $\gamma \geq \frac{\alpha+2}{3} > \frac{\alpha+1}{2}$, showing that Theorem 1.4 is false for $A > 0$ if we drop the sign condition on $f(x) - Ax$.

3. The above function f was designed in such a way as to maximize cancellations in the integral defining $G(s)$, while being non-negative, non-decreasing and $x + \Omega(x^\gamma)$. It is hard to see how one could construct a function with the properties in the above proposition, but with G extending holomorphically to a larger half-plane. (One could replace the numbers $x_i + h_i$ and $x_i - h_i$ by $x_i + h_i'$ and $x_i - h_i''$, respectively, allowing $h_i' \neq h_i''$ in the hope of achieving higher order cancellations in the above computation. But the converse happens: While the cancellation of first order terms still goes through, it breaks down in second order.)

4. While the above considerations provide only some evidence for the conjecture that follows, they prove that we cannot hope for more. □

3.3 Conjecture *Let $f : [1,\infty) \to \mathbb{R}$ be non-negative and non-decreasing. Assume that $F(s) = \int_1^\infty f(x)x^{-s-1}dx$ converges for $s > 1$ and that $F(s) - \frac{A}{s-1}$ has a holomorphic continuation to $\{\mathrm{Re}\, s > \alpha\}$, where $\alpha \in (0,1)$. Then $f(x) = Ax + O(x^{\frac{\alpha+2}{3}+\varepsilon})$.*

Equivalently: If $g : [1,\infty) \to \mathbb{R}$ is such that $x \mapsto g(x) + Ax$ is non-decreasing for some A, $G(s) = \int_1^\infty g(x)x^{-s-1}dx$ converges for $s > 1$ and $G(s)$ extends holomorphically to $\{\mathrm{Re}\, s > \alpha\}$, where $\alpha \in (0,1)$, then $g(x) = O(x^{\frac{\alpha+2}{3}+\varepsilon})$.

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