The multiplier algebra $M(A)$ as an idealizer

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Abstract

The construction of the multiplier algebra $M(A)$ is somewhat abstract, and even if A is a concrete C^*-algebra, $M(A)$ is not. For this reason the following alternative construction of the multiplier algebra is interesting:

Let $A \subseteq B(H)$ be a C^*-algebra acting non-degenerately on H, i.e. $\overline{AH} = H$. Then the *idealizer* $\widetilde{M}(A) = \{b \in B(H) \mid bA \cup Ab \subseteq A\}$ of A is a C^*-algebra and isometrically $*$-isomorphic to $M(A)$.

1 Extensions of representations of ideals

1.1 Proposition *Let B be a C^*-algebra and $I \subseteq B$ an ideal (closed 2-sided). Then*

(i) Any representation (H, π) of I extends to a representation $(H, \widehat{\pi})$ of B.

(ii) If (H, π) is non-degenerate then $\widehat{\pi}$ is unique (and non-degenerate).

(iii) If (H, π) is faithful and the ideal $I \subseteq B$ is essential then any extension $\widehat{\pi}$ is faithful.

Proof. (i) Put $K = \overline{\pi(I)H}$. If $K \subsetneq H$ then π acts as the zero representation on $K^\perp$. Thus if $p : B \to B/I$ be the quotient homomorphism then any $*$-homomorphism $\varphi : B/I \to B(K^\perp)$ gives rise to a representation $\varphi \circ p$ of B on $K^\perp$ that extends the zero representation of I. (This extension will rarely be unique.) Thus from now on we may restrict to non-degenerate representations ($K = H$), for which we will have unique extensions.

Let $\{u_\lambda\}$ be an approximate unit for I. Then for every $y = \sum_{k=1}^K \pi(a_k)x_k \in \pi(I)H$ and $b \in B$ we have $u_\lambda a_k \to a_k$ and $bu_\lambda \in I$, thus

$$\begin{aligned} \left\| \sum_{k=1}^K \pi(ba_k)x_k \right\| &= \left\| \lim_\lambda \sum_{k=1}^K \pi(bu_\lambda a_k)x_k \right\| = \left\| \lim_\lambda \sum_{k=1}^K \pi(bu_\lambda)\pi(a_k)x_k \right\| \\ &\leq \sup_\lambda \|\pi(bu_\lambda)\| \left\| \sum_{k=1}^K \pi(a_k)x_k \right\| \leq \|b\| \left\| \sum_{k=1}^K \pi(a_k)x_k \right\|, \end{aligned}$$

where we used $\|\pi(bu_\lambda)\| \leq \|bu_\lambda\| \leq \|b\|$ due to $\|\pi\| \leq 1$ (π is a $*$-homomorphism of C^*-algebras) and $\|u_\lambda\| \leq 1$.

In particular $\sum_k \pi(a_k)x_k = 0$ implies $\sum_k \pi(ba_k)x_k = 0$ for all $b \in B$. Thus if $\sum_k \pi(a_k)x_k = \sum_l \pi(a'_l)x'_l$ then $\sum_k \pi(ba_k)x_k = \sum_l \pi(ba'_l)x'_l$, so that

$$\widehat{\pi}(b) : H_0 \to H_0, \quad \sum_k \pi(a_k)x_k \mapsto \sum_k \pi(ba_k)x_k$$

is a well-defined linear operator of norm $\leq \|b\|$ on the dense subspace $H_0 = \pi(I)H$.

Due to its boundedness, $\widehat{\pi}(b)$ uniquely extends to an element of $B(H)$ that we also denote $\widehat{\pi}(b)$. It is straightforward to check that $\widehat{\pi} : B \to B(H)$ is a $*$-homomorphism.

(ii) To prove uniqueness, let $\widetilde{\pi}$ be any extension of π to B, and let $y = \sum_k \pi(a_k)x_k \in H_0$. Then

$$\widetilde{\pi}(b)y = \sum_k \widetilde{\pi}(b)\widetilde{\pi}(a_k)x_k = \sum_k \widetilde{\pi}(ba_k)x_k = \sum_k \pi(ba_k)x_k = \widehat{\pi}(b)y,$$

so that $\widetilde{\pi}(b) = \widehat{\pi}(b)$ on the dense (by non-degeneracy) subspace H_0, thus on H by continuity.

(iii) (By (ii), $\widehat{\pi}$ is the extension constructed in (i), but this is not needed.) If $\widehat{\pi}(b) = 0$ then $\pi(ba) = \widehat{\pi}(ba) = \widehat{\pi}(b)\pi(a) = 0$ for all $a \in I$. Since π is faithful this implies $ba = 0$ for all $a \in I$. Then $b = 0$ by essentiality of I. ■

1.2 Remark More generally, if (H, π) is a non-degenerate (or cyclic) representation of A, where A just is a C^*-subalgebra of B, then there is a non-degenerate (resp. cyclic) representation $(K, \widehat{\pi})$ of B such that $H \subseteq K$ and $\widehat{\pi}(A)H \subseteq H$ and $\widehat{\pi}(a) \restriction H = \pi(a)$ for all $a \in A$. (Cf. e.g. Murphy's Theorem 5.5.1.) In general, $K \supsetneq H$, but if $A \subseteq B$ is an ideal, $K = H$ and the construction reduces to the one given above. □

2 The multiplier algebra as an idealizer

2.1 Theorem *Let $A \subseteq B(H)$ be a C^*-subalgebra. Put*

$$\widetilde{M}(A) = \{b \in B(H) \mid bA \cup Ab \subseteq A\} \subseteq B(H).$$

Then

- *(i) $\widetilde{M}(A)$ is a C^*-subalgebra of $B(H)$ containing A and 1, actually the largest subalgebra of $B(H)$ containing A as an ideal, whence the idealizer of A.*
- *(ii) The following are equivalent:*
  - *(α) A acts non-degenerately on H (i.e. $\overline{AH} = H$).*
  - *(β) $A \subseteq \widetilde{M}(A)$ is an essential ideal.*
  - *(γ) There is an isometric $*$-isomorphism $\widehat{\iota} : \widetilde{M}(A) \to M(A)$ sending A to A. (More precisely, we have $\widehat{\iota} \circ \iota_1 = \iota$, where $\iota : A \to M(A), \iota_1 : A \to \widetilde{M}(A)$ are the canonical inclusions.)*

Proof. (i) This is very straightforward. If $A \subseteq B \subseteq B(H)$ and A is an ideal in B then clearly $B \subseteq \widetilde{M}(A)$.

(ii) $\alpha \Rightarrow \beta$ Assume A acts non-degenerately and $b \in \widetilde{M}(A)$ such that $bA = 0$. Then $bAH = 0$, thus $b\overline{AH} = 0$. Since $\overline{AH} = H$, this implies $b = 0$. If $Ab = 0$ then $b^*A = 0$, and the preceding argument gives $b^* = 0$, thus $b = 0$. Thus the ideal $A \subseteq \widetilde{M}(A)$ is essential.

$\beta \Rightarrow \alpha$ If A acts in a degenerate way then $K = \overline{AH}^{\perp} \neq \{0\}$. If now $0 \neq b \in B(H)$ such that $bK \subseteq K$ and $b \restriction K^{\perp} = 0$ then $ba = ab = 0 \in A$, thus $b \in \widetilde{M}(A)$, and $A \subseteq \widetilde{M}(A)$ is not essential.

$\gamma \Rightarrow \beta$ Follows from the fact that $A \subseteq M(A)$ is essential.

$\beta \Rightarrow \gamma$ Consider the following diagram:

$$\begin{tikzcd}
& M(A) \arrow[dr, hook, dotted, "\widehat{\pi}"] & \\
A \arrow[ur, hook, "\iota"] \arrow[r, hook, "\iota_1"'] & \widetilde{M}(A) \arrow[u, hook, "\widehat{\iota}"] \arrow[r, hook, "\iota_2"'] & B(H)
\end{tikzcd}$$

Here the horizontal maps are inclusion maps for subsets of $B(H)$ and π, not drawn, is the composite map $\pi = \iota_2 \circ \iota_1 : A \to B(H)$. Furthermore, $\iota : A \to M(A), a \mapsto (L_a, R_a)$ is the canonical inclusion. Since $\iota_1(A) \subseteq \widetilde{M}(A)$ is an ideal by assumption (β), we have a homomorphism $\widehat{\iota} : \widetilde{M}(A) \to M(A), b \mapsto (L_b, R_b)$, satisfying $\widehat{\iota} \circ \iota_1 = \iota$.

Since A is an ideal in $M(A)$, by Proposition 1.1 there is a representation $\widehat{\pi} : M(A) \to B(H)$ extending π in the sense $\widehat{\pi} \circ \iota = \pi$. Clearly $\widehat{\pi}(M(A))$ is a subalgebra of $B(H)$. Let $a \in A$ and $b \in M(A)$. Then $\iota(a)b \in \iota(A)$, thus

$$\pi(a)\widehat{\pi}(b) = \widehat{\pi}(\iota(a))\widehat{\pi}(b) = \widehat{\pi}(\iota(a)b) \in \widehat{\pi}(\iota(A)) = \pi(A).$$

Similarly, $\widehat{\pi}(b)\pi(a) \in \pi(A)$. Thus $\widehat{\pi}(M(A)) \subseteq B(H)$ contains $\pi(A) = A$ as an ideal. Thus by (i) we have $\widehat{\pi}(M(A)) \subseteq \widetilde{M}(A)$, to wit $\widehat{\pi}$ maps $M(A)$ into $\widetilde{M}(A)$. In view of this, $\widehat{\pi} \circ \iota = \pi$ becomes $\widehat{\pi} \circ \iota = \iota_1$. Together with $\widehat{\iota} \circ \iota_1 = \iota$, this implies $\widehat{\iota} \circ \widehat{\pi} \circ \iota = \iota$ and $\widehat{\pi} \circ \widehat{\iota} \circ \iota_1 = \iota_1$. Thus $\widehat{\iota} \circ \widehat{\pi}$ and $\widehat{\pi} \circ \widehat{\iota}$ are algebra endomorphisms of $M(A)$ and $\widetilde{M}(A)$, respectively, that leave the essential ideal A of either algebra pointwise fixed. Now the lemma below gives $\widehat{\iota} \circ \widehat{\pi} = \mathrm{id}_{M(A)}$ and $\widehat{\pi} \circ \widehat{\iota} = \mathrm{id}_{\widetilde{M}(A)}$, proving $M(A) \cong \widetilde{M}(A)$. As we know, $*$-isomorphisms are isometries. ■

2.2 Lemma *If B is C^*-algebra, $I \subseteq B$ an essential ideal and $\alpha : B \to B$ an algebra homomorphism satisfying $\alpha_{|I} = \mathrm{id}_I$ then $\alpha = \mathrm{id}_B$.*

Proof. Let $b \in B, a \in I$. Then $\alpha(b)a = \alpha(ba) = ba$ since $a, ba \in I$ and $\alpha_{|I} = \mathrm{id}_I$. Thus $(\alpha(b) - b)I = 0$, so that essentiality of I implies $\alpha(b) = b$. ■

2.3 Corollary *If $A \subseteq B(H)$ is an ideal acting non-degenerately on H then $M(A) \cong B(H)$. In particular $M(K(H)) = B(H)$.*

Proof. $A \subseteq B(H)$ being an ideal implies $\widetilde{M}(A) = B(H)$. Now Theorem 2.1 gives $M(A) \cong B(H)$. The second statement follows since $K(H) \subseteq B(H)$, which is the closure of the algebra of finite rank operators, acts non-degenerately and is an ideal, for each Hilbert space H. ■

2.4 Remark Theorem 2.1 concerns C^*-algebras acting non-degenerately on a Hilbert space. But if A is an abstract C^*-algebra, we know that A admits a faithful non-degenerate representation, e.g. the universal representation (H_u, φ_u). Now $A \cong \varphi_u(A) \subseteq B(H_u)$. It is evident that isomorphic C^*-algebras have isomorphic multiplier algebras. Thus we have an alternative construction

$$M(A) \cong M(\varphi_u(A)) \cong \widetilde{M}(\varphi_u(A)) \subseteq B(H_u)$$

of 'the' multiplier algebra $M(A)$. □