# (Almost) Real Proof of the Prime Number Theorem

Michael Müger
Institute for Mathematics, Astrophysics and Particle Physics
Radboud University Nijmegen, The Netherlands

April 2, 2017

**Abstract**

We explain a fairly simple proof of the Prime Number Theorem that uses only basic real analysis and the elementary *arithmetic* of complex numbers. This includes the $\zeta$-function (as real-differentiable function) and the Fourier transform on $\mathbb{R}$, but neither Fourier inversion nor anything from complex *analysis*.

## 1 Introduction

The first proof of the Prime Number Theorem (PNT), found in 1896 by Hadamard and de la Vallée-Poussin (independently), made massive use of complex analysis, well beyond relying on the Riemann function $\zeta(z)$ for complex $z$. Its crucial ingredient is the non-vanishing of $\zeta$ on the line $\mathrm{Re}(z) = 1$, but it requires additional estimates on $\zeta$ to ensure the existence of certain integrals. Landau gave a more conceptual proof of the PNT in terms of a Tauberian theorem, cf. [11, §241], which however still needed a growth condition. In the 1930s, Ikehara [5] used Wiener's general Tauberian theory [17, 18] to eliminate the growth condition from Landau's Tauberian theorem, thereby deducing the PNT from $\zeta(1 + it) \neq 0$ alone. Bochner [1] finally gave a simple proof of the Landau-Ikehara theorem without appealing to Wiener's Tauberian theory.

The point of this note is just to make plain that this approach to the PNT can be formulated in a way that, while keeping the complex numbers for notational convenience and using $\zeta$, avoids all notions and results from complex analysis, like holomorphicity, analytic continuation, Cauchy's theorem, etc. The resulting proof probably is the simplest imaginable. (There is, of course, nothing new about this. Cf. e.g. [9].)

## 2 The $\zeta$-function: $\mathrm{Re}\, z > 1$

For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$ we define

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{2.1}$$

In view of $|n^z| = n^{\mathrm{Re}(z)}$ and the convergence of $\sum_{n=1}^{\infty} 1/n^\alpha$ for $\alpha > 1$, it is clear that the above sum converges absolutely for $\mathrm{Re}(z) > 1$ and uniformly for $\mathrm{Re}(z) \geq \alpha > 1$. It therefore defines a 'nice' function on the open half plane $\mathrm{Re}(z) > 1$. For our purposes it will be sufficient to read 'nice' as '$C^1$'.

2.1 Remark As explained in the Introduction, the proof given below makes no use of complex analysis but does involve complex numbers. Writing $z = s + it$, we have

$$\zeta(z) = \sum_{n=1}^{\infty} e^{-z \log n} = \sum_{n=1}^{\infty} \frac{e^{-it \log n}}{n^s} = \sum_{n=1}^{\infty} \frac{\cos(t \log n)}{n^s} - i \sum_{n=1}^{\infty} \frac{\sin(t \log n)}{n^s}.$$

Since the dependence of $\zeta(s+it)$ on the real variables $s$ and $t$ is quite different, and since we do not need the holomorphicity of $\zeta$, we could interpret $\zeta$ as a function from an open half-plane in $\mathbb{R}^2$ to $\mathbb{R}^2$. But since a complex number just is a pair of real numbers, this seems pointless, in particular since multiplying complex numbers, which is inconvenient in terms of pairs of reals, is essential for the proof, cf. e.g. Lemma 2.2. Furthermore, the Fourier transform, which we use in Section 4, is much more natural for complex-valued than for real-valued functions since $\mathbb{R} \to \mathbb{C}^*, t \mapsto e^{it}$ is a homomorphism, which is not true for $\sin t, \cos t$. It really does not seem that there is a reasonable way of writing proofs of the PNT involving the $\zeta$-function in purely real terms. (As opposed to the 'elementary' proofs, cf. e.g. [13].) □

2.2 Lemma *Denoting by $\mathbb{P}$ the set of primes, for* $\mathrm{Re}(z) > 1$ *we have*

$$\zeta(z) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}. \tag{2.2}$$

*Proof.* By the definition of infinite products, the r.h.s. of the above equation is the limit $x \to \infty$ of

$$\prod_{\substack{p \in \mathbb{P} \\ p \le x}} \frac{1}{1 - p^{-z}} = \prod_{\substack{p \in \mathbb{P} \\ p \le x}} \left( \sum_{k=0}^{\infty} \frac{1}{p^z} \right) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_l=0}^{\infty} \frac{1}{(p_1^{k_1} \cdots p_l^{k_l})^z} = \sum_{\substack{n=1 \\ P^+(n) \le x}}^{\infty} \frac{1}{n^z},$$

where $\{p_1, \cdots, p_l\} = \mathbb{P} \cap [1, x]$ and $P^+(n)$ is the largest prime factor of $n$. The last identity is due to the 'fundamental theorem of arithmetic', i.e. the existence and uniqueness of the decomposition of $n$ into primes. Now, as $x \to \infty$ the summation in the last term runs over all of $\mathbb{N}$, and the expression tends to (the absolutely convergent) expression $\sum_n n^{-z} = \zeta(z)$. ■

We now want a function $H$ such that $\zeta(z) = e^{H(z)}$ for $\mathrm{Re}(z) > 1$. Here some care is required since the exponential function is not injective ('the logarithm is multi-valued').

2.3 Lemma *(i) The series $F(z) = -\sum_{k=1}^{\infty} \frac{(1-z)^k}{k}$ converges whenever $|z - 1| < 1$.*

*(ii) If $z \in (0, 2)$ then $F(z) = \log z$, the real logarithm of $z$.*

*(iii) For every $z \in \mathbb{C}$ such that $|z - 1| < 1$, we have $e^{F(z)} = z$.*

*Proof.* (i) is obvious, and (ii) is well-known. Now (ii) implies $e^{F(z)} = z$ for $z \in (0, 2)$. The exponential function and $F$ being given by power series, the power series identity $e^{F(z)} = z$ (which essentially is an algebraic fact) continues to hold for all complex $z$ satisfying $|z - 1| < 1$. ■

2.4 Proposition *Write*

$$H(z) = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{1}{k p^{kz}}. \tag{2.3}$$

*(i) The sum in (2.3) converges absolutely if* $\mathrm{Re}(z) > 1$ *and defines a nice function on this open half-plane.*

*(ii) For real $z > 1$, $H(z)$ is the (real) logarithm of $\zeta(z)$.*

*(iii) For all $z$ with* $\mathrm{Re}(z) > 1$ *we have* $\zeta(z) = \exp(H(z))$ *and thus* $\zeta(z) \neq 0$.

*Proof.* (i) It is obvious from (2.1) that $\zeta(s) > 0$ for real $s > 1$. Since also $(1 - p^{-s})^{-1} > 0$, equation (2.2), together with continuity of the logarithm (for positive argument) gives

$$\log \zeta(s) = \sum_{p \in \mathbb{P}} \log \left( \frac{1}{1 - p^{-s}} \right) = -\sum_{p \in \mathbb{P}} \log(1 - p^{-s}) = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{1}{k p^{ks}}, \tag{2.4}$$

where the last identity follows the fact that $p^{-s} < 1$ and parts (i) and (ii) of Lemma 2.3.

For $s = \mathrm{Re}(z) > 1$ we have

$$\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\left|\frac{1}{kp^{kz}}\right| = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\frac{1}{kp^{ks}} = \log\zeta(s) < \infty,$$

using (2.4). Thus the double sum in (2.3) converges absolutely and defines a 'nice' function on $\{z\,|\,\mathrm{Re}(z) > 1\}$.

(ii) This is just (2.4).

(iii) With $|p^{-z}| = p^{-s} < 1$, part (iii) of Lemma 2.3 gives $\exp[\sum_k (kp^{kz})^{-1}] = (1-p^{-z})^{-1}$ whenever $\mathrm{Re}(z) > 1$. Now $e^{H(z)} = \lim_{x\to\infty}\exp(\sum_{p\le x}\sum_{k=1}^{\infty}\frac{1}{kp^{kz}}) = \lim_{x\to\infty}\prod_{p\le x}(1-p^{-z})^{-1} = \zeta(z)$ by continuity of exp. This clearly implies $\zeta(z)\neq 0$ for $\mathrm{Re}(z) > 1$. ■

2.5 Lemma *Defining the von Mangoldt function by*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, p\in\mathbb{P}, k\in\mathbb{N} \\ 0 & \text{else} \end{cases},$$

*and* $\psi(x) = \sum_{n\le x}\Lambda(n)$, *we have for* $\mathrm{Re}(z) > 1$*:*

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z} = z\int_0^{\infty} e^{-zx}\psi(e^x)dx \tag{2.5}$$

*Proof.* Using Proposition 2.4, we have

$$-\frac{\zeta'(z)}{\zeta(z)} = -H'(z) = -\left(\sum_{p\in\mathbb{P}}\sum_{k\ge 1}\frac{1}{kp^{kz}}\right)' = \sum_{p\in\mathbb{P}}\sum_{k\ge 1}\frac{\log p}{p^{kz}} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z},$$

where the differentiation under the sum is justified by the uniform convergence of the sum of the derivatives. This is the first half of (2.5), proving also the absolute convergence of the middle sum. Convergence of the integral follows from $\psi(x) = O(x\log x) = O(x^{1+\varepsilon})$. Now

$$\begin{aligned} z\int_0^{\infty} e^{-zx}\psi(e^x)dx &= z\int_0^{\infty} e^{-zx}\left(\sum_{n\le e^x}\Lambda(n)\right)dx = z\sum_{n=1}^{\infty}\Lambda(n)\int_{\log n}^{\infty} e^{-zx}dx \\ &= z\sum_{n=1}^{\infty}\Lambda(n)\left[\frac{-1}{z}e^{-zx}\right]_{\log n}^{\infty} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z} \end{aligned}$$

proves the second half of (2.5). ■

# 3 The $\zeta$-function: $\mathrm{Re}\, z \ge 1$

3.1 Lemma *There is a* $C^1$*-function* $g : \{z\,|\,\mathrm{Re}(z) > 0\}\to\mathbb{C}$ *such that* $\zeta(z) = g(z) + \frac{1}{z-1}$ *when* $s = \mathrm{Re}(z) > 1$*. Thus* $\zeta$ *has a* $C^1$*-extension, which we also denote* $\zeta$*, to* $\{z\,|\,\mathrm{Re}(z) > 0\}\backslash\{1\}$*.*

*Proof.* For $\mathrm{Re}(z) > 1$ we have $\int_1^{\infty} t^{-z}dt = \frac{1}{z-1}$. Thus

$$g(z) = \zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty}\left(\frac{1}{n^z} - \int_n^{n+1}\frac{1}{t^z}dt\right) = \sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt. \tag{3.1}$$

Now,

$$\left|\int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt\right| = \left|z\int_n^{n+1}\int_n^t\frac{du}{u^{z+1}}dt\right| \le \sup_{u\in[n,n+1]}\left|\frac{z}{u^{z+1}}\right| = \frac{|z|}{n^{s+1}},$$

so that (3.1) converges, and thus defines $g(z)$ as a $C^1$-function on $\mathrm{Re}(z) > 0$. ■

3.2 Remark The above argument of course also shows that $\zeta(z)$ extends to a meromorphic function on $\mathrm{Re}(z) > 0$ with a simple pole of residue one at $z = 1$. This extension clearly is unique. We will have no use for this, since we will only need that $\zeta(z) - 1/(z-1)$ has a $C^1$-extension. Extensions of differentiable functions are, of course, not unique since there exist compactly supported smooth functions. This is not a problem since we only need a continuous extension of $\zeta(z) - 1/(z-1)$ to the closed half plane $\mathrm{Re}(z) \geq 1$, telling us that $\lim_{s\searrow 1} \zeta(s+it) - \frac{1}{s+it-1}$ exists for all $t$. This extension (whose existence follows from the lemma) still is unique for simple reasons of point set topology (the Hausdorff property of $\mathbb{R}^2$). □

3.3 Proposition *We have* $\zeta(z) \neq 0$ *whenever* $\mathrm{Re}(z) \geq 1$ *and* $z \neq 1$.

*Proof.* We already know that $\zeta(z) \neq 0$ when $\mathrm{Re}(z) > 1$. It remains to prove $\zeta(1+it) \neq 0$ for $t \neq 0$. With Proposition 2.4 and $z = s + it$ we obtain

$$\log|\zeta(z)| = \log|e^{H(z)}| = \mathrm{Re}(H(z)) = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty} \mathrm{Re}\left(\frac{1}{kp^{kz}}\right) = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty} \frac{\mathrm{Re}\left(e^{-ikt\log p}\right)}{kp^{ks}}. \tag{3.2}$$

Using

$$\mathrm{Re}(3 + 4e^{i\theta} + e^{2i\theta}) = 3 + 2e^{i\theta} + 2e^{-i\theta} + \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{(2 + e^{i\theta} + e^{-i\theta})^2}{2} \geq 0$$

with $\theta = -kt\log p$ and (3.2) we have

$$\log\left|\zeta^3(s)\zeta^4(s+it)\zeta(s+2it)\right| = \sum_p \sum_{k=1}^{\infty} \frac{\mathrm{Re}(3 + 4e^{-ikt\log p} + e^{-2ikt\log p})}{kp^{ks}} \geq 0,$$

thus $\left|\zeta^3(s)\zeta^4(s+it)\zeta(s+2it)\right| \geq 1$. This can be restated as

$$|(s-1)\zeta(s)|^3 \left|\frac{\zeta(s+it)}{s-1}\right|^4 |\zeta(s+2it)| \geq \frac{1}{s-1}. \tag{3.3}$$

If $t \neq 0$, the function $\zeta(z) = \zeta(s+it)$ as extended in Lemma 3.1 clearly has the partial derivative $\partial_s\zeta(s+it) \equiv \zeta'$ at $z = 1+it$. Assuming that $\zeta(1+it) = 0$, it follows that the limit

$$\lim_{s\searrow 1} \frac{\zeta(s+it)}{s-1} = \lim_{s\searrow 1} \frac{\zeta(s+it) - \zeta(1+it)}{s-1} = \zeta'(1+it)$$

exists. Together with $\lim_{s\searrow 1}(s-1)\zeta(s) = 1$, this implies that the l.h.s. of (3.3) tends to the finite limit $|\zeta'(1+it)|^4|\zeta(1+2it)|$ as $s \searrow 1$, whereas the r.h.s. of (3.3) diverges. This contradiction shows that $\zeta(1+it) \neq 0$ for all $t \neq 0$. ■

From now on, $\zeta'(z)$, where $z = s+it$, will always denote the partial derivative $\partial_s\zeta(s+it)$, as in the above proof. (Since all functions considered are holomorphic, $\zeta'(z)$ of course coincides with the complex derivative, but this will not be needed.)

3.4 Corollary *The function* $z \mapsto -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$*, well-defined on* $\mathrm{Re}(z) > 0$ *by Proposition 2.4(iii), has a continuous extension to* $\mathrm{Re}(z) \geq 1$.

*Proof.* By Lemma 3.1, there is a $C^1$-function $g$ on $\mathrm{Re}(z) > 0$ such that $\zeta(z) = g(z) + \frac{1}{z-1}$ on $\mathrm{Re}(z) > 1$. Since $\zeta, \zeta'$ are nice functions and $\zeta(z) \neq 0$ on $\{\mathrm{Re}(z) \geq 1\}\backslash\{1\}$ by Proposition 3.3, the claim is clear away from $z = 1$. To see that this also holds near $z = 1$, we compute

$$\begin{aligned} -\frac{\zeta'(z)}{\zeta(z)} &= -\frac{g'(z) - \frac{1}{(z-1)^2}}{g(z) + \frac{1}{z-1}} = \frac{-(z-1)g'(z) + \frac{1}{z-1}}{(z-1)g(z) + 1} \\ &= \frac{-(z-1)g'(z)}{(z-1)g(z)+1} + \frac{1}{z-1}\left(1 - \frac{(z-1)g(z)}{(z-1)g(z)+1}\right). \end{aligned}$$

Thus

$$-\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{z-1} + k(z), \quad \text{where} \quad k = \frac{-(z-1)g'(z) - g(z)}{(z-1)g(z)+1}.$$

The claim follows, since $k$ is continuous on $\mathrm{Re}(z) \geq 1$. ■

# 4 Ikehara's Tauberian theorem

The aim of this section is to prove the following theorem (following [3] quite closely):

4.1 Theorem *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing. If there are $a, c \geq 0$ such that the integral*

$$F(z) = \int_0^\infty f(x)e^{-zx}dx,$$

*converges for $\mathrm{Re}(z) > a$ [thus automatically defines a continuous function there] and $F(z) - \frac{c}{z-a}$ has a continuous extension to the half-plane $\mathrm{Re}(z) \geq a$ then*

$$f(x) \sim ce^{ax} \quad \text{as} \quad x \to +\infty.$$

The proof will be given using the Fourier transform on $\mathbb{R}$, concerning which we only need the definition and an easy special case of the Riemann-Lebesgue Lemma:

4.2 Lemma *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with compact support. Then the Fourier transform*

$$\widehat{f}(\xi) := \int f(x)e^{-i\xi x}dx \tag{4.1}$$

*(integrals without bounds always extend over $\mathbb{R}$) satisfies $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

*Proof.* For $\xi \neq 0$, we have

$$\widehat{f}(\xi) = -\int f(x)e^{-i\xi(x-\frac{\pi}{\xi})}dx = -\int f(x+\frac{\pi}{\xi})e^{-i\xi x}dx,$$

which leads to

$$\widehat{f}(\xi) = \frac{1}{2}\int \left(f(x) - f(x+\frac{\pi}{\xi})\right) e^{-i\xi x}dx. \tag{4.2}$$

Since $f$ is continuous with compact support, it is uniformly continuous, thus $\sup_x |f(x)-f(x+\varepsilon)| \to 0$ as $\varepsilon \to 0$. Inserting this in (4.2) and keeping in mind that $\mathrm{supp}(f)$ is bounded, the claim follows. ■

*Proof of Theorem 4.1.* As a preparatory step, pick any continuous, compactly supported even function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 1$ and $\widehat{g}(\xi) \geq 0$ $\forall \xi$, where $\widehat{g}$ is as in (4.1). (A possible choice is the tent function $g(x) = \max(1-|x|, 0)$. By an easy computation, $\widehat{g}(\xi) = 2(1-\cos\xi)/\xi^2 \geq 0$, so that all assumptions are satisfied.) Then the Fourier inversion theorem applies, thus $g(x) = (2\pi)^{-1}\int \widehat{g}(\xi)e^{i\xi x}d\xi$, and in particular we have $\int \widehat{g}(\xi)d\xi = 2\pi g(0) = 2\pi$. (This is our only use of Fourier inversion, and it can be avoided at the expense of showing $\int \widehat{g}(\xi)d\xi = 2\pi$ 'by hand', which can be done for the above choice of $g$, if somewhat arduously.[1])

Defining $g_n(x) = g(x/n)$, it is immediate that $\widehat{g_n}(\xi) = n\widehat{g}(n\xi)$ and $\int \widehat{g_n} = 2\pi$ for all $n$. For every $\delta > 0$ we have $\int_{|\xi|\geq\delta} \widehat{g_n}(\xi) = \int_{|u|\geq n\delta} \widehat{g}(u)du$, which tends to zero as $n \to 0$ since $\widehat{g_n}$ is integrable and non-negative. Thus the normalized functions $h_n = \widehat{g_n}/2\pi$ constitute an approximation of unity. (I.e. $\int fh_n \stackrel{n\to\infty}{\longrightarrow} f(0)$ for sufficiently nice $f$.)

[1] Let $a, b > 0$. Twofold partial integration gives $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$. Applying $\int_0^c \cdots db$ gives $\int_0^\infty e^{-ax}\frac{\sin cx}{x}\,dx = \arctan\frac{c}{a}$. Applying $\int_0^f \cdots dc$ gives $\int_0^\infty e^{-ax}\frac{1-\cos fx}{x^2}\,dx = f\arctan\frac{f}{a} - \frac{a}{2}\log\left(1+(\frac{f}{a})^2\right)$. (The exchanges of integration order are justified by absolute convergence and Fubini.) Taking $f = 1$ and the limit $a \searrow 0$ we obtain $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$.

Let $G$ be the continuous extension (clearly unique) of $F$ to $\mathrm{Re}(z) \geq a$. Defining $\phi(x) = e^{-ax} f(x)$, we have for all $t \in \mathbb{R}$, $\varepsilon > 0$:

$$G(a+\varepsilon+it) = F(a+\varepsilon+it) - \frac{c}{\varepsilon+it} = \int_0^\infty (\phi(x)-c)e^{-(\varepsilon+it)x}dx = \int_0^\infty \psi_\varepsilon(x)e^{-itx}dx, \quad (4.3)$$

where we abbreviate $\psi_\varepsilon(x) = e^{-\varepsilon x}(\phi(x)-c)$ to make the following computation readable. Multiplying (4.3) by $e^{iyt}g_n(t)$ and integrating over $t$, we have

$$\begin{aligned} \int e^{iyt} g_n(t) G(a+\varepsilon+it)dt &= \int e^{iyt} g_n(t) \left( \int_0^\infty \psi_\varepsilon(x) e^{-itx} dx \right) dt \\ &= \int_0^\infty \psi_\varepsilon(x) \left( \int g_n(t) e^{it(y-x)} dt \right) dx \\ &= \int_0^\infty \psi_\varepsilon(x) \widehat{g_n}(y-x) dx \\ &= \int_0^\infty e^{-\varepsilon x} \phi(x) \widehat{g_n}(y-x) dx - c \int_0^\infty e^{-\varepsilon x} \widehat{g_n}(y-x) dx. \quad (4.4) \end{aligned}$$

(The second equality holds by Fubini's theorem, since $\psi_\varepsilon$ and $g_n$ are integrable. The third identity uses that $g_n$ is even.) We now consider the limit $\varepsilon \searrow 0$ of (4.4). Since $G$ by assumption is continuous on $\mathrm{Re}(z) \geq a$, we have $G(a+\varepsilon+it) \to G(a+it)$, uniformly for $t$ in the compact support of $g_n$. Thus

$$\lim_{\varepsilon \searrow 0} \int e^{iyt} g_n(t) G(a+\varepsilon+it)dt = \int e^{iyt} g_n(t) G(a+it)dt.$$

Since $\phi$ and $\widehat{g_n}$ are non-negative, the monotone convergence theorem gives

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon x} \phi(x) \widehat{g_n}(y-x) dx &= \int_0^\infty \phi(x) \widehat{g_n}(y-x) dx, \\ \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon x} \widehat{g_n}(y-x) dx &= \int_0^\infty \widehat{g_n}(y-x) dx = \int_{-\infty}^y \widehat{g_n}(x) dx. \end{aligned}$$

Thus the $\varepsilon \searrow 0$ limit of (4.4) is

$$\int e^{iyt} g_n(t) G(a+it)dt = \int_0^\infty \phi(x) \widehat{g_n}(y-x) dx - c \int_{-\infty}^y \widehat{g_n}(x) dx. \quad (4.5)$$

Since $t \mapsto g_n(t)G(a+it)$ is continuous and compactly supported, Lemma 4.2 gives

$$\lim_{y \to +\infty} \int e^{iyt} g_n(t) G(a+it)dt = 0,$$

with which the complex numbers leave the stage. Obviously,

$$\lim_{y \to +\infty} \int_{-\infty}^y \widehat{g_n}(x)dx = \int \widehat{g_n}(x)dx = 2\pi.$$

Thus for $y \to +\infty$, (4.5) becomes (in terms of $h_n = \widehat{g_n}/2\pi$)

$$\lim_{y \to +\infty} \int_0^\infty \phi(x) h_n(y-x) dx = c. \quad (4.6)$$

It remains to show that this implies $\lim_{x \to +\infty} \phi(x) = c$, using that $\{h_n\}$ is an approximate unit and that $f(x)=e^{ax}\phi(x)$ is non-decreasing. For $\delta>0$ we have

$$\begin{aligned}\int \phi(x+\delta-y)h_n(y)dy &= \int f(x+\delta-y)e^{-a(x+\delta-y)}h_n(y)dy\\ &\geq \int_{-\delta}^{\delta} f(x+\delta-y)e^{-a(x+\delta-y)}h_n(y)dy\\ &\geq \int_{-\delta}^{\delta} f(x)e^{-a(x+2\delta)}h_n(y)dy\\ &= \phi(x)e^{-2a\delta}\int_{-\delta}^{\delta} h_n(y)dy \;=\; \phi(x)e^{-2a\delta}I_n(\delta),\end{aligned}$$

where we write $I_n(\delta)=\int_{-\delta}^{\delta}h_n(y)dy$. In view of (4.6), we have

$$\limsup_{x\to+\infty}\phi(x)\leq c\frac{e^{2a\delta}}{I_n(\delta)}.$$

Recalling that $I_n(\delta)\stackrel{n\to\infty}{\longrightarrow}1$ for each $\delta>0$, taking $n\to\infty$ gives $\limsup_{x\to+\infty}\phi(x)\leq ce^{2a\delta}$, and for $\delta\searrow 0$ we obtain $\limsup_{x\to+\infty}\phi(x)\leq c$. This also implies that $\phi$ is bounded above: $\phi(x)\leq M$ for some $M>0$.

On the other hand,

$$\begin{aligned}\int \phi(x-\delta-y)h_n(y)dy &\leq \int_{-\delta}^{\delta}\phi(x-\delta-y)h_n(y)dy + M\int_{|y|\geq\delta}h_n(y)dy\\ &= \int_{-\delta}^{\delta} f(x-\delta-y)e^{-a(x-\delta-y)}h_n(y)dy + M\left(1-\int_{-\delta}^{\delta}h_n(y)dy\right)\\ &\leq \phi(x)e^{2a\delta}I_n(\delta)+M(1-I_n(\delta)).\end{aligned}$$

This gives

$$\phi(x)\geq\frac{\int\phi(x-\delta-y)h_n(y)dy-M(1-I_n(\delta))}{e^{2a\delta}I_n(\delta)}.$$

With (4.6) we obtain

$$\liminf_{x\to\infty}\phi(x)\geq\frac{c-M(1-I_n(\delta))}{e^{2a\delta}I_n(\delta)}.$$

Taking $n\to\infty$ we find $\liminf_{x\to\infty}\phi(x)\geq\frac{c}{e^{2a\delta}}$, and $\delta\searrow 0$ gives $\liminf_{x\to\infty}\phi(x)\geq c$.

We have thus proven $\phi(x)\to c$ as $x\to+\infty$, which is equivalent to $f(x)\sim ce^{ax}$. ∎

# 5 Proof of the PNT

5.1 Theorem (Prime Number Theorem) *(i)* $\psi(x)\sim x$ *as* $x\to\infty$.

*(ii) With* $\pi(x)=\#(\mathbb{P}\cap[1,x])$ *we have* $\pi(x)\sim\frac{x}{\log x}$.

*(iii) If* $p_n$ *is the* $n$*-th prime, we have* $p_n\sim n\log n$.

*Proof.* (i) By Lemma 2.5, the integral $\int_0^\infty e^{-zx}\psi(e^x)dx$ converges for $\mathrm{Re}(z)>1$ and equals $-\frac{\zeta'(z)}{z\zeta(z)}$. By Corollary 3.4, the function $z\mapsto-\frac{\zeta'(z)}{\zeta(z)}-\frac{1}{z-1}$ has a continuous extension to $\mathrm{Re}(z)\geq 1$, thus also $-\frac{\zeta'(z)}{z\zeta(z)}-\frac{1}{z-1}=\frac{1}{z}(-\frac{\zeta'(z)}{\zeta(z)}-1-\frac{1}{z-1})$. In view of $\Lambda\geq 0$, the function $\psi(x)=\sum_{n\leq x}\Lambda(n)$ is non-decreasing. Now Theorem 4.1 gives $\psi(e^x)\sim e^x$ as $x\to\infty$, and therefore $\psi(x)\sim x$.

(ii) We compute

$$\psi(x)=\sum_{n\leq x}\Lambda(n)=\sum_{p^k\leq x}\log p=\sum_{p\leq x}\log p\left\lfloor\frac{\log x}{\log p}\right\rfloor\leq\pi(x)\log x.$$

If $1 < y < x$ then

$$\pi(x) - \pi(y) = \sum_{y<p\leq x} 1 \leq \sum_{y<p\leq x} \frac{\log p}{\log y} \leq \frac{\psi(x)}{\log y}.$$

Thus $\pi(x) \leq y + \psi(x)/\log y$. Taking $y = x/\log^2 x$ this gives

$$\frac{\psi(x)}{x} \leq \frac{\pi(x)\log x}{x} \leq \frac{\psi(x)}{x}\frac{\log x}{\log(x/\log^2 x)} + \frac{1}{\log x},$$

thus $\psi(x) \sim \pi(x)\log x$. Together with (i), this gives $\pi(x) \sim x/\log x$.

(iii) Taking logarithms of $\pi(x) \sim x/\log x$, we have $\log \pi(x) \sim \log x - \log\log x \sim \log x$ and thus $\pi(x)\log \pi(x) \sim x$. Taking $x = p_n$ and using $\pi(p_n) = n$ gives $n \log n \sim p_n$. ■

# 6 Comments

- Should one prefer complex over harmonic analysis (why??), a weak version of Ikehara's Tauberian theorem, sufficient for obtaining the PNT, can be proven using complex analysis, as shown by Newman [14] (and presented efficiently in [19]). The proof is not simpler than the one given here and gives rather less insight.
- Without too much more work, cf. [10], the proof of the PNT given above can be strengthened to the more quantitative statement

  $$\psi(x) = x + O\left(\frac{x}{\log^n x}\right) \quad \forall n \in \mathbb{N},$$

  which implies

  $$\pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{\log^n x}\right) \quad \forall n, \quad \text{where} \quad \mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

  In order to do so, one needs some estimates on $\zeta(1+it)$ and its derivatives, but no information on $\zeta(z)$ for $\mathrm{Re}(z) < 1$.
- In order to prove stronger results on the error $\psi(x)-x$ (or $\pi(x)-\mathrm{Li}(x)$) one needs information on the non-vanishing of $\zeta(z)$ for $\mathrm{Re}(z) < 1$. It is known that

  $$\zeta(s+it) \neq 0 \text{ when } s > 1 - \frac{c}{\log^\alpha(|t|+1)}, \tag{6.1}$$

  where $\alpha \in (0,1)$, is equivalent to

  $$\psi(x) - x = O(xe^{-d\log^\beta x}), \quad \text{where} \quad \beta = \frac{1}{\alpha+1} < 1.$$

  With $\beta = 1$ this would become $\psi(x) - x = O(xe^{-d\log x}) = x^{1-d}$. It is known that

  $$\zeta(s+it) \neq 0 \text{ when } s > \gamma \quad \Leftrightarrow \quad \psi(x) - x = O(x^{\gamma+\varepsilon}) \ \forall \varepsilon > 0.$$

  Note that for any $\beta, \gamma \in (0,1)$ and $n \in \mathbb{N}$ we have

  $$\frac{x^\gamma}{xe^{-d\log^\beta x}} \to 0 \qquad \text{and} \qquad \frac{xe^{-d\log^\beta x}}{\frac{x}{\log^n x}} \to 0.$$

  Since there are (infinitely many) $t \in \mathbb{R}$ such that $\zeta(\frac{1}{2} + it) = 0$, the best possible result is $\psi(x) - x = O(x^{\frac{1}{2}+\varepsilon})$ (which then actually implies $\psi(x) - x = O(\sqrt{x}\log^2 x)$). However, the best current proven result is (6.1) with $\alpha = 2/3$, implying an error estimate of the form $O(xe^{-d\log^{3/5} x})$.

- Most proofs of the PNT do not work directly with $\pi(x)$, but rather with $\psi(x)$ or with the (only superficially) simpler function $\theta(x) = \sum_{p\le x} \log p$. See however [8, 2] for a deviation from this rule, following a Fourier analytic approach similar to the present one (but more sophisticated due to the use of distributional language and less Tauberian, thus perhaps less conceptual.)
- The approach to the PNT discussed above combines Tauberian theorems that have no number-theoretic content themselves with the number-theoretic information contained in the non-vanishing of $\zeta(1+it)$. A beautiful alternative approach due to Landau [11, §160] and Ingham [6] uses information about $\zeta$ to prove strong Tauberian theorems which, while not looking particularly number theoretical, actually 'contain' the PNT in the sense of implying it with very little additional work, not appealing to the zeta-function again. Similarly to the history of the Landau-Ikehara theorem, Landau's proof used information on $\zeta$ stronger than $\zeta(1+it) \neq 0$, which Ingham eliminated using Wiener's theory. However, a simpler early result of Wiener [16], cf. also [12], is sufficient. The latter has much in common with the proof of Ikehara's theorem given here.
- In 1948, 50 years after the first proofs of the PNT, Selberg and Erdös found a proof of the PNT that avoids not only complex analysis, but also harmonic analysis and every use of $\mathbb{C}$. Not much later, Karamata showed that this approach leads to an elementary proof of a version of Ingham's (first) Tauberian theorem, sufficient for the PNT. [Despite the fact that different elementary proofs of the PNT have appeared since, this author thinks that the one of Karamata still is the simplest and most conceptual to date. Cf. [13] for an exposition.]
- For some insights on the relationship between the traditional $\zeta$-based and 'elementary' proofs of the PNT cf. [7], [4, Chap. 3, §4] and [15]. But there still is much room for a better understanding.

# References


[1] S. Bochner: Ein Satz von Laudau und Ikehara. Math. Zeit. **37**, 1-9 (1933).

[2] J.-B. Bost: Le théorème des nombres premiers et la transformation de Fourier. *In:* N. Berline, C. Sabbah (eds.): *La fonction zeta*. Ecole Polytechnique, 2003.

[3] D. Choimet, H. Queffelec: *Twelve landmarks of twentieth-century analysis.* Cambridge University Press, 2015.

[4] W. & F. Ellison: *Prime numbers.* Wiley, Hermann, 1985. (Original French version: W. Ellison, M. Mendès France: *Les nombres premiers.* Hermann, 1975.)

[5] S. Ikehara: An extension of Landau's Theorem in the analytic theory of numbers. Journ. Math. Phys. M.I.T. **10**, 1-12 (1931).

[6] A. E. Ingham: Some Tauberian theorems connected with the prime number theorem. J. Lond. Math. Soc. **20**, 171-180 (1945).

[7] A. E. Ingham: Review of the papers of Selberg and Erdös. Mathem. Reviews, MR0029411 (10,595c)

[8] J.-P. Kahane: Une formule de Fourier sur les nombres premiers. Gaz. Math. **67**, 3-9 (1996).

[9] A. Kienast: Über die Unabhängigkeit des Beweises des Primzahlsatzes vom Begriff der analytischen Funktion einer komplexen Variable. Comm. Math. Helv. **8**, 130-141 (1935) and **9**, 40-41 (1936) (erratum).

[10] A. Kienast: Beweis des Satzes $x^{-1}\log^q x(\psi(x)-x) \to 0$ ohne Überschreitung der Geraden $\sigma = 1$. Math. Zeit. **43**, 113-119 (1938).

[11] E. Landau: *Handbuch der Lehre von der Verteilung der Primzahlen.* Teubner, 1909.

[12] N. Levinson: On the elementary character of Wiener's general tauberian theorem. J. Math. Anal. Appl. **42**, 381-396 (1973).

[13] M. Müger: On Karamata's proof of the Landau-Ingham Tauberian theorem. L'Ens. Math. **61**, 45-69 (2015).

[14] D. J. Newman: Simple analytic proof of the prime number theorem. Amer. Math. Monthly **87**, 693-696 (1980).

[15] T. Tao: Fourier-analytic proofs of the prime number theorem. `http://www.math.ucla.edu/~tao/preprints/Expository/prime.dvi`

[16] N. Wiener: A new method in Tauberian theorems. Journ. Math. Phys. M.I.T. **7**, 161-184 (1928).

[17] N. Wiener: Tauberian theorems. Ann. Math. **33**, 1-100 (1932).

[18] N. Wiener: *The Fourier integral and certain of its applications*. Cambridge University Press, 1933.

[19] D. Zagier: Newman's short proof of the prime number theorem. Amer. Math. Monthly **104**, 705-708 (1997).