The Prime Number Theorem without Euler products

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Abstract

We give a simple proof of the Prime Number Theorem (PNT), the main aims being to minimize and isolate the input from number theory and to avoid the Euler factorization of the zeta function. We will use some very basic complex and harmonic analysis, proving all we need of the latter.

1 The Prime Number Theorem

Part (ii) of the following lemma is the only place where the fundamental theorem of arithmetic and related number theoretic reasoning play a rôle. $\mathbb{P}$ denotes the set of primes.

1.1 Lemma *(i) There is a unique arithmetic function $\Lambda : \mathbb{N} \to \mathbb{R}$ satisfying*

$$\sum_{d|n} \Lambda(d) = \log n \quad \forall n \in \mathbb{N}. \tag{1.1}$$

(ii) This solution is given by the 'von Mangoldt function' defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \in \mathbb{P},\ k \in \mathbb{N} \\ 0 & \text{else} \end{cases} \tag{1.2}$$

(iii) We have $0 \le \Lambda(n) \le \log n$ for all $n \in \mathbb{N}$.

Proof. (i) For $n = 1$ we have $\Lambda(1) = \sum_{d|1} \Lambda(d) = \log 1 = 0$. For $n \ge 2$ we are forced to inductively define

$$\Lambda(n) = \log n - \sum_{\substack{d|n \\ d<n}} \Lambda(d). \tag{1.3}$$

This shows that (1.1) has a unique solution. (ii) By the fundamental theorem of arithmetic, every $n \in \mathbb{N}$ has a unique prime factorization $n = p_1^{k_1} \cdots p_m^{k_m}$, and the divisors of n are of the form $p_1^{\ell_1} \cdots p_m^{\ell_m}$ where $0 \le \ell_i \le k_i$. The only divisors d of n for which the Λ of (1.2) is non-zero are $\{p_i^k \mid i = 1, \ldots, m,\ k = 1, \ldots, k_m\}$. Thus

$$\sum_{d|n} \Lambda(n) = \sum_{i=1}^m \sum_{k=1}^{k_i} \log p_i = \sum_{i=1}^m k_i \log p_i = \log \prod_{i=1}^m p_i^{k_i} = \log n,$$

proving that (1.2) is a, thus *the*, solution of (1.1). Now (iii) is an obvious consequence of (ii). ■

For use in the following proofs, we recall one fact from basic complex analysis: If $f : \Omega \to \mathbb{C}$ is meromorphic, then for every $z_0 \in \Omega$ there is a unique $n(z_0) \in \mathbb{Z}$ such that $f(z) = (z - z_0)^{n(z_0)} g_{z_0}(z)$, where g_{z_0} is holomorphic in some neighborhood of z_0 and $g_{z_0}(z_0) \ne 0$. Furthermore, the function $z \mapsto \frac{f'(z)}{f(z)}$ is meromorphic on Ω, has a pole of order one and residue $n(z)$ whenever f has a zero or pole at z, and is holomorphic elsewhere. Thus $n(z) = \lim_{\varepsilon \to 0} \varepsilon (f'/f)(z + \varepsilon)$.

1.2 Proposition *The series*

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z} \tag{1.4}$$

defines a holomorphic function on the open halfplane $\mathrm{Re}\, z>1$*. It there satisfies* $\zeta(z)\neq 0$ *and*

$$\zeta'(z) = -\sum_{n=1}^{\infty}\frac{\log n}{n^z}, \tag{1.5}$$

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}. \tag{1.6}$$

Proof. In view of $|n^z|=n^{\mathrm{Re}\, z}$ and the convergence of $\sum_{n=1}^{\infty}1/n^{\alpha}$ for $\alpha>1$, it is clear that (1.4) converges absolutely for $\mathrm{Re}\, z>1$ and uniformly for $\mathrm{Re}\, z\geq\alpha>1$. It therefore defines a holomorphic function on the open half plane $\mathrm{Re}\, z>1$. By basic complex analysis, we may differentiate under the sum and obtain (1.5). Using the consequence $|\Lambda(n)|\leq\log n$ of Lemma 1.1(iii), we similarly see that $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}$ defines a holomorphic function on $\mathrm{Re}\, z>1$. Since the series (1.4) and (1.6) converge absolutely for $\mathrm{Re}\, z>1$, we may compute

$$\begin{aligned}\zeta(z)\cdot\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z} &= \sum_{n=1}^{\infty}\frac{1}{n^z}\cdot\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^z}=\sum_{n,m=1}^{\infty}\frac{\Lambda(m)}{(nm)^z}\\ &= \sum_{n=1}^{\infty}\frac{1}{n^z}\sum_{d|n}\Lambda(d)=\sum_{n=1}^{\infty}\frac{\log n}{n^z}=-\zeta'(z),\end{aligned}$$

where we used (1.1) and (1.5). Since ζ clearly does not vanish identically, it has only isolated zeros, and for $\zeta(z)\neq 0$ we have proven (1.6). But since ζ'/ζ has poles at all zeros of ζ, whereas the r.h.s. of (1.6) is holomorphic on $\mathrm{Re}\, z>1$, we have proven that $\zeta(z)\neq 0$ for $\mathrm{Re}\, z>1$. ■

1.3 Lemma *There is a holomorphic function* $g:\{z\,|\,\mathrm{Re}\, z>0\}\to\mathbb{C}$ *such that* $\zeta(z)=g(z)+\frac{1}{z-1}$ *when* $\mathrm{Re}\, z>1$*. Thus* ζ *has a meromorphic extension, which we also denote* ζ*, to* $\{z\,|\,\mathrm{Re}\, z>0\}\backslash\{1\}$*.*

Proof. For $\mathrm{Re}\, z>1$ we have $\int_1^{\infty}t^{-z}dt=\frac{1}{z-1}$. Thus

$$g(z)=\zeta(z)-\frac{1}{z-1}=\sum_{n=1}^{\infty}\left(\frac{1}{n^z}-\int_n^{n+1}\frac{1}{t^z}dt\right)=\sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^z}-\frac{1}{t^z}\right)dt. \tag{1.7}$$

Now,

$$\left|\int_n^{n+1}\left(\frac{1}{n^z}-\frac{1}{t^z}\right)dt\right|=\left|z\int_n^{n+1}\int_n^t\frac{du}{u^{z+1}}dt\right|\leq\sup_{u\in[n,n+1]}\left|\frac{z}{u^{z+1}}\right|=\frac{|z|}{n^{\mathrm{Re}(z)+1}},$$

so that (1.7) converges, and thus defines $g(z)$ as a holomorphic function on $\mathrm{Re}\, z>0$. ■

1.4 Proposition (i) *We have* $\zeta(1+it)\neq 0$ *for all* $0\neq t\in\mathbb{R}$*.*

(ii) *The function* $z\mapsto-\frac{\zeta'}{\zeta}(z)-\frac{1}{z-1}$ *extends continuously to* $\mathrm{Re}\, z\geq 1$*.*

Proof. (i) We have $\zeta(z)\in\mathbb{R}$ for real $z>1$, implying $\zeta(\overline{z})=\overline{\zeta(z)}$, thus $n(1+it)=n(1-it)$. By Lemma 1.3 we have $\lim_{\varepsilon\searrow 0}\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon)=n(1)=-1$ and, assuming $t\neq 0$,

$$\lim_{\varepsilon\searrow 0}\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon\pm it)=n(1\pm it)=:\mu,\quad \lim_{\varepsilon\searrow 0}\varepsilon\frac{\zeta'}{\zeta}(1+\varepsilon\pm 2it)=n(1\pm 2it)=:\nu,$$

where $\mu, \nu \geq 0$ since $\zeta(z)$ is holomorphic at $1+it$ for $t \neq 0$. With (1.6) we have

$$-\sum_{r=-2}^{2} \left(\begin{array}{c} 4 \\ 2+r \end{array} \right) \frac{\zeta'}{\zeta}(1+\varepsilon+irt) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\varepsilon}} (n^{it/2}+n^{-it/2})^4 \geq 0,$$

since $\Lambda \geq 0$ and $n^{it/2}+n^{-it/2} \in \mathbb{R}$. Multiplying by ε and taking $\varepsilon \searrow 0$ gives $6-8\mu-2\nu \geq 0$. (The 5th row $\binom{4}{\bullet}$ of Pascal's triangle is $1,4,6,4,1$.) Thus $n(1+it) = \mu = 0$, which means $\zeta(1+it) \neq 0$.

(ii) By Lemma 1.3, ζ is holomorphic on $\operatorname{Re} z > 0$ with the exception of the simple pole at $z=1$. By (i), $\zeta(1+it) \neq 0$ for $t \neq 0$. Thus $-\zeta'/\zeta$ is holomorphic at $1+it$ for $t \neq 0$ and has a simple pole of residue 1 at $z=1$. The latter is cancelled by the subtraction of the term $(z-1)^{-1}$. ■

1.5 Lemma *Defining $\psi(x) = \sum_{n \leq x} \Lambda(n)$, we have*

$$z \int_0^\infty e^{-zx} \psi(e^x) dx = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)} \quad \textit{for} \quad \operatorname{Re}(z) > 1. \tag{1.8}$$

Proof. In view of Lemma 1.1, we have $\psi(x) = O(x \log x) = O(x^{1+\varepsilon})$, thus the integral converges absolutely. Now

$$\begin{aligned} z \int_0^\infty e^{-zx} \psi(e^x) dx &= z \int_0^\infty e^{-zx} \left(\sum_{n \leq e^x} \Lambda(n) \right) dx = z \sum_{n=1}^{\infty} \Lambda(n) \int_{\log n}^{\infty} e^{-zx} dx \\ &= z \sum_{n=1}^{\infty} \Lambda(n) \left[\frac{-1}{z} e^{-zx} \right]_{\log n}^{\infty} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} \end{aligned}$$

proves the first identity of (1.8). The second identity was proven in Proposition 1.2. ■

The following Tauberian theorem, due to Ikehara and not involving number theory, will be proven in the Appendix:

1.6 Theorem *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing. If there are $a, c \geq 0$ such that the integral*

$$F(z) = \int_0^\infty f(x) e^{-zx} dx,$$

converges absolutely for $\operatorname{Re}(z) > a$ [thus automatically defines a continuous function there] and $F(z) - \frac{c}{z-a}$ has a continuous extension to the half-plane $\operatorname{Re}(z) \geq a$ then

$$f(x) \sim ce^{ax} \quad \text{as} \quad x \to +\infty.$$

1.7 Proposition *$\psi(x) \sim x$ as $x \to \infty$.*

Proof. By Lemma 1.5, the integral $\int_0^\infty e^{-zx}\psi(e^x)dx$ converges for $\operatorname{Re}(z) > 1$ and equals $-\frac{\zeta'(z)}{z\zeta(z)}$. By Proposition 1.4(ii), the function $z \mapsto -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$ has a continuous extension to $\operatorname{Re}(z) \geq 1$, thus also $-\frac{\zeta'(z)}{z\zeta(z)} - \frac{1}{z-1} = \frac{1}{z}(-\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1} - 1)$. In view of $\Lambda \geq 0$, the function $\psi(x) = \sum_{n \leq x} \Lambda(n)$ is non-decreasing. Now Theorem 1.6 gives $\psi(e^x) \sim e^x$ as $x \to \infty$, and therefore $\psi(x) \sim x$. ■

1.8 Theorem (Prime Number Theorem) *Let $\pi(x) = \#(\mathbb{P} \cap [1,x])$, and let p_n denote the n-th prime. Then*

$$\pi(x) \sim \frac{x}{\log x} \qquad \textit{and} \qquad p_n \sim n \log n \qquad \textit{as } x, n \to \infty.$$

Proof. Using Lemma 1.1(ii), we compute

$$\psi(x)=\sum_{n\le x}\Lambda(n)=\sum_{p^k\le x}\log p=\sum_{p\le x}\log p\left\lfloor\frac{\log x}{\log p}\right\rfloor\le\sum_{p\le x}\log x=\pi(x)\log x.$$

If $1<y<x$ then

$$\pi(x)-\pi(y)=\sum_{y<p\le x}1\le\sum_{y<p\le x}\frac{\log p}{\log y}\le\frac{\psi(x)}{\log y}.$$

Thus $\pi(x)\le y+\psi(x)/\log y$. Taking $y=x/(\log x)^2$ this gives

$$\frac{\psi(x)}{x}\le\frac{\pi(x)\log x}{x}\le\frac{\psi(x)}{x}\,\frac{\log x}{\log(x/(\log x)^2)}+\frac{1}{\log x}.$$

In view of $\log(x/(\log x)^2)=\log x-2\log\log x\sim\log x$ and $\psi(x)\sim x$ from Proposition 1.7 we have $\pi(x)\sim x/\log x$.

Taking logarithms of $\pi(x)\sim x/\log x$, we have $\log\pi(x)\sim\log x-\log\log x\sim\log x$ and thus $\pi(x)\log\pi(x)\sim x$. Taking $x=p_n$ and using $\pi(p_n)=n$ gives $n\log n\sim p_n$. ■

1.9 Remark

- We avoided the Euler factorization of the zeta-function (which at any rate just is an analytic restatement of the unique prime factorization) and the power series expansion of the logarithm. Instead we directly arrived at the Dirichlet series (1.6) for the logarithmic derivative ζ'/ζ, using the very basic theory of Λ and the relation between Dirichlet convolution of arithmetic functions and pointwise multiplication of the associated Dirichlet series.
- Arithmetic (or number-theoretic) reasoning is confined to the proof of part (ii) of Lemma 1.1. That result is used only via its consequence (iii). More precisely, to prove $\zeta(z)\ne 0$ for $\mathrm{Re}\,z>1$ we only needed the slow growth of Λ. The positivity of Λ gave the monotonicity of ψ, needed to apply Theorem 1.6, and went into the proof of $\zeta(1+it)\ne 0$.
- Actually, $\Lambda(n)\le\log n\ \forall n$ is clearly implied by the elementary identity (1.3) together with $\Lambda\ge 0$. Thus the latter statement is the *only* arithmetic input needed for the proof.
- But eliminating the use of $\Lambda\ge 0$ or proving it without use of the fundamental theorem of arithmetic seems impossible: Even the simplest 'elementary' (i.e. ζ-free) proof of the PNT uses $\Lambda\ge 0$. Circumventing this by taking (1.2) as the definition of Λ is no solution, since then one needs the fundamental theorem of arithmetic for proving that Λ satisfies $\Lambda\star\mathbf{1}=\log$.

□

A Proof of Ikehara's Tauberian theorem

The proof of Theorem 1.6 will be given using the Fourier transform on $\mathbb{R}$, concerning which we only need the definition and an easy special case of the Riemann-Lebesgue Lemma:

A.1 Lemma *Let $f:\mathbb{R}\to\mathbb{C}$ be continuous with compact support. Then the Fourier transform*

$$\widehat{f}(\xi):=\int f(x)e^{-i\xi x}dx \tag{A.1}$$

(integrals without bounds always extend over $\mathbb{R}$) satisfies $\widehat{f}(\xi)\to 0$ as $|\xi|\to\infty$.

Proof. For $\xi\ne 0$, we have

$$\widehat{f}(\xi)=-\int f(x)e^{-i\xi(x-\frac{\pi}{\xi})}dx=-\int f(x+\frac{\pi}{\xi})e^{-i\xi x}dx,$$

which leads to

$$\widehat{f}(\xi)=\frac{1}{2}\int\left(f(x)-f(x+\frac{\pi}{\xi})\right)e^{-i\xi x}dx. \tag{A.2}$$

Being continuous with compact support, f is uniformly continuous, thus $\sup_x |f(x)-f(x+\varepsilon)|\to 0$ as $\varepsilon\to 0$. Inserting this in (A.2) and keeping in mind that $\mathrm{supp}(f)$ is bounded, the claim follows. ■

Proof of Theorem 1.6. As a preparatory step, pick any continuous, compactly supported even function $g:\mathbb{R}\to\mathbb{R}$ such that $g(0)=1$ and $\widehat{g}(\xi)\geq 0\ \forall\xi$, where $\widehat{g}$ is as in (A.1). (A possible choice is the tent function $g(x)=\max(1-|x|,0)$. By an easy computation, $\widehat{g}(\xi)=2(1-\cos\xi)/\xi^2\geq 0$, so that all assumptions are satisfied.) Then the Fourier inversion theorem applies, thus $g(x)=(2\pi)^{-1}\int\widehat{g}(\xi)e^{i\xi x}d\xi$, and in particular we have $\int\widehat{g}(\xi)d\xi=2\pi g(0)=2\pi$. (This is our only use of Fourier inversion, and it can be avoided at the expense of showing $\int\widehat{g}(\xi)d\xi=2\pi$ 'by hand', which can be done for the above choice of g, if somewhat arduously.[1])

Defining $g_n(x)=g(x/n)$, it is immediate that $\widehat{g_n}(\xi)=n\widehat{g}(n\xi)$ and $\int\widehat{g_n}=2\pi$ for all n. For every $\delta>0$ we have $\int_{|\xi|\geq\delta}\widehat{g_n}(\xi)=\int_{|u|\geq n\delta}\widehat{g}(u)du$, which tends to zero as $n\to 0$ since $\widehat{g_n}$ is integrable and non-negative. Thus the normalized functions $h_n=\widehat{g_n}/2\pi$ constitute an approximation of unity. (I.e. $\int fh_n\stackrel{n\to\infty}{\longrightarrow}f(0)$ for sufficiently nice f.)

Let $G(z)$ be the continuous extension (clearly unique) of $F(z)-\frac{c}{z-a}$ to $\mathrm{Re}(z)\geq a$. Defining $\phi(x)=e^{-ax}f(x)$, we have for all $t\in\mathbb{R}$, $\varepsilon>0$:

$$G(a+\varepsilon+it)=F(a+\varepsilon+it)-\frac{c}{\varepsilon+it}=\int_0^\infty(\phi(x)-c)e^{-(\varepsilon+it)x}dx=\int_0^\infty\psi_\varepsilon(x)e^{-itx}dx, \tag{A.3}$$

where we abbreviate $\psi_\varepsilon(x)=e^{-\varepsilon x}(\phi(x)-c)$ to make the following computation readable. Multiplying (A.3) by $e^{iyt}g_n(t)$ and integrating over t, we have

$$\begin{aligned}\int e^{iyt}g_n(t)G(a+\varepsilon+it)dt &= \int e^{iyt}g_n(t)\left(\int_0^\infty\psi_\varepsilon(x)e^{-itx}dx\right)dt\\ &= \int_0^\infty\psi_\varepsilon(x)\left(\int g_n(t)e^{it(y-x)}dt\right)dx\\ &= \int_0^\infty\psi_\varepsilon(x)\widehat{g_n}(y-x)dx\\ &= \int_0^\infty e^{-\varepsilon x}\phi(x)\widehat{g_n}(y-x)dx-c\int_0^\infty e^{-\varepsilon x}\widehat{g_n}(y-x)dx. \end{aligned}\tag{A.4}$$

(The second equality holds by Fubini's theorem, since ψ_ε and g_n are integrable. The third identity uses that g_n is even.) We now consider the limit $\varepsilon\searrow 0$ of (A.4). Since G by assumption is continuous on $\mathrm{Re}(z)\geq a$, we have $G(a+\varepsilon+it)\to G(a+it)$, uniformly for t in the compact support of g_n. Thus

$$\lim_{\varepsilon\searrow 0}\int e^{iyt}g_n(t)G(a+\varepsilon+it)dt=\int e^{iyt}g_n(t)G(a+it)dt.$$

Since ϕ and $\widehat{g_n}$ are non-negative, the monotone convergence theorem gives

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_0^\infty e^{-\varepsilon x}\phi(x)\widehat{g_n}(y-x)dx &= \int_0^\infty\phi(x)\widehat{g_n}(y-x)dx,\\ \lim_{\varepsilon\searrow 0}\int_0^\infty e^{-\varepsilon x}\widehat{g_n}(y-x)dx &= \int_0^\infty\widehat{g_n}(y-x)dx=\int_{-\infty}^y\widehat{g_n}(x)dx.\end{aligned}$$

Thus the $\varepsilon\searrow 0$ limit of (A.4) is

$$\int e^{iyt}g_n(t)G(a+it)dt=\int_0^\infty\phi(x)\widehat{g_n}(y-x)dx-c\int_{-\infty}^y\widehat{g_n}(x)dx. \tag{A.5}$$

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[1] Let $a,b>0$. Twofold partial integration gives $\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$. Applying $\int_0^c\cdots db$ gives $\int_0^\infty e^{-ax}\frac{\sin cx}{x}\,dx=\arctan\frac{c}{a}$. Applying $\int_0^f\cdots dc$ gives $\int_0^\infty e^{-ax}\frac{1-\cos fx}{x^2}\,dx=f\arctan\frac{f}{a}-\frac{a}{2}\log\left(1+(\frac{f}{a})^2\right)$. (The exchanges of integration order are justified by absolute convergence and Fubini.) Taking $f=1$ and the limit $a\searrow 0$ we obtain $\int_0^\infty\frac{1-\cos x}{x^2}\,dx=\frac{\pi}{2}$.

Since $t \mapsto g_n(t)G(a+it)$ is continuous and compactly supported, Lemma A.1 gives

$$\lim_{y\to+\infty} \int e^{iyt} g_n(t)G(a+it)dt = 0,$$

with which the complex numbers leave the stage. Obviously,

$$\lim_{y\to+\infty} \int_{-\infty}^{y} \widehat{g_n}(x)dx = \int \widehat{g_n}(x)dx = 2\pi.$$

Thus for $y \to +\infty$, (A.5) becomes (in terms of $h_n = \widehat{g_n}/2\pi$)

$$\lim_{y\to+\infty} \int_0^\infty \phi(x)h_n(y-x)dx = c. \tag{A.6}$$

It remains to show that this implies $\lim_{x\to+\infty}\phi(x) = c$, using that $\{h_n\}$ is an approximate unit and that $f(x) = e^{ax}\phi(x)$ is non-decreasing. For $\delta > 0$ we have

$$\begin{aligned}
\int \phi(x+\delta-y)h_n(y)dy &= \int f(x+\delta-y)e^{-a(x+\delta-y)}h_n(y)dy \\
&\geq \int_{-\delta}^{\delta} f(x+\delta-y)e^{-a(x+\delta-y)}h_n(y)dy \\
&\geq \int_{-\delta}^{\delta} f(x)e^{-a(x+2\delta)}h_n(y)dy \\
&= \phi(x)e^{-2a\delta}\int_{-\delta}^{\delta} h_n(y)dy \;=\; \phi(x)e^{-2a\delta}I_n(\delta),
\end{aligned}$$

where we write $I_n(\delta) = \int_{-\delta}^{\delta} h_n(y)dy$. By (A.6), the l.h.s. tends to c as $y \to \infty$, so that

$$\limsup_{x\to+\infty} \phi(x) \leq c\frac{e^{2a\delta}}{I_n(\delta)} \qquad \forall n \in \mathbb{N}, \delta > 0.$$

Recalling that $I_n(\delta) \stackrel{n\to\infty}{\longrightarrow} 1$ for each $\delta > 0$, taking $n \to \infty$ gives $\limsup_{x\to+\infty}\phi(x) \leq ce^{2a\delta}$ for every $\delta > 0$. Taking $\delta \searrow 0$ we obtain $\limsup_{x\to+\infty}\phi(x) \leq c$. This also implies that ϕ is bounded above: $\phi(x) \leq M$ for some $M > 0$.

On the other hand,

$$\begin{aligned}
\int \phi(x-\delta-y)h_n(y)dy &\leq \int_{-\delta}^{\delta} \phi(x-\delta-y)h_n(y)dy + M\int_{|y|\geq\delta} h_n(y)dy \\
&= \int_{-\delta}^{\delta} f(x-\delta-y)e^{-a(x-\delta-y)}h_n(y)dy + M\left(1 - \int_{-\delta}^{\delta} h_n(y)dy\right) \\
&\leq \phi(x)e^{2a\delta}I_n(\delta) + M(1 - I_n(\delta)).
\end{aligned}$$

This gives

$$\phi(x) \geq \frac{\int \phi(x-\delta-y)h_n(y)dy - M(1-I_n(\delta))}{e^{2a\delta}I_n(\delta)}.$$

With (A.6) we obtain

$$\liminf_{x\to\infty} \phi(x) \geq \frac{c - M(1-I_n(\delta))}{e^{2a\delta}I_n(\delta)}.$$

Taking $n \to \infty$ we find $\liminf_{x\to\infty}\phi(x) \geq \frac{c}{e^{2a\delta}}$, and $\delta \searrow 0$ gives $\liminf_{x\to\infty}\phi(x) \geq c$.

We have thus proven $\phi(x) \to c$ as $x \to +\infty$, which is equivalent to $f(x) \sim ce^{ax}$. ■