On trace class operators
(and Hilbert-Schmidt operators)

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Abstract

We give a short treatment of the facts on trace class operators needed for von Neumann algebra theory. We do this without using Hilbert-Schmidt operators, but we also prove all basic results about the latter and their relationship to trace class operators.

1 Introduction

At some point in the discussion of von Neumann algebras, one needs trace class operators and their properties. Many books on functional analysis, e.g. [2, 7], only briefly touch upon this material in a few exercises. Books on operator algebras, like [5, 4], often mix up the discussion of trace class operators with that of Hilbert-Schmidt operators. While also the latter are important (also for operator algebras), I don't like this approach. For this reason I've written up the main facts on trace class operators in a way that does not use Hilbert-Schmidt operators, similar to the treatment in [6] but with more detail.

Nevertheless, there is a Section on Hilbert-Schmidt operators that emphasizes that they form a Hilbert space, a fact that Murphy [4] inexplicably fails to mention.

2 Trace of positive operators. Definition of $L^p(H)$

If $a \in B(H)$ is positive and E is an orthonormal base of H, we define

$$\mathrm{Tr}_E(a) = \sum_{e\in E} \langle ae, e\rangle \in [0, \infty].$$

As a sum of non-negative reals, the order of summation does not matter, but the sum may be $+\infty$.

2.1 Lemma *(i) For every $a \in B(H)$ and ONB E we have $\mathrm{Tr}_E(a^*a) = \mathrm{Tr}_E(aa^*)$.*

(ii) If $a \geq 0$ and u is unitary then $\mathrm{Tr}_E(uau^*) = \mathrm{Tr}_E(a)$.

(iii) If $a \geq 0$ then $\mathrm{Tr}_E(a)$ is independent of the ONB E. We therefore just write $\mathrm{Tr}(a)$.

(iv) For $a, b \in B(H)^+$ and $\lambda \geq 0$ we have $\mathrm{Tr}(a + b) = \mathrm{Tr}(a) + \mathrm{Tr}(b)$ and $\mathrm{Tr}(\lambda a) = \lambda \mathrm{Tr}(a)$.

Proof. (i) We have

$$\begin{aligned}\mathrm{Tr}_E(a^*a) &= \sum_{e\in E}\langle a^*ae, e\rangle = \sum_{e\in E}\langle ae, ae\rangle = \sum_{e\in E}\|ae\|^2 = \sum_{e\in E}\sum_{e'\in E}|\langle ae, e'\rangle|^2 \\ &= \sum_{e'\in E}\sum_{e\in E}|\langle ae, e'\rangle|^2 = \sum_{e'\in E}\sum_{e\in E}|\langle e, a^*e'\rangle|^2 = \sum_{e'\in E}\|a^*e'\|^2 = \mathrm{Tr}_E(aa^*),\end{aligned}$$

where the exchange of summation is justified since all summands are non-negative.

(ii) Put $b = ua^{1/2}$. Then $\mathrm{Tr}_E(a) = \mathrm{Tr}_E(b^*b) \overset{(i)}{=} \mathrm{Tr}_E(bb^*) = \mathrm{Tr}_E(uau^*)$.

(iii) Let $a \geq 0$ and let E, F be ONBs for H. Since E, F have the same cardinality, we can pick a bijection $\alpha : F \to E$. The latter extends to a unitary operator $u : H \to H$. Thus by (ii),

$$\mathrm{Tr}_E(a) = \mathrm{Tr}_E(uau^*) = \sum_{e\in E} \langle uau^*e, e\rangle = \sum_{e\in E} \langle au^*e, u^*e\rangle = \sum_{f\in F} \langle af, f\rangle = \mathrm{Tr}_F(a).$$

(iv) The second statement is evident, and the first follows from the fact that a sum of non-negative numbers is independent of the order or bracketing. ■

The map $\mathrm{Tr} : B(H)^+ \to [0, \infty]$ will be extended to some non-positive operators later on.

2.2 Definition *If H is a Hilbert space and $1 \leq p < \infty$, we define*

$$\|a\|_p = (\mathrm{Tr}(|a|^p))^{1/p} \in [0, \infty], \qquad L^p(H) = \{a \in B(H) \mid \|a\|_p < \infty\}.$$

One can show that $L^p(H) \subseteq B(H)$ is an ideal for each $p \in [1, \infty)$. Cf. e.g. [8]. (Since $L^p(H) \subseteq K(H)$ for all p and $L^p(H) \subseteq L^q(H)$ if $p \leq q$, it makes good sense to put $L^\infty(H) = K(H)$.) We will only study the most important cases $p = 1$ and $p = 2$.

3 $L^1(H)$: Trace class operators

Specializing Definition 2.2 to $p = 1$ we have

3.1 Definition *We have $L^1(H) = \{a \in B(H) \mid \|a\|_1 < \infty\}$, where $\|a\|_1 = \mathrm{Tr}|a| \in [0, \infty]$. The elements of $L^1(H)$ are called trace class operators.*

We will need a fact concerning the polar decomposition that is not always mentioned:

3.2 Lemma *If $a = u|a|$ is the polar decomposition then $|a| = u^*a$.*

Proof. u^*u is the source projection of the partial isometry u, i.e. the orthogonal projection onto $(\ker u)^\perp$. By definition of the polar decomposition, $\ker u = \ker a = \ker |a|$. And since $|a|$ is self-adjoint, $\overline{|a|H} \subseteq (\ker |a|)^\perp$. Combining these facts, we see that u^*u acts as the identity on $\overline{|a|H}$, so that $u^*u|a| = |a|$. Now $u^*a = u^*u|a| = |a|$. ■

3.3 Theorem *Let H be any Hilbert space. Then*

(i) $\|a\| \leq \|a\|_1$ *for all* $a \in B(H)$.

(ii) $\|a^*\|_1 = \|a\|_1$ *for all* $a \in B(H)$. *Thus* $L^1(H)$ *is self-adjoint.*

(iii) For all $\lambda \in \mathbb{C}$, $a, b \in B(H)$ *we have* $\|\lambda a\|_1 = |\lambda|\|a\|_1$ *and* $\|a + b\|_1 \leq \|a\|_1 + \|b\|_1$. *If* $0 \neq a \in L^1(H)$ *then* $\|a\|_1 > 0$. *Thus* $(L^1(H), \|\cdot\|_1)$ *is a normed vector space.*

(iv) For all $a, b \in B(H)$ *we have* $\|ab\|_1 \leq \|a\|\|b\|_1$ *and* $\|ab\|_1 \leq \|a\|_1\|b\|$. *Thus* $L^1(H) \subseteq B(H)$ *is a two-sided ideal.*

(v) $F(H) \subseteq L^1(H) \subseteq K(H)$.

(vi) $\overline{F(H)}^{\|\cdot\|_1} = L^1(H)$.

(vii) The normed space $(L^1(H), \|\cdot\|_1)$ *is complete, thus a Banach space.*

(viii) $(L^1(H), \|\cdot\|_1)$ *is a Banach* $*$*-algebra.*

(ix) For $a \in B(H)$, the following are equivalent:

  (α) $a \in L^1(H)$, i.e. $\mathrm{Tr}|a| < \infty$.

  (β) a is a finite linear combination of positive operators with finite trace.

  (γ) $\sum_e |\langle vae, e\rangle| < \infty$ for each ONB E and each unitary v.[1]

  (δ) $\sum_e |\langle vae, e\rangle| < \infty$ for some ONB E and each unitary v. Under this condition, $\|a\|_1 \leq \sup_{v \in U(H)} \sum_e |\langle vae, e\rangle|$.

  (ϵ) $\sum_e |\langle ae, e\rangle| < \infty$ for each ONB E.

(x) For each ONB E,

$$\mathrm{Tr} : L^1(H) \to \mathbb{C}, \ a \mapsto \sum_{e \in E} \langle ae, e\rangle$$

is absolutely convergent[2] *and independent of the choice of E, defining a linear functional on $L^1(H)$.*

(xi) For all $a \in L^1(H)$ we have $\mathrm{Tr}(a^*) = \overline{\mathrm{Tr}(a)}$.

(xii) If $a \in B(H), b \in L^1(H)$ then $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ and $|\mathrm{Tr}(ab)| \leq \|a\|\|b\|_1$.
In particular $|\mathrm{Tr}(b)| \leq \|b\|_1$, thus $\mathrm{Tr} \in (L^1(H), \|\cdot\|_1)^*$.

Proof. (i) Let $b \geq 0$ and $x \in H$ a unit vector. If E is an ONB containing x, we have $\langle bx, x\rangle \leq \mathrm{Tr}_E(b)$. Since b is positive, we have $\|b\| = \sup_{x, \|x\|=1} \langle bx, x\rangle$, thus $\|b\| \leq \mathrm{Tr}(b)$. If now $a \in B(H)$ with polar decomposition $a = u|a|$, applying the above to $b = |a|$ and using $\|u\| \leq 1$ gives

$$\|a\| = \|u|a|\,\| \leq \|\,|a|\,\| \leq \mathrm{Tr}|a| = \|a\|_1.$$

(ii) Let $a \in B(H)$ with polar decomposition $a = u|a|$. Using $u^*u|a| = |a|$ from the proof of Lemma 3.2, we have

$$(u|a|u^*)^2 = u|a|u^*u|a|u^* = u|a|^2u^* = (u|a|)(u|a|)^* = aa^* = |a^*|^2.$$

Since $|a^*|$ and $u|a|u^*$ are both positive, taking roots gives $u|a|u^* = |a^*|$. Chosing an ONB E such that each $e \in E$ is either in $\ker u^*$ or in $(\ker u^*)^\perp$, thus $u^*e = 0$ or $u^*e = e$, we find

$$\|a^*\|_1 = \mathrm{Tr}_E|a^*| = \mathrm{Tr}_E(u|a|u^*) = \sum_e \langle |a|u^*e, u^*e\rangle \leq \sum_e \langle |a|e, e\rangle = \mathrm{Tr}|a| = \|a\|_1.$$

Replacing a by a^* gives the opposite inequality so that $\|a^*\|_1 = \|a\|_1$.

(iii) The first statement follows from $|\lambda a| = |\lambda||a|$. For the second, let $a, b \in B(H)$ with polar decompositions $a = u|a|, \ b = v|b|, \ a + b = w|a+b|$. If E is an ONB and $F \subseteq E$ is finite,

$$\begin{aligned}\sum_{e \in F} \langle |a+b|e, e\rangle &= \sum_{e \in F} \langle w^*(a+b)e, e\rangle = \sum_{e \in F} (\langle w^*u|a|e, e\rangle + \langle w^*v|b|e, e\rangle) \\ &\leq \sum_{e \in F} |\langle w^*u|a|e, e\rangle| + \sum_{e \in F} |\langle w^*v|b|e, e\rangle|. \qquad (1)\end{aligned}$$

Focusing on the first term of the r.h.s., we have

$$\begin{aligned}\sum_{e \in F} |\langle w^*u|a|e, e\rangle| &= \sum_{e \in F} |\langle |a|^{1/2}e, |a|^{1/2}u^*we\rangle| \ \leq \ \sum_{e \in F} \|\,|a|^{1/2}e\| \, \||a|^{1/2}u^*we\| \\ &\leq \left(\sum_{e \in F} \|\,|a|^{1/2}e\|^2\right)^{1/2} \left(\sum_{e \in F} \||a|^{1/2}u^*we\|^2\right)^{1/2}, \qquad (2)\end{aligned}$$

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[1] This is the same as $\sum_{i \in I} |\langle ae_i, f_i\rangle| < \infty$ for any ONBs $\{e_i\}_{i \in I}, \{f_i\}_{i \in I}$.

[2] An unordered sum (as opposed to a series, which is an ordered sum) is convergent if and only if it is absolutely convergent! Cf. e.g. [1, Theorem 5.5.2]. Thus the 'absolutely' is redundant and is only included in order not to use the fact just mentioned.

where the first $\leq$ comes from applying Cauchy-Schwarz to the inner product $\langle |a|^{1/2}e, |a|^{1/2}u^*we\rangle$ in H, the second $\leq$ from Cauchy-Schwarz in $\mathbb{C}^{|F|}$.

The argument of the first square root in the r.h.s. of (2) is bounded above by $\sum_{e\in E}\| \,|a|^{1/2}e\|^2 = \mathrm{Tr}|a|$, and for the argument of the second root we have

$$\sum_{e\in F}\| |a|^{1/2}u^*we\|^2 = \sum_{e\in F}\langle |a|^{1/2}u^*we, |a|^{1/2}u^*we\rangle = \sum_{e\in F}\langle w^*u|a|u^*we, e) \leq \mathrm{Tr}(w^*u|a|u^*w\rangle.$$

Now, picking an ONB E such that each $e \in E$ is either in $\ker w$ or in $(\ker w)^\perp$, we find $\mathrm{Tr}(w^*u|a|u^*w) \leq \mathrm{Tr}(u|a|u^*)$. Repeating the argument with u, we have $\mathrm{Tr}(u|a|u^*) \leq \mathrm{Tr}|a|$. Thus $\mathrm{Tr}(w^*u|a|u^*w) \leq \mathrm{Tr}|a|$, so that $\sum_{e\in F}\| |a|^{1/2}u^*we\|^2 \leq \mathrm{Tr}|a|$. Inserting this in (2), we find

$$\sum_{e\in F}|\langle w^*u|a|e, e\rangle| \leq \mathrm{Tr}|a| = \|a\|_1.$$

Analogously one proves the bound $\sum_{e\in F}|\langle w^*v|b|e, e\rangle| \leq \mathrm{Tr}|b| = \|b\|_1$ for the other summand in (1). Now taking the limit $F \nearrow E$ we have $\|a+b\|_1 \leq \|a\|_1 + \|b\|_1$. In view of this, it is clear that $L^1(H)$ is a vector space.

(iv) Let $b = u|b|$ and $ab = v|ab|$ be the polar decompositions of b, ab, respectively. Using Lemma 3.2, we have $|ab| = v^*ab = v^*au|b| = w|b|$, where $w = v^*au$. In view of $\|u\|, \|v\| \leq 1$ we have $\|w\| \leq \|a\|$. Using $w^*w \leq \|w^*w\|\mathbf{1} = \|w\|^2\mathbf{1}$ and [4, Theorem 2.2.5(2)] we have

$$|ab|^2 = |ab|^*|ab| = |b|w^*w|b| \leq \|w\|^2|b|^2.$$

In view of $0 \leq a \leq b \Rightarrow a^{1/2} \leq b^{1/2}$, cf. [4, Theorem 2.2.6], this implies $|ab| \leq \|w\|\,|b|$. Thus $\|ab\|_1 = \mathrm{Tr}|ab| \leq \|w\|\mathrm{Tr}|b| = \|w\|\|b\|_1 \leq \|a\|\|b\|_1$.

The other inequality follows by $\|ab\|_1 = \|(ab)^*\|_1 = \|b^*a^*\|_1 \leq \|b^*\|\|a^*\|_1 = \|a\|_1\|b\|$, where we used the bound just proven and (ii). That $L^1(H)$ is an ideal now is obvious.

(v) We have $(x \otimes y)^* = (y \otimes x)$, thus $(x\otimes y)^*(x\otimes y) = \|x\|^2(y\otimes y) = \|x\|^2\|y\|^2(e\otimes e)$ with $e = y/\|y\|$, so that taking roots gives $|x\otimes y| = \|x\|\|y\|(e\otimes e)$, which clearly is in $L^1(H)$. Since every element of $F(H)$ is a finite linear combination of such $x \otimes y$, we have $F(H) \subseteq L^1(H)$. [Alternatively use the obvious fact $(x\otimes x) \in L^1(H)$ together with with [4, Theorem 2.4.6], saying that these span $F(H)$.]

If $a \in L^1(H)$ the then $|a|^2 = a^*a \in L^1(H)$ since $L^1(H)$ is an ideal. Thus for any ONB E we have

$$\sum_{e\in E}\|\,ae\|^2 = \sum_e\langle a^*ae, e\rangle = \sum_e\langle |a|^2e, e\rangle = \mathrm{Tr}_E(|a|^2) < \infty.$$

Let $F \subseteq E$ be finite and $x \in F^\perp$ with $\|x\| = 1$. Then $F \cup \{x\}$ is an orthonormal set and can be completed to an ONB E. Thus $\sum_{e\in F}\|\,ae\|^2 + \|\,ax\|^2 \leq \mathrm{Tr}(|a|^2)$, or

$$\|ax\|^2 \leq \mathrm{Tr}(|a|^2) - \sum_{e\in F}\|ae\|^2.$$

Since the r.h.s. goes to zero as $F \nearrow E$, we have

$$\sup\Big\{\|ax\| \mid x \in F^\perp, \|x\| = 1\Big\} \overset{F\nearrow E}{\longrightarrow} 0. \tag{3}$$

If p_F is the orthogonal projection onto $\mathrm{span}_{\mathbb{C}}F$ then $a_F = ap_F$ is a finite rank operator that converges in norm to a by (3). Thus $a \in \overline{F(H)}^{\|\cdot\|} = K(H)$, proving $L^1(H) \subseteq K(H)$.

(vi) Assume first $a \in L^1(H)^+$. By (v), a is compact. Compact self-adjoint operators can be diagonalized, thus there is an ONB E such that $a = \sum_{e\in E}\lambda_e\, e\otimes e$. (Cf. e.g. [4, Theorem 2.4.4] or [5, Theorem 3.3.8].) In our case, $\lambda_e \geq 0$ for all $e \in E$ and $\sum_e \lambda_e = \mathrm{Tr}(a) < \infty$. For a finite subset $F \subseteq E$ define $a_F = \sum_{e\in F}\lambda_e\, e\otimes e$, which is finite rank. Now $a - a_F = \sum_{e\in E\backslash F}\lambda_e\, e\otimes e \geq 0$.

Thus $\|a-a_F\|_1 = \mathrm{Tr}(a-a_F) = \sum_{e\in E\backslash F}\lambda_e$. With $\sum_e \lambda_e < \infty$ this implies $\|a-a_F\|_1 \to 0$ as $F \nearrow E$, thus $a \in \overline{F(H)}^{\|\cdot\|_1}$.

Let now $a \in L^1(H)$ with polar decomposition $a = u|a|$. Then $|a| \in L^1(H)$, thus by the above for each $\varepsilon > 0$ there is $b \in F(H)$ with $\| \, |a| - b\|_1 < \varepsilon$. With (iv) and $\|u\| \le 1$ this implies $\|a - ub\|_1 = \|u(|a|-b)\|_1 \le \||a|-b\|_1 < \varepsilon$. Since $F(H)$ is an ideal, we have $ub \in F(H)$, finishing the proof of $L^1(H) \subseteq \overline{F(H)}^{\|\cdot\|_1}$. The converse is clear.

(vii) This can be proven directly, using $L^1(H) \subseteq K(H)$, cf. e.g. [5, Theorem 3.4.12]. Since we don't need the result soon, we will follow Murphy in deducing it later from the isometric isomorphism $L^1(H) \cong B(H)^*$ of normed spaces and completeness of the dual space $B(H)^*$.

(viii) It only remains to prove submultiplicativity: If $a, b \in L^1(H)$ then $\|ab\|_1 \le \|a\|\|b\|_1 \le \|a\|_1\|b\|_1$, where we used (i) and (iv).

(ix) $(\beta) \Rightarrow (\alpha)$: This is trivial since $L^1(H)$ is a vector space by (iii) and obviously contains the positive operators of finite trace.

$(\alpha) \Rightarrow (\beta)$ Assume $a \in L^1(H)$. By (ii), $a^* \in L^1(H)$ so that (iii) implies $\mathrm{Re}(a), \mathrm{Im}(a) \in L^1(H)$. If $a = a^* \in L^1(H)$, let $a = a_+ - a_-$ be the canonical decomposition with $a_\pm \ge 0$ and $a_+a_- = 0$. Then $|a| = a_+ + a_-$, so that $\mathrm{Tr}(a_\pm) \le \mathrm{Tr}|a| = \|a\|_1 < \infty$, implying $a_\pm \in L^1(H)$. Thus every trace class operator is a linear combination of four (or less) positive trace class operators.

$(\beta) \Rightarrow (\gamma)$ Let $a \in L^1(H)$ and E an ONB for H. By (β), we have $a = \sum_{k=1}^K \lambda_k a_k$ with $a_k \in L^1(H)^+ \ \forall k$. Now $|\langle ae, e\rangle| \le \sum_{k=1}^K |\lambda_k| \langle a_k e, e\rangle$ for all $e \in E$, so that $a_k \in L^1(H)^+ \ \forall k$ implies $\sum_e |\langle ae, e\rangle| < \infty$. Since $L^1(H)$ is an ideal, the same conclusion if a is replaced by va.

$(\gamma) \Rightarrow (\delta) + (\epsilon)$ is trivial.

$(\delta) \Rightarrow (\alpha)$ Let $a = u|a|$ be the polar decomposition of $a \in B(H)$. Recall that u maps $\overline{|a|H} \subseteq H$ isometrically to $\overline{aH} \subseteq H$ and sends $\overline{|a|H}^\perp$ to zero. Let $v = u^*$ on $\overline{aH} = u\overline{|a|H} \subseteq H$. Since the closed subspaces $\overline{|a|H}$ and $\overline{aH}$ are unitarily equivalent, they have the same dimension, thus also $\overline{|a|H}^\perp$ and $\overline{aH}^\perp$ have the same dimension. Define $v : \overline{aH}^\perp \to \overline{|a|H}^\perp$ to be (any) isometry. Then v is unitary. Since $vu|a|e = u^*u|a|e = |a|e$ for all e, we have

$$\sum_{e\in E} |\langle vu|a|e, e\rangle| = \sum_{e\in E} |\langle |a|e, e\rangle| = \sum_{e\in E} \langle |a|e, e\rangle = \mathrm{Tr}|a|.$$

By assumption, the l.h.s. is finite, thus $\mathrm{Tr}|a| < \infty$. The bound on $\|\cdot\|_1$ is obvious in view of the preceding computation.

$(\epsilon) \Rightarrow (\alpha)$ If a has property (ϵ) then the same clearly holds for a^* and thus for $b = \mathrm{Re}(a)$ and $c = \mathrm{Im}(a)$. Property (ϵ) clearly implies $\lim\langle be, e\rangle \to \infty$. (Thus for each $\varepsilon > 0$ the set $\{e \in E \mid |\langle be, e\rangle| \ge \varepsilon\}$ is finite.) This implies that $b = b^*$ is compact (cf. [3, Lemma 16.17] for the proof, which isn't hard but lengthy), thus diagonalizable, i.e. $b = \sum_{f\in F} \lambda_f \, f \otimes f$ for a certain ONB F and $\lambda_f \in \mathbb{R}$. Now (ϵ) clearly implies $\|b\|_1 = \mathrm{Tr}|b| = \sum_f |\lambda_f| < \infty$, so that $b \in L^1(H)$. Similarly, $c \in L^1(H)$, thus also $a = b + ic$ is trace-class by (iii).

(x) Let E be an ONB. By (ix)(γ), the sum $\sum_{e\in E}\langle ae, e\rangle$ is absolutely convergent for each $a \in L^1(H)$. This proves that $\mathrm{Tr}_E : L^1(H) \to \mathbb{C}$ is well-defined and linear. It remains to show that Tr_E is independent of E. By (ix)(β), we have $a = \sum_{k=1}^K \lambda_k a_k$ where K is finite and $a_k \in L^1(H)^+ \ \forall k$. If now F is another ONB, we have

$$\mathrm{Tr}_E(a) = \sum_k \lambda_k \mathrm{Tr}_E(a_k) = \sum_k \lambda_k \mathrm{Tr}_F(a_k) = \mathrm{Tr}_F(a),$$

where we used the linearity of Tr_E and Tr_F and the fact that $\mathrm{Tr}_E(a_k) = \mathrm{Tr}_F(a_k)$ by $a_k \ge 0$ and Lemma 2.1(ii). This proves $\mathrm{Tr}_E = \mathrm{Tr}_F$.

(xi) If $a \in L^1(H)$ and E is an ONB, we have

$$\mathrm{Tr}(a^*) = \sum_e \langle a^*e, e\rangle = \sum_e \langle e, ae\rangle = \sum_e \overline{\langle ae, e\rangle} = \overline{\mathrm{Tr}(a)},$$

where the last identity comes from absolute convergence and continuity of complex conjugation.

(xii) Let $b \in L^1(H)$ throughout. If u is unitary then $bu, ub \in L^1(H)$ and

$$\mathrm{Tr}_E(bu) = \sum_{e\in E} \langle bue, e\rangle = \sum_{e\in E} \langle ubue, ue\rangle = \sum_{f\in F} \langle ubf, f\rangle = \mathrm{Tr}_F(ub),$$

where $F = uE$ is another ONB. Since Tr does not depend on the ONB, we have $\mathrm{Tr}(ub) = \mathrm{Tr}(bu)$. If now $a \in B(H)$ is arbitrary, we have $a = \sum_{l=1}^4 \lambda_l u_l$ where $\lambda_l \in \mathbb{C}$ and the u_l are unitaries. Then $\mathrm{Tr}(ab) = \sum_l \lambda_l \mathrm{Tr}(u_l b) = \sum_l \lambda_l \mathrm{Tr}(bu_l) = \mathrm{Tr}(ba)$, thus the first claim.

Let $a \in L^1(H)$ with polar decomposition $a = u|a|$. Then

$$\begin{aligned} |\mathrm{Tr}(a)| &= |\sum_e \langle u|a|e, e\rangle| = |\sum_e \langle |a|^{1/2}e, |a|^{1/2}u^*e\rangle| \\ &\leq \sum_e \| \, |a|^{1/2}e\| \, \| \, |a|^{1/2}u^*e\| \\ &\leq \left(\sum_{e\in E} \| \, |a|^{1/2}e\|^2\right)^{1/2} \left(\sum_{e\in E} \| \, |a|^{1/2}u^*e\|^2\right)^{1/2}. \end{aligned}$$

Now, as in the proof of (iii), $\sum_{e\in E} \| \, |a|^{1/2}e\|^2 = \mathrm{Tr}|a|$ and $\sum_{e\in E} \| \, |a|^{1/2}u^*e\|^2 \leq \mathrm{Tr}|a|$. This proves $|\mathrm{Tr}(a)| \leq \|a\|_1$ for all $a \in L^1(H)$.

If now $a \in L^1(H), b \in B(H)$ then $|\mathrm{Tr}(ab)| \leq \|ab\|_1 \leq \|a\|_1 \|b\|$ by (iv). ■

Let H be a Hilbert space. By Theorem 3.3(xii), we have maps $\alpha : L^1(H) \to B(H)^*, a \mapsto \mathrm{Tr}(a\cdot)$ and $\beta : B(H) \to L^1(H)^*, a \mapsto \mathrm{Tr}(a\cdot)$ such that $\|\alpha(a)\| \leq \|a\|_1$ and $\|\beta(a)\| \leq \|a\|$. We will follow Murphy quite closely in proving that the maps $\beta : (B(H), \|\cdot\|) \to (L^1(H), \|\cdot\|_1)^*$ and $\alpha_K : (L^1(H), \|\cdot\|_1) \to (K(H), \|\cdot\|)^*$ are isometric bijections. However:

3.4 PROPOSITION *The map $\alpha : (L^1(H), \|\cdot\|_1) \to (B(H), \|\cdot\|)^*$ is isometric, but it is not surjective if H is infinite dimensional.*

Proof. Since $\|\alpha(a)\| \leq \|a\|_1$ for all $a \in L^1(H)$ and $\|\alpha(a) \restriction K(H)\| = \|a\|_1$ it is clear that also $\|\alpha(a)\| = \|a\|_1$, thus α is isometric.

For each $x, y \in H$ we have $x \otimes y \in K(H)$, and $\mathrm{Tr}(a(x \otimes y)) = \mathrm{Tr}((ax) \otimes y) = (ax, y)$. Thus if $\alpha(a) = \mathrm{Tr}(a\cdot)$ vanishes on the compact operators then $a = 0$, thus $\alpha(a) = 0$.

However, if H is infinite dimensional, $K(H) \subsetneq B(H)$ is a proper closed ideal, and the quotient C^*-algebra $C(H) = B(H)/K(H)$ (known as the Calkin algebra) is non-trivial. Thus it admits a bounded non-zero functional ψ by Hahn-Banach. If $p : B(H) \to C(H)$ is the quotient map, $\psi \circ p$ is a non-zero norm-continuous functional on $B(H)$ that vanishes on $K(H)$. Such a functional cannot be of the form $\mathrm{Tr}(a\cdot)$ with $a \in L^1(H)$, proving that α is not surjective. ■

We will later see that $\alpha(L^1(H)) \subsetneq B(H)^*$ consists precisely of the linear functionals that are not only norm-continuous but also ultra-weakly continuous (or, equivalently, normal).

4 $L^2(H)$: Hilbert-Schmidt operators

Among the 'classical' spaces $L^p(X, \mathcal{A}, \mu)$, the case $p = 2$ is the 'nicest' since it is the only one to lead to a Hilbert space. The same holds for $L^p(H)$.

4.1 PROPOSITION *Let H be a Hilbert space. Then*

(i) $\|a\|_2 = (\mathrm{Tr}(a^*a))^{1/2}$.

(ii) $\|a\| \leq \|a\|_2 = \|a^*\|_2$ *for all* $a \in B(H)$. *Thus* $L^2(H)$ *is self-adjoint.*

(iii) For every $a, b \in L^2(H)$, $\mathrm{Tr}(b^*a) = \sum_e (b^*ae, e)$ is absolutely convergent and independent of the ONB E. Now $(L^2(H), (\cdot,\cdot))$ is a Hilbert space, in particular complete.

(iv) For all $a, b \in B(H)$ we have $\|ab\|_2 \le \|a\|\|b\|_2$ and $\|ab\|_2 \le \|a\|_2\|b\|$. Thus $L^2(H) \subseteq B(H)$ is a two sided ideal.

Proof. (i) This is immediate in view of Definition 2.2 and $|a| = (a^*a)^{1/2}$.

(ii) If $x \in H$ is a unit vector, pick an ONB E containing x. Then $\|ax\|^2 = \langle a^*ax, x\rangle \le \mathrm{Tr}_E(a^*a) = \|a\|_2^2$. Thus $\|ax\| \le \|a\|_2$ whenever $\|x\| = 1$, proving the inequality. And Lemma 2.1(i) gives $\|a^*\|_2^2 = \mathrm{Tr}(aa^*) = \mathrm{Tr}(a^*a) = \|a\|_2^2$.

(iii) If E is any ONB (whose choice does not matter) for H, we have (as before)

$$\mathrm{Tr}(a^*a) = \sum_{e\in E}\langle a^*ae, e\rangle = \sum_{e\in E}\langle ae, ae\rangle = \sum_{e\in E}\|ae\|^2 = \sum_{e\in E}\sum_{e'\in E}|\langle ae, e'\rangle|^2.$$

Thus $L^2(H)$ is the set of $a \in B(H)$ for which the matrix elements $\langle ae, e'\rangle$ (w.r.t. the ONB E) are absolutely square summable. We therefore have a map

$$\alpha : L^2(H) \to \ell^2(E\times E),\ a \mapsto \{\langle ae, e'\rangle\}_{(e,e')\in E^2}$$

that clearly is injective. (Recall that $\ell^2(S) = L^2(S,\mu)$, where μ is the counting measure.) To show surjectivity of α, let $f = \{f_{ee'}\} \in \ell^2(E\times E)$. Define a linear operator $a : H \to H$ by $a : e \mapsto \sum_{e'} f_{ee'}e'$. For each e, the r.h.s. is in H by square summability of f. If $x \in H$ then

$$\begin{aligned}\|ax\|^2 &= \left\|\sum_e \langle x, e\rangle \sum_{e'} f_{ee'}e'\right\|^2 = \left\|\sum_{e'}\left(\sum_e \langle x, e\rangle f_{ee'}\right)e'\right\|^2\\ &= \sum_{e'}\left|\sum_e \langle x, e\rangle f_{ee'}\right|^2 \le \|x\|^2\sum_{e,e'}|f_{ee'}|^2,\end{aligned}$$

where the change of summation order is allowed since $f \in \ell^2(E\times E) \subseteq \ell^1(E\times E)$ and $|\langle x, e\rangle| \le \|x\|\ \forall e \in E$. This computation shows that $\|a\| \le (\sum_{e,e'}|f_{ee'}|^2)^{1/2} < \infty$. Thus $a \in B(H)$ and $\alpha(a) = f$, so that α is surjective. Thus $\alpha : L^2(H) \to \ell^2(E\times E)$ is a linear bijection. Now $\ell^2(E\times E)$ is a Hilbert space (in particular complete) with inner product $\langle f, g\rangle = \sum_{e,e'} f_{ee'}\overline{g_{ee'}}$, and pulling this inner product back to $L^2(H)$ along α we have

$$\begin{aligned}\langle a, b\rangle &= \sum_{(e,e')\in E^2}\langle ae, e'\rangle\overline{\langle be, e'\rangle} = \sum_{(e,e')\in E^2}\langle ae, e'\rangle\langle e', be\rangle\\ &= \sum_e \langle ae, be\rangle = \sum_e \langle b^*ae, e\rangle = \mathrm{Tr}(b^*a),\end{aligned}$$

where all sums converge absolutely. Lemma 2.1(ii) implies that $\langle a, a\rangle = \mathrm{Tr}(a^*a)$ is independent of the chosen ONB, and for general $\langle a, b\rangle$ this follows by the polarization identity. (Later we'll prove $a^*b \in L^1(H)$, but it seems wrong to have the simple and pretty L^2 theory depend on the harder and ugly L^1 theory.)

(iv) For any ONB E we have

$$\|ab\|_2^2 = \mathrm{Tr}(b^*a^*ab) = \sum_{e\in E}\|abe\|^2 \le \|a\|^2\sum_{e\in E}\|be\|^2 = \|a\|^2\mathrm{Tr}(b^*b) = \|a\|^2\|b\|_2^2,$$

proving $\|ab\|_2 \le \|a\|\|b\|_2$. And $\|ab\|_2 = \|(ab)^*\|_2 = \|b^*a^*\|_2 \le \|b^*\|\|a^*\|_2 = \|a\|_2\|b\|$, where we used the fact just proven and $\|a^*\|_2 = \|a\|_2$. The conclusion is obvious. ■

Notice that the above proofs are much 'cleaner' than those for $L^1(H)$ since they don't require the polar decomposition. The existence of the inner product and competeness can also be proven without using the unitary equivalence $L^2(H) \cong \ell^2(E \times E)$, cf. e.g. [5], but then one effectively redoes the well known proofs of completeness and existence of the inner product for the ℓ^2 spaces.

Up to now our developments of $L^1(H)$ and $L^2(H)$ were independent. Now we explore the various connections between $L^1(H)$ and $L^2(H)$:

4.2 PROPOSITION *Let H be any Hilbert space. Then*

(i) If $a, b \in L^2(H)$ then $ab \in L^1(H)$ and $\|ab\|_1 \le \|a\|_2 \|b\|_2$ for all $a, b \in B(H)$.

(ii) For every $c \in L^1(H)$ there are $a, b \in L^2(H)$ such that $c = ab$.

(iii) $F(H) \subseteq L^1(H) \subseteq L^2(H) \subseteq K(H)$ and $\|a\| \le \|a\|_2 \le \|a\|_1$ for all $a \in B(H)$. The spaces $L^1(H), L^2(H), K(H)$ are the closures of $F(H)$ w.r.t. $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|$, respectively.

Proof. (i) Let $a, b \in L^2(H)$. By (4) we have $\sum_e \langle abe, e\rangle = \mathrm{Tr}(ab) = (b, a^*)$. Recall the Cauchy-Schwarz inequality in the form $\sum_{s\in S} |f(s)g(s)| \le \|f\|\|g\|$ for $f, g \in \ell^2(S)$. Applying this to the Hilbert space $(L^2(H), \langle\cdot,\cdot\rangle)$ gives $\sum_e |\langle abe, e\rangle| \le |\langle b, a^*\rangle| \le \|a\|_2\|b\|_2$ (for each ONB E). If v is any unitary, $\|va\|_2 = \|a\|_2$, thus also $\sum_e |\langle vabe, e\rangle| \le \|a\|_2\|b\|_2$. Now the implication $(\delta) \Rightarrow (\alpha)$ of Theorem 3.3(vii) gives $ab \in L^1(H)$ and $\|ab\|_1 \le \|a\|_2\|b\|_2$.

(ii) Let $c \in L^1(H)$, thus $\mathrm{Tr}|c| < \infty$. If $c = u|c|$ is the polar decomposition then $\|\,|c|^{1/2}\|_2 = \mathrm{Tr}(|c|) < \infty$, thus $b = |c|^{1/2} \in L^2(H)$. Then by Proposition 4.1(vi) also $a = u|c|^{1/2} \in L^2(H)$. Now $c = u|c|^{1/2}|c|^{1/2} = ab$ proves the claim.

(iii) The inclusion $F(H) \subseteq L^1(H)$ and $\overline{F(H)}^{\|\cdot\|_1} = L^1(H)$ were proven in Section 2. It is well known that $K(H) = \overline{F(H)}^{\|\cdot\|}$ (sometimes taken as definition of $K(H)$).

If $c \in L^1(H)$, by (ii) we have $c = ab$ with $a, b \in L^2(H)$. Since $L^2(H) \subseteq B(H)$ is a subalgebra, we have $c = ab \in L^2(H)$, thus $L^1(H) \subseteq L^2(H)$.

If $a \in L^2(H)$ and F is a finite subset of the ONB E, define $p_F = \sum_{e\in F} e \otimes e$ and $a_F = ap_F$. Then $a_F \in F(H)$ and

$$\|a - a_F\|_2^2 = \|a(1 - p_F)\|_2^2 = \sum_{(e,e')\in(E\backslash F)\times E} |\langle ae, e'\rangle|^2.$$

This implies $\|a - a_F\|_2 \to 0$ as $F \nearrow E$, so that $L^2(H) = \overline{F(H)}^{\|\cdot\|_2}$.

Furthermore, Proposition 4.1(iii) implies $\|a - a_F\| \to 0$. Thus $a \in \overline{F(H)}^{\|\cdot\|} = K(H)$, proving $L^2(H) \subseteq K(H)$.

We turn to the proof of $\|a\|_2 \le \|a\|_1$, to wit $(\mathrm{Tr}(a^*a))^{1/2} \le \mathrm{Tr}((a^*a)^{1/2})$, for all a. In view of $a^*a \ge 0$ and the fact that $B(H)^+ \to B(H)^+, b \mapsto b^2$ is a bijection, it is equivalent to prove $\mathrm{Tr}(b^2) \le (\mathrm{Tr}(b))^2$ for all $b \ge 0$. We may also assume $\mathrm{Tr}(b) < \infty$. Then b is compact by $L^1(H) \subseteq K(H)$ and therefore diagonalizable: There is an ONB E and numbers $\lambda_e \ge 0$ such that $b = \sum_{e\in E} \lambda_e (e \otimes e)$. Then $b^2 = \sum_{e\in E} \lambda_e^2 (e \otimes e)$, and

$$\mathrm{Tr}(b^2) = \sum_e \lambda_e^2 \le \left(\sum_e \lambda_e\right)^2 = (\mathrm{Tr}(b))^2.$$

■

The connection between $L^1(H)$ and $L^2(H)$ established by Proposition 4.2 can be used to give alternative proofs for some of the facts from Section 2, in particular the construction of the map $\mathrm{Tr} : L^1(H) \to \mathbb{C}$, as is done in [4, 5]. But, as said before, I very much prefer direct arguments as in Section 2.

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