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Superselection Structure of Quantum Field Theories in $1+1$ Dimensions

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Abstract

The subject of this dissertation is the superselection structure of quantum field theories in $1+1$ dimensions in the framework of Algebraic QFT, or Local Quantum Physics. The present work decomposes into two parts. In the first one, Chapter 2, which is concerned with massless models, we investigate the phenomenon of 'degenerate statistics characters', which is equivalent to non-invertibility of Verlinde's matrix S. We give a sufficient criterion for the absence of degenerate sectors. Then we prove Rehren's conjecture to the effect that an application of the construction of charged fields due to Doplicher and Roberts yields a non-degenerate theory. The methods which we use for doing so also lead to new results pertaining to the 'classical' situation in $\geq 2+1$ dimensions.

The larger part of this work is concerned with massive models which satisfy the split property for wedges (SPW). In Chapter 3 we prove in a model independent way that nets of local observables, which satisfy Haag duality and the SPW in the vacuum representation, are quite rigid. They permit neither nontrivial DHR sectors nor representations which are equivalent to the vacuum upon restriction to wedge regions. Furthermore, Haag duality holds in all irreducible locally normal representations. These results are applied to soliton sectors, which by the above are the only remaining representations of physical relevance.

In Chapter 4 we start from the well-known phenomenon that in $1+1$ dimensions the fixpoint net of a massive theory $\mathcal{F}$ under the action of an inner symmetry group G violates Haag duality. Using disorder operators we construct a non-local extension $\hat{\mathcal{F}}$ of the field net $\mathcal{F}$, which is interesting in two respects. Firstly, it allows for an explicit computation of the dual net $\mathcal{A}^d$ of the fixpoint net $\mathcal{A}$. On the other hand, $\hat{\mathcal{F}}$ carries an action of Drinfel'd's quantum double $D(G)$ as an inner symmetry. This symmetry, which describes the spacelike commutation relations via the R-matrix, is non-degenerate in the above sense but spontaneously broken. In analogy to Roberts' treatment of spontaneously broken group symmetries, the violation of Haag duality by $\mathcal{A}$ can be interpreted as a consequence of this spontaneously broken 'hidden' symmetry.

Zusammenfassung

Gegenstand dieser Dissertation ist die Superauswahlstruktur von Quantenfeldtheorien in $1+1$ Raum-Zeit Dimensionen im Rahmen der Algebraischen Quantenfeldtheorie. Die vorliegende Untersuchung zerfällt in zwei Teile. Im ersten Teil, Kapitel 2, der sich auf masselose Modelle bezieht, wird das Phänomen der 'Entartung von Statistikcharakteren', d.h. der Nicht-Invertierbarkeit der Verlinde-Matrix S, näher untersucht. Zunächst wird ein hinreichendes Kriterium für die Abwesenheit entarteter Sektoren angegeben. Anschließend wird Rehrens Vermutung bewiesen, daß man durch Anwendung der Konstruktion geladener Felder von Doplicher und Roberts eine nichtentartete Theorie erhält. Hierzu werden Methoden entwickelt, die auch in der 'klassischen' Situation von $\geq 2+1$ Dimensionen neue Aussagen erlauben.

Der größere Teil dieser Arbeit widmet sich massiven Modellen, die die Split-Eigenschaft für Keilgebiete (SPW) erfüllen. In Kapitel 3 wird in modellunabhängiger Weise gezeigt, daß Netze lokaler Observablen, die Haag-Dualität und die SPW in der Vakuum-Darstellung erfüllen, eine sehr rigide Struktur besitzen. Sie lassen weder nichttriviale DHR-Sektoren zu, noch Darstellungen, die eingeschränkt auf Keilgebiete äquivalent zum Vakuum sind. Darüber hinaus gilt Haag-Dualität in allen irreduziblen, lokal normalen Darstellungen. Anwendungen auf Soliton-Sektoren, die somit die einzig möglichen Darstellungen von physikalischer Relevanz sind, werden gegeben.

Als Ausgangspunkt für Kapitel 4 dient das bekannte Phänomen, daß das Fixpunktnetz $\mathcal{A}$ einer massiven Theorie $\mathcal{F}$ unter einer inneren Symmetriegruppe G in $1+1$ Dimensionen die Haag-Dualität verletzt. Unter Verwendung von Disorder-Operatoren wird eine nichtlokale Erweiterung $\hat{\mathcal{F}}$ des Feldnetzes $\mathcal{F}$ konstruiert, die zwei interessante Eigenschaften besitzt. Zum einen erlaubt sie die explizite Berechnung des zum Fixpunktnetz gehörenden dualen Netzes $\mathcal{A}^d$. Zum anderen trägt $\hat{\mathcal{F}}$ eine Aktion von Drinfel'ds Quantendoppel $D(G)$ als innerer Symmetrie. Diese Symmetrie, die vermittels der R-Matrix auch die raumartigen Vertauschungsrelationen beschreibt, ist nicht-entartet im obigen Sinne, aber spontan gebrochen. In Analogie zu Roberts' Behandlung spontan gebrochener Gruppensymmetrien kann man die Verletzung der Haag-Dualität von $\mathcal{A}$ als Konsequenz dieser spontan gebrochenen 'verborgenen' Symmetrie interpretieren.

Contents

Chapter 1

Introduction and Prerequisites

1.1 Why should anyone want to study QFT in $1+1$ dimensions?

The present dissertation is concerned with several aspects of the superselection structure (=representation theory) for quantum field theories in $1+1$ dimensional Minkowski space. This class of theories has been studied for several decades despite the fact that the lessons they provide for the study of QFTs in 3+1 dimensions may appear limited. In constructive quantum field theory the investigation of theories in less than three space dimensions has been motivated by the less violent nature of the ultraviolet divergences in low dimensional toy worlds. Thus, one begins with low dimensional theories and proceeds to (ultimately) four dimensions via successive improvements of one's technical arsenal. This argument does, however, not apply to general quantum field theory, where one anyhow circumvents the thorny task of proving the existence of a specific model by studying large classes of models which satisfy a reasonably small set of physically meaningful axioms. (To be sure, one of the ultimate aims of general QFT, namely classifying *all* reasonable quantum field theories, is at least as difficult as the construction of individual models.) As it turns out, this axiomatic analysis exhibits a number of phenomena which do not appear in higher dimensions, like braid group statistics, quantum symmetries and exact integrability. In the case of massive models these peculiarities can be traced back to the fact that the spacelike complement of a bounded region (e.g. a double cone $\mathcal{O}$) decomposes into two connected components. Stated alternatively, there is a Lorentz invariant distinction between left and right, which in particular explains the existence of soliton representations. For conformally covariant models, which live on a compactified spacetime manifold, this is not true. There the non-simply connectedness of the latter is responsible for the appearance of the above mentioned phenomena.

The importance of topological peculiarities or 'pathologies' of two dimensional space-time for the respective models might lead practically minded physicists to the judgment that there is little to be learnt from them for four dimensional physics. In the author's opinion, this point of view is not warranted. In fact, most aspects of two dimensional theories show up in some way in four dimensional gauge theories, albeit in a much more complicated form. The nontrivial commutation relations between 'order' and 'disorder' operators (cf. Chapter 4), e.g., reappear in the guise of commutation relations between Wilson and 't Hooft loops, whereas solitons become magnetic monopoles, etc. Furthermore, we mention the recent upsurge of interest of QCD phenomenologists in conformally covariant and/or exactly integrable two dimensional models, e.g. [79]. This is due to the fact that 4d QCD effectively turns into a two dimensional theory in certain asymptotic regimes.

The aim of the present work is to extend the existing body of knowledge on QFT in $1+1$ dimensions in several respects. A short overview of our results will be given in the final section of this introductory chapter. Since our work takes place in the framework of general QFT, more specifically Algebraic QFT, the next section will provide a short introduction to the concepts of the latter as well as some well known results as far as they are independent on the dimension of spacetime. In order to prepare the stage, the third section is devoted to those results which have been proved for theories in $\geq 2+1$ dimensions (or $\geq 3+1$, depending on certain localization properties). They are of relevance since several techniques which have been developed for the high dimensional analysis will be brought to bear on two dimensional models later on. In the next-to-last section we will survey the results on two dimensional models which are rigorously known. It will be apparent that there are still many open questions.

1.2 Algebraic Quantum Field Theory

Conceptually, Algebraic Quantum Field Theory is based on two fundamental ingredients. On the technical or mathematical side it is characterized by its reliance on topological (C^* and von Neumann) $*$-algebras of *bounded* operators instead of the unbounded operator valued distributions of the older Wightman framework [104]. This is motivated by the desire to circumvent the physically meaningless domain problems which plagued the latter. Since every self-adjoint operator is uniquely characterized by its family of (bounded) spectral projections, no information is lost by this shift of perspective. Furthermore, algebras of bounded operators are *much* more convenient when the action of symmetries on a quantum field theory are studied, as is made plain, e.g., by the solution of the reconstruction problem for charged fields in [47], see also Chapter 2. In particular in the context of conformal QFT it is occasionally said that CQFT is somehow related to Galois theory. While there may in fact be relations to the Galois theory for fields, it is seldom appreciated that it is mainly the Galois theory for algebras which is relevant for QFT, irrespective of conformal covariance. Hopefully, our results in Chapter 4 contribute to establishing this insight.

The second pillar on which Algebraic Quantum Field Theory, or *Local Quantum Physics* [71], rests is its emphasis on observable quantities, however problematic this notion may be. The history of QFT has shown that unobservable objects, introduced purely for technical convenience, are by no means unique since they can be submitted to transformations à la Klein, Jordan-Wigner, Bogolubov etc. Furthermore, the existence of superselection sectors is nicely interpreted [70] in terms of inequivalent representations of an (abstract) algebra of observables. This point of view, while put forward as early as 1964 and pursued vigorously in [37, 39], has, unfortunately, become widely accepted only in the 80s in the context of CQFT (where one speaks of the chiral algebra).

We now turn to a short overview of some aspects of local quantum physics, for more detailed introductions see [71, 9, 74]. We consider a quantum field theory to be defined by its net of observables:

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}). \tag{1.2.1}$$

This is a map which assigns to each subset $\mathcal{O}$ of Minkowski space a C^*-algebra $\mathcal{A}(\mathcal{O})$ of observables measurable in $\mathcal{O}$. Commonly it is sufficient to restrict oneself to certain classes of regions, mostly the open double cones $\mathcal{O} \in \mathcal{K}$, which are intersections of forward and backward lightcones:

$$\mathcal{K} = \{V_+ + x \cap V_- + y \mid y - x \in V_+\}. \tag{1.2.2}$$

The assignment (1.2.1) is inclusion preserving (isotonous)

$$\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \tag{1.2.3}$$

which allows to define the quasilocal algebra by

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}. \tag{1.2.4}$$

The algebra $\mathcal{A}(G)$ associated to an arbitrary subset of Minkowski space is then understood to be the subalgebra of $\mathcal{A}$ generated (as a C^*-algebra) by all $\mathcal{A}(\mathcal{O})$ where $G \supset \mathcal{O} \in \mathcal{K}$. The principle of Einstein causality is implemented by requiring the net to be local in the sense that

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\} \tag{1.2.5}$$

if $\mathcal{O}_1, \mathcal{O}_2$ are spacelike to each other. Furthermore, the Poincaré group $\mathcal{P}$ acts on $\mathcal{A}$ by automorphisms $\alpha_{\Lambda,a}$ such that

$$\alpha_{\Lambda,x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x) \;\forall \mathcal{O}. \tag{1.2.6}$$

Up to now our algebraic definitions have been abstract, insofar as no specific representations in some Hilbert space have been involved. This approach is particularly useful if there is more than one vacuum representation.

Now, the basic physical idea of AQFT is that the physical content of any quantum field theory resides in the observables (and their vacuum representation(s)). All other physically relevant representations as well as unobservable charged fields interpolating between those and the vacuum sector should be constructed from the observable data. This requires the specification of a class of representations which are considered as physically significant. Of a reasonable representation π of $\mathcal{A}$ one requires that at least the translations (Lorentz invariance might be broken) are unitarily implemented

$$\pi \circ \alpha_x(A) = U_\pi(x)\pi(A)U_\pi(x)^*, \tag{1.2.7}$$

the generators of the representation $x \mapsto U(x)$, i. e. the energy-momentum operators, satisfying the spectrum condition. Vacuum representations are then characterized by the existence of a unique (up to a phase) translation invariant vector Ω. Furthermore, they should be irreducible, i.e. $\pi_0(\mathcal{A})' = \mathbb{C}\mathbf{1}$ *.

The requirement of positive energy (stability) is, however, not sufficiently selective and an early attempt by Borchers was plagued by technical problems. Only in 1982, Buchholz and Fredenhagen succeeded in analyzing a large class of physically relevant representations from first principles. Improving on earlier ideas of Swieca they proved [18] for every massive one-particle representation (i.e. there is a mass gap in the spectrum followed by an isolated one-particle hyperboloid) in $\geq 2+1$ dimensions the existence of a vacuum representation π_0 such that

$$\pi \restriction \mathcal{A}(\mathcal{C}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{C}') \quad \forall \mathcal{C}. \tag{1.2.8}$$

($\cong$ means unitary equivalence.) The $\mathcal{C}$'s are spacelike cones which we do not need to define precisely; essentially they are half-lines which thicken as they go to infinity. In $1+1$ dimensions (here the spacelike cones reduce to wedges, i. e. translates of $W_R = \{x \in \mathbb{R}^2 \mid x^1 > |x^0|\}$ and the spacelike complement $W_L = W_R'$) things are more complicated. In this case the arguments in [18] allow only to conclude the existence of two a priori different vacuum representations π_0^L, π_0^R such that the restriction of π to left handed wedges (translates of W_L) is equivalent to π_0^L and similarly for the right handed ones. Such representations are of course well known as soliton representations. In analogy to the situation in $\geq 2+1$ dimensions we will speak of 'BF representations' should the vacua π_0^L, π_0^R happen to coincide. Since the analysis of statistics which builds upon the BF localization or, alternatively, on the earlier DHR criterion is strongly dependent on the dimensionality of spacetime we defer its further exposition to the subsequent sections.

With the exception of our discussion of soliton sectors in Chapter 3 we pick some definite vacuum representation π_0 and identify the local algebras with their images in $\mathcal{B}(\mathcal{H}_0)$: $\mathcal{A}(\mathcal{O}) \equiv \pi_0(\mathcal{A}(\mathcal{O}))$. Since the quasilocal algebra is simple [3], all representations are faithful and no information is lost in this way. From now on we will mostly omit the symbol π_0. In algebraic QFT one usually may assume that any two representations π_1, π_2 are unitarily equivalent upon restriction to every double cone $\mathcal{O}$ (local normality). It thus makes sense to require $\pi(\mathcal{A}(\mathcal{O}))$ to be a von Neumann algebra (i.e. ultraweakly closed) for all π under consideration and all $\mathcal{O} \in \mathcal{K}$. For infinite regions like spacelike complements $\mathcal{O}'$ and wedges W, one must, however, carefully distinguish between the C^*-subalgebras $\mathcal{A}(W) \equiv \pi_0(\mathcal{A}(W))$ of $\pi_0(\mathcal{A})$ and their weak closures $\mathcal{R}(W) = \mathcal{A}(W)''$.

For reasons which will become clear later, one usually further assumes [69, 70] that vacuum representations satisfy *Haag duality*:

$$\pi_0(\mathcal{A}(\mathcal{O}))' = \pi_0(\mathcal{A}(\mathcal{O}'))'' \quad \forall \mathcal{O} \in \mathcal{K}. \tag{1.2.9}$$

This property, which strengthens the locality postulate in the sense that the local algebras cannot be enlarged without violating spacelike commutativity, plays a prominent role in superselection theory. In the context of conformally covariant theories [16] in suitably compactified [84] Minkowski space ($\geq 1+1$ dimensions), as well as for Möbius covariant

*In general $\mathcal{M}' = \{X \in \mathcal{B}(\mathcal{H}) | XY = YX \, \forall Y \in \mathcal{M}\}$ denotes the algebra of all bounded operators commuting with all operators in $\mathcal{M}$.

theories on the circle [64], Haag duality can be proved from first principles. As to non-conformal models, this duality property (or its adaption for fermions, see below) has up to now only been proved for free massive and massless fields (scalar [5] and Dirac [35]) in $\geq 1+1$ dimensions (apart from the massless scalar field in two dimensions) as well as for several interacting theories $(P(\phi)_2, Y_2)$.

Therefore it is advisable to be prepared for the case where Haag duality does not hold in the first place. Introducing the *dual net* by

$$\mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')', \tag{1.2.10}$$

one has $\mathcal{A}^d(\mathcal{O}) \supset \mathcal{A}(\mathcal{O})$ and Haag duality amounts to $\mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O})\ \forall \mathcal{O} \in \mathcal{K}$. $\mathcal{A}$ is said to satisfy *essential duality* if the extended net $\mathcal{A}^d$ is still local. In this case one can easily prove $\mathcal{A}^d$ to satisfy Haag duality: $\mathcal{A}^{dd}(\mathcal{O}) = \mathcal{A}^d(\mathcal{O})$. Now, for nets of local observables which arise from a Wightman theory one can prove [12] another duality property, namely *wedge duality*:

$$\mathcal{R}(W)' = \mathcal{R}(W') \quad \forall W \in \mathcal{W}. \tag{1.2.11}$$

Here $\mathcal{W}$ denotes the set of all wedges, i.e. regions arising from the standard wedges W_L, W_R (defined exactly as above for the two dimensional case) via Poincaré transformations. Wedge duality can be shown to imply essential duality, and the dual net is given by the intersection of all wedge algebras containing a given double cone:

$$\mathcal{A}^d(\mathcal{O}) = \bigcap_{W \in \mathcal{W}, W \supset \mathcal{O}} \mathcal{R}(W). \tag{1.2.12}$$

Thus the property of Haag duality can always be obtained. A priori, it is, however, unclear how the superselection structures of the nets $\mathcal{A}$ and $\mathcal{A}^d$ are related. This question will play an important role in this work.

In the rest of this section we introduce another technical property which will play an equally important role in our investigations as Haag duality does. This is motivated by the observation that the postulates of local quantum physics as described up to now, i.e. isotony, locality, covariance etc., are not sufficient to guarantee that a quantum field theory is physically acceptable insofar as particle states exist and there is a reasonable thermodynamical behavior. Therefore Borchers (unpublished) proposed the *split property* which formalizes the idea that the local algebras of two regions which are separated by a finite spacelike distance are statistically independent in a sense which goes beyond local commutativity. For a detailed review of the numerous concepts which exist in the literature we refer to [106]. The notion which will be of relevance for us is the one of W^*-independence. We say that the split property (for double cones) holds in a representation π, provided that

$$\pi(\mathcal{A}(\mathcal{O}_1)) \vee \pi(\mathcal{A}(\mathcal{O}_2'))'' \simeq \pi(\mathcal{A}(\mathcal{O}_1)) \otimes \pi(\mathcal{A}(\mathcal{O}_2'))'', \tag{1.2.13}$$

whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$, which is an abbreviation for $\overline{\mathcal{O}_1} \subset \mathcal{O}_2^0$. This means that the von Neumann algebra on $\mathcal{H}_\pi$ generated by $\pi(\mathcal{A}(\mathcal{O}_1))$ and $\pi(\mathcal{A}(\mathcal{O}_2'))''$ is algebraically isomorphic to the tensor product $\pi(\mathcal{A}(\mathcal{O}_1)) \otimes \pi(\mathcal{A}(\mathcal{O}_2'))''$ (which, of course, lives on $\mathcal{H}_\pi \otimes \mathcal{H}_\pi$). Again, we will typically omit the symbol π whenever we mean the vacuum representation.

The following obvious consequence should make plain the physical relevance of this property and justify the interpretation in terms of statistical independence. Given any pair of normal (ultraweakly continuous) states ϕ_1, ϕ_2 on $\pi(\mathcal{A}(\mathcal{O}_1)),\ \pi(\mathcal{A}(\mathcal{O}_2'))''$, there is a normal state ϕ on $\pi(\mathcal{A}(\mathcal{O}_1)) \vee \pi(\mathcal{A}(\mathcal{O}_2'))''$ which restricts to ϕ_i on the respective subalgebras. In [4] several other characterizations of the split property are given.

Now, in the vacuum sector of quantum field theories one has more structure, in that the vacuum vector Ω (in fact every vector which is analytic for the energy) is cyclic and separating for the local algebras by the Reeh-Schlieder theorem (see, e.g., [3]). In particular, Ω is cyclic and separating for $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ and the relative commutant $\mathcal{A}(\mathcal{O}_2) \cap \mathcal{A}(\mathcal{O}_1)'$, rendering $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ a 'standard split inclusion' [41]. This entails that the isomorphism (1.2.13) is spatial, i.e. unitarily implemented. There is even a canonical choice of this implementer, cf. [41, 20] and Subsection 4.3.6. Given an inclusion of double cones $\Lambda = (\mathcal{O}_1, \mathcal{O}_2),\quad \mathcal{O}_1 \subset\subset \mathcal{O}_2$ there is a unitary $Y^\Lambda : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ such that

$$Y^\Lambda\, A_1 A_2\, Y^{\Lambda *} = A_1 \otimes A_2 \quad \forall A_1 \in \mathcal{A}(\mathcal{O}_1), A_2 \in \mathcal{A}(\mathcal{O}_2')''. \tag{1.2.14}$$

Under this unitary mapping we have the following spatial isomorphisms:

$$\begin{array}{rcccc} \mathcal{A}(\mathcal{O}_1) & \cong & \mathcal{A}(\mathcal{O}_1) & \otimes & \mathbf{1}, \\ \mathcal{A}(\mathcal{O}_2')'' & \cong & \mathbf{1} & \otimes & \mathcal{A}(\mathcal{O}_2')'', \\ \mathcal{A}(\mathcal{O}_2) & \cong & \mathcal{B}(\mathcal{H}_0) & \otimes & \mathcal{A}(\mathcal{O}_2), \end{array} \tag{1.2.15}$$

where we have used Haag duality to obtain the last identity. In particular, one can see that there is a type I factor (i.e. a von Neumann algebra, which is isomorphic to $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ appropriate) $N = Y^{\Lambda *}(\mathcal{B}(\mathcal{H}_0) \otimes \mathbf{1}) Y^\Lambda$ sitting between $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$. This fact is well known to imply hyperfiniteness of the local algebras, and it will play an important role also in our investigations. One of the most important applications of the split property is the construction of local charges [20] for field nets, which gives rise to a version of Noether's theorem in general quantum field theory. This will play an important role in Chapter 4.

The split property has been proved for free massive [17] and massless [21] scalar fields as well as for massive Dirac fields in two and four dimensions. For the superrenormalizable theories in two dimensions it follows then by local normality. Physically intuitive as the postulate of statistical independence for spacelike separated regions, i.e. the split property, is, one would like to derive it from a more fundamental notion. To this purpose the 'nuclearity criterion' has been put forward by Buchholz and Wichmann [19], strengthening an earlier compactness criterion due to Haag and Swieca. Besides implying the split property, this condition entails [23] the existence of KMS- (i.e. thermal) states for all temperatures. This is particularly gratifying as thermodynamic considerations had played an important role [19] in motivating the nuclearity criterion. For further developments see, e.g., [24, 25, 26]). Since in Chapters 3-4 we will make use of a stronger version of the split property than the one described above, we will in Appendix B give an adapted nuclearity-type criterion which implies this variant.

1.3 Superselection Theory in $\geq 2+1$ Dimensions

Despite the fact that in this work we are concerned with quantum field theories in low spacetime dimensions (mainly $1+1$), some familiarity with the results pertaining to higher dimensions is indispensable. This is due to the fact that, of course, part of the analysis for the physical spacetime ($d = 3+1$) carries over to the situation under study. On the other hand, this knowledge is necessary in order to appreciate the specifically low-dimensional phenomena. Therefore, this section is intended as a review of the most important known facts concerning the superselection theory of quantum field theories in at least two space dimensions. For such theories there is a large amount of beautiful results, which nicely exemplify the harmony between ideas arising in quantum field theory on one hand and in the mathematical areas of operator algebras and abstract harmonic analysis on the other. Particular instances will be the statistical dimension [37, 56] vs. Jones index [81] and fusion of sectors [37, 39] vs. abstract group duality [46, 47]. It should, however, be mentioned that there are very important unsolved problems, in particular the superselection structure of theories containing massless particles, and even the notion of local gauge invariance is ill understood. (It should perhaps be emphasized that these problems are not just shortcomings of the algebraic approach. Quite to the contrary, the number of rigorous results on these matters is scarce whatever framework is used for the analysis, e. g. constructive quantum field theory.)

More than ten years before the results on massive one-particle representations [18] were established, Doplicher, Haag, and Roberts initiated a first approach to superselection theory, implicitly assuming the physical $3+1$-dimensional spacetime. The starting point for the preliminary analysis conducted in [35] is a net of field algebras $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ acted upon by a compact group G of inner symmetries (gauge group of the first kind):

$$\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O}) \ \forall g \in G. \tag{1.3.1}$$

The field algebra acts irreducibly on a Hilbert space $\mathcal{H}$ and the gauge group is unbroken, i.e. represented by unitary operators $U(g)$ in a strongly continuous way: $\alpha_g(F) = Ad\,U(g)(F)$. (Compactness of G need in fact not be postulated, as it follows by [41, Theorem 3.1] if the field net satisfies the split property.)

The field net is supposed to fulfill Bose-Fermi commutation relations, i.e. any local operator decomposes into a bosonic and a fermionic part $F = F_+ + F_-$ such that for spacelike separated F and G we have

$$[F_+, G_+] = [F_+, G_-] = [F_-, G_+] = \{F_-, G_-\} = 0. \tag{1.3.2}$$

The above decomposition is achieved by

$$F_\pm = \frac{1}{2}(F \pm \alpha_k(F)), \tag{1.3.3}$$

where k is an element of order 2 in the center of the group G. $V \equiv U_k$ is the unitary operator which acts trivially on the space of bosonic vectors and like $-\mathbf{1}$ on the fermionic ones. To formulate this locality requirement in a way more convenient for later purposes we introduce the twist operation $F^t = ZFZ^*$, where

$$Z = \frac{1+iV}{1+i}, \quad (\Rightarrow Z^2 = V). \tag{1.3.4}$$

This leads to $ZF_+Z^* = F_+,\ ZF_-Z^* = iVF_-$, implying $[F, G^t] = 0$. The (twisted) locality postulate (1.3.2) can now be stated simply as

$$\mathcal{F}(\mathcal{O})^t \subset \mathcal{F}(\mathcal{O}')'. \tag{1.3.5}$$

In analogy to the bosonic case, this can be strengthened to twisted duality:

$$\mathcal{F}(\mathcal{O})^t = \mathcal{F}(\mathcal{O}')'. \tag{1.3.6}$$

The observables are now defined as the fixpoints under the action of G:

$$\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G = \mathcal{F}(\mathcal{O}) \cap U(G)'. \tag{1.3.7}$$

The Hilbert space $\mathcal{H}$ decomposes as follows:

$$\mathcal{H} = \bigoplus_{\xi \in \hat{G}} \mathcal{H}_\xi \otimes \mathbb{C}^{d_\xi}, \tag{1.3.8}$$

where ξ runs through the equivalence classes of finite dimensional continuous unitary representations of G and d_ξ is the dimension of ξ. The observables and the group G act reducibly according to

$$\begin{array}{rclccc} A & = & \bigoplus_{\xi \in \hat{G}} & \pi_\xi(A) & \otimes & \mathbf{1}, \\ U(g) & = & \bigoplus_{\xi \in \hat{G}} & \mathbf{1}_{\mathcal{H}_\xi} & \otimes & U_\xi(g), \end{array} \tag{1.3.9}$$

where π_ξ and U_ξ are irreducible representations of $\mathcal{A}$ and G, respectively. As a consequence of twisted duality for the fields, the restriction of the observables $\mathcal{A}$ to a simple sector (subspace $\mathcal{H}_\xi$ with $d_\xi = 1$), in particular the vacuum sector, satisfies Haag duality. Since the unitary representation of the Poincaré group commutes with G, the restriction of $\mathcal{A}$ to $\mathcal{H}_0$ satisfies all requirements for a net of observables in the vacuum representation in the sense of Section 2.

One can furthermore prove that the representations π_ξ, though mutually inequivalent, are 'strongly locally equivalent', i.e. unitarily equivalent when restricted to the algebra $\mathcal{A}(\mathcal{O}')$ of the causal complement of any double cone. In particular they satisfy the 'DHR-criterion':

$$\pi_\xi \restriction \mathcal{A}(\mathcal{O}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{O}') \quad \forall \mathcal{O} \in \mathcal{K}. \tag{1.3.10}$$

In [37, 39] the perspective was reversed, starting with a net of observables in a vacuum representation satisfying Haag duality (1.2.9). A first class of interesting representations was singled out by imposing the DHR criterion (1.3.10). In the case of a fixpoint net as above it is not necessarily true that the representations π_ξ in (1.3.9) exhaust all equivalence classes of DHR representations (as is seen by taking, e.g., $\mathcal{F} = \mathcal{A}$ and $G = \{e\}$). This is, however, the case [98] whenever the field net does not possess nontrivial localized representations (equivalently, $\mathcal{F}$ has 'quasitrivial 1-cohomology'). It is clear that the DHR criterion is more restrictive than the BF localization property (1.2.8), which in turn does not cover charged representations in QED due to Gauss' law. On the other hand, for conformally covariant theories in compactified Minkowski space ($\geq 1+1$ dimensions), as well as on the circle one can in fact prove [22] that *all* positive energy representations are of the DHR type. Essentially, this is due to local normality in conjunction with the fact that the spacelike complement of a double cone or interval is again a double cone or interval, respectively. Combined with the automatic validity of Haag duality in conformal models this fact shows that at least these models fit perfectly into the DHR framework. In this introduction, we restrict ourselves to indicating a few steps of the DHR analysis in $\geq 2+1$ dimensions, referring to [37, 39] for the whole story.

Let π be a DHR representation and $\mathcal{O} \in \mathcal{K}$ a double cone. By (1.3.10) there is a unitary operator $X^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_\pi$ such that $X^{\mathcal{O}}\,\pi_0(A) = \pi(A)\,X^{\mathcal{O}} \;\; \forall A \in \mathcal{A}(\mathcal{O}')$. Thus the equivalent representation

$$\rho(\cdot) = X^{\mathcal{O}*}\,\pi(A)\,X^{\mathcal{O}}, \tag{1.3.11}$$

which lives on $\mathcal{H}_0$, satisfies

$$\rho(A) = A \;\; \forall A \in \mathcal{A}(\mathcal{O}'). \tag{1.3.12}$$

The importance of Haag duality in the vacuum representation π_0 arises from the following easy consequence: Let $\mathcal{O}_1$ be a double cone containing $\mathcal{O}$. Then ρ maps $\mathcal{A}(\mathcal{O}_1)$ into itself, which implies that ρ is an endomorphism of the quasilocal algebra $\mathcal{A}$, localized in the sense of (1.3.12) . Furthermore, given another double cone $\mathcal{O}_2$ there is an endomorphism ρ_2 of $\mathcal{A}$ which is localized in $\mathcal{O}_2$ and which is equivalent to ρ: There is a unitary T such that $T\rho(A) = \rho_2(A)T \;\forall A \in \mathcal{A}$. By Haag duality, T is contained in $\mathcal{A}(\hat{\mathcal{O}})$ whenever $\mathcal{K} \ni \hat{\mathcal{O}} \supset \mathcal{O} \cup \mathcal{O}_2$. In this brief discussion we have ignored the subtleties which arise in conformal theories due to the compactified spacetime, cf. [57].

The importance of these findings stems from the fact that endomorphisms can be composed, providing the set of DHR representations with a monoidal (product) structure. Physically the composition of morphisms amounts to an 'addition' of charges. One can prove that two localized endomorphisms ρ_1, ρ_2 commute if their localization regions are mutually spacelike. Even if this is not the case, $\rho_1\rho_2$ and $\rho_2\rho_1$ are equivalent up to an inner automorphism, i.e., there is a unitary $\varepsilon(\rho_1, \rho_2) \in \mathcal{A}$ such that

$$\varepsilon(\rho_1, \rho_2)\,\rho_1\rho_2(\cdot) = \rho_2\rho_1(\cdot)\,\varepsilon(\rho_1, \rho_2), \tag{1.3.13}$$

or $\varepsilon(\rho_1, \rho_2) \in (\rho_1\rho_2, \rho_2\rho_1)$. These *statistics operators* are defined as follows. Let ρ_1', ρ_2' be mutually spacelike localized, thus commuting, endomorphisms which are equivalent to ρ_1, ρ_2, respectively, with intertwiners $U_i \in (\rho_i, \rho_i')$. Then a short computation shows that the unitary operator $\rho_2(U_1^*)U_2^*U_1\rho_1(U_2)$ intertwines $\rho_1\rho_2$ and $\rho_2\rho_1$ as required. This operator is canonically defined in that it is independent of the choice of the auxiliary objects ρ_i', U_i'. Furthermore, it satisfies the following important identities (among others):

$$\varepsilon(\rho, id) = \varepsilon(id, \rho) \;\; = \;\; \mathbf{1}, \tag{1.3.14}$$

$$\varepsilon(\rho_1, \rho_2)\,\varepsilon(\rho_2, \rho_1) \;\; = \;\; \mathbf{1}, \tag{1.3.15}$$

$$\varepsilon(\rho_1\rho_2, \rho_3) = \rho_1(\varepsilon(\rho_2, \rho_3))\,\varepsilon(\rho_1, \rho_3). \tag{1.3.16}$$

Defining $\sigma_i = \rho^{i-1}(\varepsilon(\rho, \rho)),\; i \in \mathbb{N}$ one has $\sigma_i^2 = \mathbf{1}$, $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}\;\forall i$ and $[\sigma_i, \sigma_j] = 0,\; |i-j| \geq 2$, such that the σ_i constitute a presentation of the infinite permutation group S_∞ by unitary operators in $\mathcal{A}$.

Whereas ρ is not invertible if it is a true endomorphism, one can prove the existence of left inverses ϕ_ρ such that $\phi_\rho \circ \rho = id$. Then $\rho \circ \phi_\rho$ is a conditional expectation of

$\mathcal{A}$ onto $\rho(\mathcal{A})$. Using the left inverse one obtains, for ρ irreducible, a number λ_ρ via $\phi(\varepsilon(\rho,\rho)) = \lambda_\rho \mathbf{1}$. As a consequence of permutation group statistics, the absolute value of λ_ρ turns out to be zero or the inverse of an integer, the statistical dimension d_ρ, whereas the modulus, the statistics phase ω_ρ, equals $\pm\mathbf{1}$ and distinguishes representations with bosonic and fermionic character. The statistical dimension measures the deviation from Haag duality in the representation π and has an interpretation in terms of the Jones index of certain inclusions of von Neumann algebras due to Longo [81]:

$$d_\rho^2 = [\pi(\mathcal{A}(\mathcal{O}'))' : \pi(\mathcal{A}(\mathcal{O}))] = [\mathcal{A}(\mathcal{O}) : \rho(\mathcal{A}(\mathcal{O}))], \tag{1.3.17}$$

where in the first equation $\mathcal{O} \in \mathcal{K}$ is arbitrary and in the second equation $\rho \cong \pi$ is an endomorphism localized in $\mathcal{O}$. Representations with infinite statistics (i.e. $\lambda_\rho = 0$) are considered pathological, and in fact massive one particle representations have finite statistics [55].

In the case of observables arising as group fixpoints and a DHR-representation π_ξ obtained as in (1.3.9), d_ρ equals the dimension d_ξ of the representation U_ξ in (1.3.8) and, equivalently, the multiplicity of the representation π_ξ in the Hilbert space $\mathcal{H}$. Thus, d_ρ measures the degree of parastatistics. Furthermore [38], ρ is the restriction to $\mathcal{A}$ of an inner endomorphism of $\mathcal{F}$. An inner endomorphism is of the form

$$\rho(\cdot) = \sum_{i \in I} \psi_i \cdot \psi_i^*, \tag{1.3.18}$$

where $\{\psi_i,\ i \in I\}$ is a multiplet of isometries with support one, i.e. satisfies

$$\psi_i^* \psi_j = \delta_{i,j} \mathbf{1}, \tag{1.3.19}$$

$$\sum_{i \in I} \psi_i \psi_i^* = \mathbf{1}. \tag{1.3.20}$$

In [38] it was conjectured that the fixpoint situation is generic in the following sense. Let a net of observables $\mathcal{A}$ satisfying Haag duality be given. Then the strict symmetric monoidal C^*-category of DHR sectors together with their intertwiners is equivalent to the representation category of a compact group G. One can construct a field net $\mathcal{F}$ with normal (i.e. Bose/Fermi) commutation relations acted upon by G such that the decomposition (1.3.8, 1.3.9) contains all equivalence classes of DHR representations of $\mathcal{A}$. The charged fields in $\mathcal{F}$ implement the DHR endomorphisms in the above way. In the case where all DHR representations correspond to localized automorphisms, G thus being abelian, this was proved already in [36]. In complete generality the proof turned out quite difficult, and constitutes one of the triumphs of the algebraic approach. From the mathematical point of view it amounts to a new duality theory for compact groups [46] which considerably improves on the old Tannaka-Krein theory. A crucial role was played by the Cuntz algebra $\mathcal{O}_d$, which is the abstract C^*-algebra generated by d isometries satisfying (1.3.19,1.3.20), see [43]. We refrain from going into details and refer to the series of papers by Doplicher and Roberts [42, 43, 44, 45, 46, 47] for the whole story and to the first two sections of [47] for a relatively nontechnical introduction.

As stated above, the fixpoint net $\mathcal{A} = \mathcal{F}^G$ of a (twisted) dual field net satisfies Haag duality in the vacuum sector if the group G is unbroken, i.e. $\omega_0 \circ \alpha_g = \omega_0\ \forall g \in G$

where $\omega_0 = (\Omega, \cdot\,\Omega)$. If only a subgroup $G_0 \subset G$ is unbroken, the same argument yields duality for $B(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{G_0}$ in restriction to $\mathcal{H}^{G_0}$. Clearly, the smaller net $\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G$ violates duality, but essential duality is still true [95] and $\mathcal{B}(\mathcal{O}) = \mathcal{A}^d(\mathcal{O})$ (on $\mathcal{H}^{G_0}$). Based on these observations, one interprets the weaker property of essential duality as a sign of spontaneous symmetry breakdown even in the case of nets of observables which are not known (a priori) to arise from a field net. In this situation it is known [98, 99] that one can extend every DHR representation of $\mathcal{A}$ in a unique way to a DHR representation of the dual net $\mathcal{A}^d$. Applying the DR reconstruction to the dual net one obtains the field net and the unbroken part G_0 of the symmetry group G. The full group G is then obtained as the group of all local symmetries of $\mathcal{F}$ which leave the observables $\mathcal{A}$ pointwise invariant, cf. [47, 29].

Concerning the split property, it is known that it is fulfilled by the observables provided it is true for the fields. The conjecture that the converse is also true has, however, been proved only for the case of finite abelian gauge groups [40].

All of the above results hold in at least $2+1$ spacetime dimensions. The DR construction of charged fields works also for the less restrictive BF criterion (1.2.8), provided the number of space dimensions is at least three. (Due to the weaker localization properties the transition to braid group statistics and the loss of group symmetry, cf. the next section, occur already in $2+1$ dimensions, see [62].)

1.4 Known Results on Low Dimensions

As explained in the first section, quantum field theories in $1+1$ dimensions permit phenomena which differ markedly from those in higher dimensions. It is well known [61] that the possibility of a Lorentz invariant distinction between left and right allows the commutation relations between charge carrying fields to depend in an essential way on the ordering of the fields. The same fact is responsible for the existence of solitons which, in contrast to DHR- and BF- representations, cannot be considered as localizable excitations with respect to a single vacuum representation. Solitons will be discussed in more detail below.

For the moment we will stick to the study of localized charges. The analysis of representations satisfying the DHR criterion [56, 91, 57] proceeds in analogy to the higher dimensional case up to the definition of statistics operators. Due to the fact that the spacelike complement of a bounded region has two components, there are two choices for the relative localization of the auxiliary morphisms ρ_1', ρ_2'. For definiteness, one defines $\varepsilon(\rho_1, \rho_2)$ by choosing ρ_1' to be localized to the right of ρ_2'. Then $\varepsilon(\rho_1, \rho_2)$ and $\varepsilon(\rho_2, \rho_1)^*$ are two a priori different statistics operators. In particular, $\varepsilon(\rho, \rho)^2 = \mathbf{1}$ will typically be false, such that the σ_i only give a presentation of the braid group $\mathcal{B}_\infty$. Under these circumstances the result of [37], according to which the statistical dimension d_ρ must be an integer, is no more true, yet the relation (1.3.17) stays valid. It is still not known by which structure the compact group appearing in the higher dimensional situation has to be replaced, if a completely general solution to this question exists at all.

The deviation from permutation group statistics can be measured by the monodromy operators

$$\varepsilon_M(\rho_1, \rho_2) = \varepsilon(\rho_1, \rho_2)\varepsilon(\rho_2, \rho_1). \tag{1.4.1}$$

An irreducible morphism ρ is said to be degenerate if $\varepsilon_M(\rho, \sigma) = \mathbf{1}$ for all σ. Given two irreducible morphisms ρ_1, ρ_2 one obtains the $\mathbb{C}$-number valued *statistics character* [91] via

$$Y_{ij}\mathbf{1} = d_i d_j \, \phi_j(\varepsilon_M(\rho_i, \rho_j)^*). \tag{1.4.2}$$

(The factor $d_i d_j$ has been introduced for later convenience.) These numbers depend only on the sectors and satisfy the following identities:

$$Y_{0i} = Y_{i0} = d_i, \tag{1.4.3}$$

$$Y_{ij} = Y_{ji} = Y^*_{i\bar{j}} = Y_{\bar{\imath}\bar{j}}, \tag{1.4.4}$$

$$Y_{ij} = \sum_k N_{ij}^k \frac{\omega_i \omega_j}{\omega_k} d_k, \tag{1.4.5}$$

$$\frac{1}{d_j} Y_{ij} Y_{kj} = \sum_m N_{ik}^m Y_{mj}. \tag{1.4.6}$$

Here $[\rho_{\bar{\imath}}]$ is the conjugate morphism of $[\rho_i]$ and $N_{ij}^k \in \mathbb{N}_0$ is the multiplicity of $[\rho_k]$ in the decomposition of $[\rho_i \rho_j]$ into irreducible morphisms. The matrix of statistics characters is of particular interest if the theory is *rational*, i.e. has only a finite number of inequivalent irreducible representations. Then, as proved by Rehren [91], the matrix Y is invertible iff there is no degenerate morphism besides the trivial one which corresponds to the vacuum representation. In the nondegenerate case the number $\sigma = \sum_i d_i^2 \omega_i^{-1}$ satisfies $|\sigma|^2 = \sum_i d_i^2$ and the matrices

$$S = |\sigma|^{-1} Y, \quad T = \left(\frac{\sigma}{|\sigma|}\right) Diag(\omega_i) \tag{1.4.7}$$

are unitary and satisfy the relations

$$S^2 = (ST)^3 = C, \quad TC = CT = T, \tag{1.4.8}$$

where $C_{ij} = \delta_{i,\bar{j}}$ is the charge conjugation matrix. I.e., S and T constitute a representation of the modular group $SL(2, \mathbb{Z})$. Furthermore, the 'fusion coefficients' N_{ij}^k are given by the Verlinde relation [113]

$$N_{ij}^k = \sum_m \frac{S_{im} S_{jm} S^*_{km}}{S_{0m}}. \tag{1.4.9}$$

As was emphasized in [91], these relations hold independently of conformal covariance in every (nondegenerate) two dimensional theory with finitely many DHR sectors. This is remarkable, since the equation (1.4.9) first appeared in the context of conformal quantum field theory on the torus, where the S-matrix has the additional (and in [113], defining) property of describing the behavior of the conformal characters $tr_{\pi_i} e^{-\tau L_0}$ under the inversion $\tau \to -1/\tau$.

The equations (1.4.8, 1.4.9) do not hold if the matrix Y fails to be invertible, i.e. when there are degenerate sectors. On can show that the set of degenerate sectors is stable under composition and reduction into irreducibles (Lemma 2.4.2). It thus constitutes a closed subcategory of the category of DHR endomorphisms to which one can apply the DR construction of charged fields. It was conjectured in [91] that the resulting 'field' net is nondegenerate, the above Verlinde-type analysis thus being applicable (provided the enlarged theory is rational). These matters will be examined in detail in Chapter 2. The apparent generality of the analysis in [56, 91] can lead (and so it did) to the impression that, from the point of view of superselection theory, there are no essential differences between conformal and massive models in $1+1$ dimensions. In Chapter 3 we will, however, see that there are considerable differences.

Besides the appearance of braid group statistics another well known consequence of the topology of low dimensional Minkowski space is the existence of soliton sectors. Rigorous treatments in the frameworks of constructive and general quantum field theory were first given in [60] and [58, 59], respectively. A short review of Fredenhagen's analysis of solitons in the algebraic framework will be given in Section 3.6. As for solitons an operation of composition can only be defined if the 'vacua fit together' [58], there is in general no such thing as permutation or braid group statistics.

1.5 Overview of the Present Work

In this section we give an overview of the results obtained in this thesis. Our results can be divided naturally into two parts. The first part consists of several studies which are primarily of relevance for conformally covariant models in $1+1$ dimensions whereas the second is concerned with massive models, as characterized by the split property for wedges. Whereas the usual split property (for double cones) as defined in Section 2 makes perfect sense in conformally covariant models, the analogous statement for wedges is considerably stronger and can hold only in massive models. The reader is advised to keep this fact in mind, in order not to confuse the results in the Chapters 2 and 3-4, respectively.

In Chapter 2 we begin with the proof of a few new results on the Doplicher-Roberts construction of charged fields in $\geq 2+1$ dimensions. They are of independent interest since they contribute to the clarification of the relation between observable and field algebras. We then turn to $1+1$ dimensional theories, giving a characterization of degenerate endomorphisms and proving a sufficient condition for the absence of degenerate sectors. In sufficiently regular theories this condition is probably also necessary, but this remains conjectural. Finally we give a proof of Rehren's conjecture that the net which is obtained by applying the DR construction to the degenerate sectors is nondegenerate. This result shows that the symmetry of a two dimensional theory 'factorizes' into a 'classical', i.e. group-, part and a purely quantum one. In Chapter 4 we will see that the role of the latter can be played by modular C^*-Hopf algebras. There, however, the quantum symmetry is spontaneously broken.

Chapter 3 is devoted to the proof of a number of structural results on quantum field theories which satisfy the split property for wedges. In particular we show that a net of observables satisfying the latter condition and Haag duality (HD)

- satisfies several nice additivity properties, in particular n-regularity [68] and the time slice axiom.

- does not admit DHR and BF sectors. More precisely, every representation which is equivalent to π_0 upon restriction to every wedge is equivalent to a multiple of π_0.

- The relative commutants $\mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$ for $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ are minimal, i.e. equal to $\mathcal{A}(\hat{\mathcal{O}} \cap \mathcal{O}')$.

- Every irreducible locally normal representation, in particular every soliton representation, satisfies Haag duality.

- Soliton representations are characterized, up to unitary equivalence, by their left and right vacua.

- The extension of a DHR representation of a wedge-dual net to the dual net is at best a soliton representation. The localization properties of these extensions are studied.

In short, these results show that the only interesting non-vacuum representations of a net which satisfies HD+SPW are soliton representations. Thus, assuming the SPW, DHR sectors can exist only for nets which satisfy only wedge duality. For these, however, one loses the one-to-one correspondence between (equivalence classes of) DHR representations and endomorphisms of the quasilocal algebra. A general theory which takes this fact into account does not yet exist.

In Chapter 4, the largest part of this work, we consider 'orbifold' nets $\mathcal{A}$, i.e. nets of algebras which arise as fixpoint nets of a Haag dual net $\mathcal{F}$ under the action of a compact group G of inner symmetries. (This is just the scenario which is generic in the classical case ($\geq 2+1$ dimensions) by the work of Doplicher and Roberts.) Assuming the larger (or 'field'-) net to satisfy the SPW we know that it has no sectors. For the fixpoint net this is not true, since it does not satisfy Haag duality, in contrast to the higher dimensional situation. It is then quite natural to ask what the dual net looks like, and in analogy to the situation in higher dimensions one may even ask whether the failure of Haag duality for the fixpoint net has an interpretation in terms of some spontaneous symmetry breakdown. Both questions will be resolved by the introduction of *disorder operators*, which play an important role in statistical mechanics. Using these operators we will consider a nonlocal extension $\hat{\mathcal{F}}$ of the field net and show that there is a canonical action of Drinfel'd's quantum double $D(G)$ on the enlarged theory. We will explicitly exhibit a local subnet $\hat{\mathcal{A}}$ which coincides with the dual net in every abelian sector. If G is abelian, things are particularly transparent, for then we obtain a canonical action of the dual group $\hat{G}$ on the dual net. Since this action is spontaneously broken it must give rise to the existence of soliton sectors due to the results of [101]. Establishing the relation to the more general discussion in Chapter 3, we show that these soliton sectors are just the extensions of the DHR sectors of $\mathcal{A}$ to the dual net $\mathcal{A}^d = \hat{\mathcal{A}} \restriction \mathcal{H}_0$. Finally, we partially extend our analysis, which in the case of non-abelian groups was confined to finite ones, to the compact case. The technique of disorder operators also allows for a transparent discussion of bosonization, i.e. the passage from bosonic nets satisfying duality to fermionic nets with twisted duality, and vice versa, on the same Hilbert space. We conclude this chapter with a short sketch of how similar methods can be brought to bear on chiral models on the circle.

In Appendix A we prove for quantum field theories in $1+1$ dimensions the commutativity of inner symmetries with the translations, but not necessarily the Lorentz boosts. Appendix B is devoted to a derivation of the SPW from a modified nuclearity condition, where the time translations are replaced by the Lorentz group. Appendix C gives a very short introduction to the terminology pertaining to quantum groups and quantum doubles which we need in Chapter 4.

Chapter 2

On the Construction of Field Nets and a Conjecture by Rehren

2.1 Introduction

As mentioned in Chapter 1, a few years ago a long standing problem in local quantum physics was solved in [47]. There the conjecture [35, 38] was proved that the superselection structure of the local observables can always be described in terms of a compact group. This group (gauge group of the first kind) acts by automorphisms on a net of field algebras which generate the charged sectors from the vacuum and obey normal Bose and Fermi commutation relations. Here superselection structure refers to DHR representations in $\geq 2+1$ dimensions or BF representations in $\geq 3+1$ dimensions. In this chapter we consider only DHR representations.

In the next section we prove a few complementary results concerning the DR-construction. In particular, we show that the DR field net does not possess localized superselection sectors provided it is complete, i.e. contains charged fields generating all sectors of the observables. While this may appear to be an obvious consequence of the uniqueness result [47] for the complete normal field net, the proof relies upon a number of preparatory facts which are of independent interest. Our main result, however, pertains to the low dimensional case to which we turn now.

As we will see in the next chapter, sufficiently regular massive models in $1+1$ dimensions do not have DHR sectors at all if one insists in the assumption of Haag duality, which, of course, is needed for the DHR analysis. Therefore, the considerations in this chapter are aimed primarily at conformally covariant models, where the situation is quite different. For these it is known [22] that all positive-energy representations are automatically of the DHR type, while Haag duality holds in the vacuum sector [16]. Nevertheless, the conformal covariance will play no role in the following investigations. In particular, everything takes place in the usual uncompactified Minkowski space, cf. the remarks at the end of this chapter.

As explained in Chapter 1, DHR theory in $1+1$ dimensions differs markedly from that in higher dimensions. Yet, the more familiar structures known from the Doplicher-Roberts analysis can also be of relevance in two dimensions. This is the case whenever there exist degenerate sectors. Before we enter into the discussion of this possibility we give in Section 3 a sufficient criterion for the absence of degenerate sectors.

In Section 4 we prove Rehren's conjecture to the effect that the field net obtained by applying the DR construction to the degenerate sectors does no more possess degenerate sectors. This presumably constitutes a precise version of the claim that the Verlinde S-matrix is invertible 'if the symmetry algebra is maximally extended', which can be found in the CQFT folklore. In the attempt to understand the quantum symmetries one can thus restrict oneself to the nondegenerate situation.

2.2 On the Reconstruction of Fields from Observables

Our first aim in this section will be to prove the intuitively reasonable fact that a complete field net associated (in $\geq 2+1$ dimensions) to a net of observables does not possess localized superselection sectors. This result, which may not be too important in itself, will be the basis of our proof of a conjecture by Rehren (Theorem 2.4.4). Furthermore, we show that the construction of the complete field net 'can be done in steps'. I.e. one also obtains the complete field net by applying the DR construction to an intermediate, thus incomplete, field net and its DHR sectors. For the sake of simplicity we defer the treatment of the general case for a while and begin with the purely bosonic case.

2.2.1 Purely Bosonic Case

The superselection theory of a net of observables is called purely bosonic if all DHR sectors have statistics phase $+1$. In this case the charged fields which generate these sectors from the vacuum are local and the fields associated with different sectors can be chosen to be relatively local. Then the Doplicher-Roberts construction [47] gives rise to a local field net $\mathcal{F}$, which in addition satisfies Haag duality. Thus it makes sense to consider the DHR sectors of $\mathcal{F}$ and to apply the DR construction to these. (In analogy to [37, 39] one needs $\mathcal{F}$ to satisfy the technical 'property B' [37], which can be derived [3] from standard assumptions, in particular positive energy. Since a DR field net is Poincaré covariant with positive energy [47, Sect. 6], provided this is true for the vacuum sector and the DHR representations of the observables, we may take the property B for granted also for $\mathcal{F}$.)

We cite the following definitions from [47].

Definition 2.2.1 *Given a net $\mathcal{A}$ of observables and a vacuum representation π_0, a normal field system with gauge symmetry, $\{\pi, \mathcal{F}, G\}$, consists of a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ containing π_0 as a subrepresentation on $\mathcal{H}_0 \subset \mathcal{H}$, a compact group G of unitaries on $\mathcal{H}$ leaving $\mathcal{H}_0$ pointwise fixed and a net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ of von Neumann algebras such that*

α) *the $g \in G$ induce automorphisms α_g of $\mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ with $\pi(\mathcal{A}(\mathcal{O}))$ as fixed-point algebra,*

β) *the field net $\mathcal{F}$ is irreducible,*

γ) *$\mathcal{H}_0$ is cyclic for $\mathcal{F}(\mathcal{O})$ $\forall \mathcal{O} \in \mathcal{K}$,*

δ) *there is an element k in the center of G with $k^2 = e$ such that the net $\mathcal{F}$ obeys graded local commutativity for the $\mathbb{Z}_2$-grading defined by k, cf. (1.3.2, 1.3.3).*

Definition 2.2.2 *A field system with gauge symmetry $\{\pi, \mathcal{F}, G\}$ is complete if each equivalence class of irreducible representations of $\mathcal{A}$ satisfying (1.3.10) and having finite statistics is realized as a subrepresentation of π, i.e. π describes all relevant superselection sectors.*

For a given net of observables $\mathcal{A}$ we denote by Δ the set of all transportable localized morphisms with finite statistics. Let Γ be a closed semigroup of localized *bosonic* endomorphisms and let $\mathcal{F}$ be the associated local field net. Now let Σ be a closed semigroup of localized endomorphisms of $\mathcal{F}$. After iterating the DR construction again we are faced with the following situation. There are three nets $\mathcal{A}$, $\mathcal{F}$, $\tilde{\mathcal{F}}$ acting faithfully and irreducibly on the Hilbert spaces $\mathcal{H}_0 \subset \mathcal{H} \subset \tilde{\mathcal{H}}$, respectively, such that Haag duality holds (twisted duality in the case of $\tilde{\mathcal{F}}$). The nets $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are normal field nets with respect to the nets $\mathcal{F}$ and $\mathcal{A}$, respectively, in the sense of Definition 2.2.1. Thus there are representations π of $\mathcal{A}$ on $\mathcal{H}$ and $\tilde{\pi}$ of $\mathcal{F}$ on $\tilde{\mathcal{H}}$, respectively, such that $\tilde{\pi} \circ \pi(\mathcal{A}) \subset \tilde{\pi}(\mathcal{F}) \subset \tilde{\mathcal{F}}$. Furthermore, there are compact groups G and $\tilde{G}$ acting on $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively, such that $\mathcal{F}(\mathcal{O})^G = \pi(\mathcal{A}(\mathcal{O}))$ and $\tilde{\mathcal{F}}(\mathcal{O})^{\tilde{G}} = \tilde{\pi}(\mathcal{F}(\mathcal{O}))$. The following result is crucial:

Lemma 2.2.3 *The net $\tilde{\mathcal{F}}$ is a normal field net w.r.t. the observables $\mathcal{A}$. In particular, there is a compact group $\overline{G}$ acting on $\tilde{\mathcal{F}}$ and containing $\tilde{G}$ as a closed normal subgroup such that $\tilde{\mathcal{F}}(\mathcal{O})^{\overline{G}} = \tilde{\pi} \circ \pi(\mathcal{A}(\mathcal{O}))$.*

Proof. Since the groups G and $\tilde{G}$ consist of *all* automorphisms of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ which leave $\mathcal{A}$ and $\mathcal{F}$, respectively, pointwise stable, we are precisely in the situation studied in [29, Sec. 3]. Thus we can apply [29, Prop. 3.1] and obtain a short exact sequence

$$\mathbf{1} \to \tilde{G} \to \overline{G} \to G \to \mathbf{1}, \tag{2.2.1}$$

where $\overline{G}$ is just the group of all symmetries of $\tilde{\mathcal{F}}$ which leaves the net $\mathcal{A}$ pointwise stable. Furthermore, $\overline{G}$ is unitarily implemented since G and $\tilde{G}$ are (on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively). It remains to prove the requirements $\beta) - \delta)$ of Definition 2.2.1. Now, $\beta)$ and $\delta)$ are automatically true by [47, Thm. 3.5]. Finally, $\gamma)$, viz. the cyclicity of $\mathcal{H}_0$ for $\tilde{\mathcal{F}}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ is also easy: in application to $\mathcal{H}_0$, $\tilde{\pi}(\mathcal{F}(\mathcal{O})) \subset \tilde{\mathcal{F}}(\mathcal{O})$ gives a dense subset of $\mathcal{H}$, the image of which under the action of the charged (w.r.t. $\mathcal{F}$) fields in $\tilde{\mathcal{F}}$ is dense in $\tilde{\mathcal{H}}$. ■

Let now $\Gamma = \Delta$, the set of all transportable localized morphisms with finite statistics. Using the above lemma it is easy to prove the following.

Theorem 2.2.4 *The complete (local) field net $\mathcal{F}$ associated with a purely bosonic theory has no DHR sectors with finite statistics.*

Proof. Assuming the converse, the above lemma gives us a field net $\tilde{\mathcal{F}}$ on a larger Hilbert space $\tilde{\mathcal{H}}$, which obviously is also complete, since the representation π of $\mathcal{A}$ on $\mathcal{H}$ is a subrepresentation of $\tilde{\pi} \circ \pi$. Thus, by [47, Thm. 3.5] both field systems are equivalent.

I.e., there is a unitary operator $W : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $W\pi(A) = \tilde{\pi} \circ \pi(A)W \ \forall A \in \mathcal{A}$ etc. In view of the decomposition

$$\pi = \bigoplus_{\xi \in \hat{G}} d_\xi \, \pi_\xi, \tag{2.2.2}$$

where the irreducible representations π_ξ are mutually inequivalent, and similarly for $\tilde{\pi} \circ \pi$, π and $\tilde{\pi}$ can be unitary equivalent only if $G = \tilde{G}$ and thus $\mathcal{F} = \tilde{\mathcal{F}}$. ■

We have thus, in the purely bosonic case, reached our first goal. Before we turn to the general situation we show that the construction of the complete field net 'can be done in steps'. I.e., one also obtains the complete field net by applying the DR construction to an intermediate field net and its DHR sectors, again assuming that the intermediate net is local (this is not required for the complete field net).

The following lemma is more or less obvious and is stated here since it does not appear explicitly in [45, 47].

Lemma 2.2.5 *Let Γ_1, Γ_2 be subsemigroups of Δ which are both closed under direct sums, subobjects and conjugates and let $\mathcal{F}_i, \ i = 1, 2$ be the associated normal field nets on the Hilbert spaces $\mathcal{H}_i$ with symmetry groups G_i and π_i the representations of $\mathcal{A}$. If $\Gamma_1 \subset \Gamma_2$ then there is an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$V\pi_1(A) = \pi_2(A)V, \quad A \in \mathcal{A}, \tag{2.2.3}$$

$$VG_1V^* = G_2E, \tag{2.2.4}$$

$$V\mathcal{F}_1V^* = (\mathcal{F}_2 \cap \{E\}')E, \tag{2.2.5}$$

where $E = VV^*$. Furthermore, there is a closed normal subgroup N of G_2 such that E is the projection onto the subspace of N-invariant vectors in $\mathcal{H}_2$ and $\{\pi_1, G_1, \mathcal{F}_1\}$ is equivalent to $\{\pi_2^N, G_2/N, \mathcal{F}_2^N\}$.

Proof. As usual the field theory $\mathcal{F}_2$ is constructed by applying [47, Cor. 6.] to the quadruple $(\mathcal{A}, \Delta_2, \varepsilon, \pi_0)$ and by defining $\mathcal{F}(\mathcal{O})$ to be the von Neumann algebra on $\mathcal{H}_2$ generated by the Hilbert spaces $H_\rho, \ \rho \in \Delta_2(\mathcal{O})$. Let E be the projection $[\mathcal{B}_1\mathcal{H}_0]$ where $\mathcal{B}_1$ is the C^*-algebra generated by $H_\rho, \ \rho \in \Delta_1$. Trivially, $\mathcal{B}_1$ maps $E\mathcal{H}_2$ into itself. $\mathcal{B}_1$ is stable under G_2 as each of the Hilbert spaces H_ρ is. This implies that G_2 leaves $E\mathcal{H}_2$ stable. Restricting $\mathcal{B}_1$ and G_2 to $E\mathcal{H}_2$ the system $(E\mathcal{H}_2, E\pi_2(\cdot)E, EU_2E, \rho \in \Delta_1 \to EH_\rho E)$ which satisfies a) to g) of [47, 6.2]. With the exception of g) all of these are trivially obtained as restrictions. Property g) follows by appealing to [44, Lemma 2.4]. We can thus conclude from the uniqueness result of [47, Cor. 6.2] that $(E\mathcal{H}_2, E\pi_2E, EU_2E, \rho \in \Delta_1 \to H_\rho)$ is equivalent to the system $(\mathcal{H}_1, \pi_1, U_1, \rho \in \Delta_1 \to H_\rho)$ obtained from the quadruple $(\mathcal{A}, \Delta_2, \varepsilon, \pi_0)$. I.e. there is a unitary V from $\mathcal{H}_1$ to $E\mathcal{H}_2$ such that $V\pi_1(A) = \pi_2(A)V, \ V\mathcal{F}_1 = \mathcal{B}_1V, \ VU_1 = U_2V$. Interpreting V as an isometry mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ we have (2.2.3-2.2.5). The rest follows from [47, Prop. 3.17]. ■

Lemma 2.2.6 *Let $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ be the field net associated to a subsemigroup Γ of Δ, closed under direct sums, subobjects and conjugates. Then every localized endomorphism $\eta \in \Delta$ of $\mathcal{A}$ extends to an endomorphism $\tilde{\eta}$ of $\mathcal{F}$ commuting with the action of the gauge group. If η is localized in $\mathcal{O}$ the same holds for $\tilde{\eta}$.*

Remark. This result is of interest only if $\eta \notin \Gamma$. Otherwise we already know that η extends to an inner endomorphism of $\mathcal{F}$ by definition of the field algebra.

Proof. By the preceding result we know that the field net $\mathcal{F} = \mathcal{F}_\Gamma$ is equivalent to a subnet of the complete field net $\overline{\mathcal{F}} = \mathcal{F}_\Delta$. We identify $\mathcal{F}$ with this subnet. By construction every localized endomorphism $\eta \in \Delta(\mathcal{O})$ of $\mathcal{A}$ extends to an inner endomorphism of $\overline{\mathcal{F}}$. I.e. there is a multiplet of isometries $\psi_i \in \overline{\mathcal{F}}(\mathcal{O}),\ i = 1, \ldots, d$ satisfying $\sum_i \psi_i\psi_i^* = \mathbf{1},\ \psi_i^*\psi_j = \delta_{i,j}\mathbf{1}$ such that $\hat{\eta} \circ \pi(A) = \pi \circ \eta(A)$ where

$$\hat{\eta}(\cdot) = \sum_i \psi_i \cdot \psi_i^*. \tag{2.2.6}$$

$\hat{\eta}$ commuting with the action of G, it is easy to verify that η leaves $\mathcal{F} = \overline{\mathcal{F}}^N$ stable and thus restricts to an endomorphism of $\mathcal{F}$ which extends η. This extension is not necessarily local, for $\hat{\eta}(F) = -F$ if F is a fermionic operator localized spacelike to $\mathcal{O}$ and η is a fermionic endomorphism. This defect is easily remedied by defining

$$\tilde{\eta} = \begin{cases} \hat{\eta} & \text{if } \omega(\eta) = 1 \\ Ad\, V \circ \hat{\eta} & \text{if } \omega(\eta) = -1 \end{cases} \tag{2.2.7}$$

Clearly, $\tilde{\eta}$ has the desired localization properties and coincides with η on $\mathcal{A}$. Transportability of $\tilde{\eta}$ is automatic as $W \in (\eta, \eta')$ implies $\pi(W) \in (\tilde{\eta}, \tilde{\eta}')$. Finally the statistical dimensions of η and $\tilde{\eta}$ coincide as is seen using, e.g., the arguments in [81]. ■

Remark. The preceding lemmas do not depend on the restriction to bosonic families Γ of endomorphisms.

Lemma 2.2.7 *Let Γ be a semigroup of bosonic endomorphisms and let $\mathcal{F}$ be the associated (incomplete) local field net. Let Σ be the semigroup of all localized endomorphisms of $\mathcal{F}$. Then the associated DR-field net $\tilde{\mathcal{F}}$ is a complete field net with respect to $\mathcal{A}$.*

Proof. Let η be a localized endomorphism of $\mathcal{A}$. By the preceding lemma there is an extension (typically reducible) to a localized endomorphism $\tilde{\eta}$ of $\tilde{\mathcal{F}}$. By completeness of $\tilde{\mathcal{F}}$ with respect to endomorphisms of $\mathcal{F}$, $\tilde{\eta}$ is implemented by a Hilbert space in $\tilde{\mathcal{F}}$ and there is a subspace $\mathcal{H}_{\tilde{\eta}}$ of $\tilde{\mathcal{H}}$ such that $\tilde{\pi} \restriction \mathcal{H}_{\tilde{\eta}} \cong \tilde{\eta}$ as a representation of $\mathcal{F}$. Restricting to $\mathcal{A}$ and choosing an irreducible subspace $\mathcal{H}_\eta$ we have $\pi_\Sigma \restriction \mathcal{H}_{\tilde{\eta}} \cong \pi_0 \circ \eta$. Thus $\tilde{\mathcal{F}}$ is a complete field net for $\mathcal{A}$. ■

Theorem 2.2.8 *Let $\mathcal{A}$ be a net of observables and let Γ be a bosonic subsemigroup of Δ with the associated field net $\mathcal{F}_\Gamma$. Then the complete normal field net $\mathcal{F}_{\Gamma,\Sigma}$ obtained from the net $\mathcal{F}_\Gamma$ and its semigroup Σ of all localized endomorphisms is equivalent to the complete normal field net $\mathcal{F}_\Delta$. In particular the group $\overline{G}$ obtained in Lemma 2.2.3 is isomorphic to the one belonging to $\mathcal{F}_\Delta$.*

Proof. By Lemmas 2.2.3 and 2.2.7, $\mathcal{F}_{\Gamma,\Sigma}$ is a complete normal field net for $\mathcal{A}$. The same trivially holding for $\mathcal{F}_\Delta$ we are done since two such nets are isomorphic by [47, Thm. 3.5]. ■

2.2.2 General Case, Including Fermions

In the attempt to prove generalizations of Theorem 2.2.4 for theories possessing fermionic sectors and of Theorem 2.2.8 for fermionic intermediate nets $\mathcal{F}$ we are faced with the problem that it is not entirely obvious what these generalizations should be. We would like to show the representation theory of a complete normal field net, which is now assumed to comprise Fermi fields, to be trivial in some sense. It is not clear a priori that the methods used in the purely bosonic case lead to more than, at best, a partial solution. Yet we will adopt a conservative strategy and try and adapt the DHR/DR theory to $\mathbb{Z}_2$-graded nets. The fermionic version of Theorem 2.2.8 will vindicate this approach.

Clearly, the criterion (1.3.10) makes sense also for $\mathbb{Z}_2$-graded nets. Since things are complicated by the spacelike anticommutativity of fermionic operators, the assumption of twisted duality for $\mathcal{F}$ is, however, not sufficient to deduce that representations satisfying (1.3.10) are equivalent to (equivalence classes) of transportable endomorphisms of $\mathcal{F}$. To make this clear, assume π satisfies (1.3.10), and let $X^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_\pi$ be such that $X^{\mathcal{O}}A = \pi(A)X^{\mathcal{O}} \quad \forall A \in \mathcal{F}(\mathcal{O}')$. We would like to show that $\rho(A) \equiv X^{\mathcal{O}*}\pi(A)X^{\mathcal{O}}$ maps $\mathcal{F}(\mathcal{O}_1)$ into itself if $\mathcal{O}_1 \supset \mathcal{O}$. Now, let $x \in \mathcal{F}(\mathcal{O}_1)$, $y \in \mathcal{F}(\mathcal{O}_1')^t$, which implies $xy = yx$. We would like to apply ρ on both sides and use $\rho(y) = y$ to conclude $\rho(\mathcal{F}(\mathcal{O}_1)) \subset \mathcal{F}(\mathcal{O}_1')^{t\prime} = \mathcal{F}(\mathcal{O}_1)$. As it stands, this argument does not work, since π and thus ρ are defined only on the quasilocal algebra $\mathcal{F}$, but not on the operators $VF_- \in \mathcal{F}^t$ which result from the twisting operation. Assume, for a moment, that the representation ρ lifts to an endomorphism $\hat{\rho}$ of the C^*-algebra $\hat{\mathcal{F}}$ on $\mathcal{H}$ generated by $\mathcal{F}$ and the unitary V, such that $\hat{\rho}(V) = V$ or, alternatively, $\hat{\rho}(V) = -V$. Using triviality of ρ in restriction to $\mathcal{F}(\mathcal{O}_1')$ we then obtain $\rho(\mathcal{F}(\mathcal{O}_1')^t) = \mathcal{F}(\mathcal{O}_1')^t$, which justifies the above argument. Now, in order for $\hat{\rho}(V) = \pm V$ to be consistent, we must have

$$\rho \circ \alpha_k(A) = \rho(VAV) = \hat{\rho}(V)\rho(A)\hat{\rho}(V) = V\rho(A)V = \alpha_k \circ \rho(A), \tag{2.2.8}$$

i.e. $\rho \circ \alpha_k = \alpha_k \circ \rho$. In view of $\rho(A) = X^{\mathcal{O}*}\pi(A)X^{\mathcal{O}}$ we can now claim:

Lemma 2.2.9 *There is a one-to-one correspondence between equivalence classes of*

a) *Representations of $\mathcal{F}$ which are, for every $\mathcal{O} \in \mathcal{K}$, unitarily equivalent to a representation ρ on $\mathcal{H}_0$ such that $\rho \restriction \mathcal{F}(\mathcal{O}') = id$ and $\rho \circ \alpha_k = \alpha_k \circ \rho$ (where $Aut\,\mathcal{B}(\mathcal{H}_0) \ni \alpha_k \equiv Ad\,V$).*

b) *Transportable localized endomorphisms of $\mathcal{F}$ commuting with α_k.*

Remark. In a) covariance of π with respect to α_k is not enough. We need that upon transferring the representation to the vacuum Hilbert space via $\rho(A) = X^{\mathcal{O}*}\pi(A)X^{\mathcal{O}}$, α_k is implemented by the grading operator V.

Proof. The direction b)$\Rightarrow$a) is trivial. As to the converse, by the above all that remains to prove is extendibility of ρ to $\hat{\rho}$. By the arguments in [105, p. 121] the C^*-crossed product (covariance algebra) $\mathcal{F} \rtimes_{\alpha_k} \mathbb{Z}_2$ is simple such that the actions of $\mathcal{F}$ and $\mathbb{Z}_2$ on $\mathcal{H}_0$ and $\mathcal{H}_\pi$ via $\pi_0 = id$, V and π, V_π can be considered as faithful representations of the crossed product. Thus there is an isomorphism between $C^*(\mathcal{F}, V)$ and $C^*(\pi(\mathcal{F}), V_\pi)$ which maps $F \in \mathcal{F}$ into $\pi(F)$ and V into V_π. ■

Definition 2.2.10 *DHR-Representations and transportable endomorphisms are called even iff they satisfy a) and b) of Lemma 2.2.9, respectively.*

We have thus singled out a class of representations which gives rise to localized endomorphisms of the field algebra $\mathcal{F}$. But this class is still too large in the sense that unitarily equivalent even representations need not be inner equivalent. Let ρ be an even endomorphism of $\mathcal{F}$, localized in $\mathcal{O}$. Then $\sigma = Ad_{UV} \circ \rho$ with $U \in \mathcal{F}_-(\mathcal{O})$ is even and equivalent to ρ as a representation, but $(\rho, \sigma) \cap \mathcal{F} = \{0\}$, which precludes an extension of the DHR analysis of permutation statistics etc. Furthermore, ρ and σ, while equivalent as representations of $\mathcal{F}$, restrict to inequivalent endomorphisms of $\mathcal{F}_+$. This observation leads us to confine our attention to the following class of representations.

Definition 2.2.11 *An even DHR representation of $\mathcal{F}$ is called bosonic if it restricts to a bosonic DHR representation (in the conventional sense) of the even subnet $\mathcal{F}_+$.*

A better understanding of this class of representations is gained by the following lemma.

Lemma 2.2.12 *There is a bijective correspondence between the equivalence classes of bosonic even DHR representations of $\mathcal{F}$ and bosonic DHR representations of $\mathcal{F}_+$. I.e., equivalent bosonic even DHR representations of $\mathcal{F}$ restrict to equivalent bosonic DHR representations of $\mathcal{F}_+$. Conversely, every bosonic DHR representation of $\mathcal{F}_+$ extends uniquely to a bosonic even DHR representation of $\mathcal{F}$.*

Remark. It will become clear in Theorem 2.2.14 that nothing is lost by considering only representations which restrict to bosonic sectors of $\mathcal{F}_+$.
Proof. Clearly, the restriction of a bosonic even DHR representation of $\mathcal{F}$ to $\mathcal{F}_+$ is a bosonic DHR representation. Let ρ, σ be irreducible even DHR morphisms of $\mathcal{F}$, localized in $\mathcal{O}$, and let $T \in (\rho, \sigma)$. Twisted duality implies $T \in \mathcal{F}(\mathcal{O})^t$, i.e., $T = T_+ + T_- V$ where $T_\pm \in \mathcal{F}_\pm$. Now both sides of

$$\sigma(F) = T_+ \rho(F) T_+^* + T_- \alpha_k \circ \rho(F) T_-^* + T_+ \rho(F) V T_-^* + T_- V \rho(F) T_+^* \tag{2.2.9}$$

must commute with α_k. The first two terms on the right hand side obviously having this property, we obtain $T_+ \rho(F) V T_-^* + T_- V \rho(F) T_+^* = 0 \; \forall F \in \mathcal{F}$. For $F = F^*$ this reduces to $T_+ \rho(F) T_-^* = 0$, which can be true only if $T_+ = 0$ or $T_- = 0$ since ρ is irreducible. The case $T = T_- V$ is ruled out by the requirement that the restrictions of ρ and σ to $\mathcal{F}_+$ are both bosonic. Thus we conclude $T \in \mathcal{F}_+(\mathcal{O})$ and the restrictions ρ_+ and σ_+ are equivalent.

As to the converse, a bosonic DHR representation π_+ of $\mathcal{F}_+$ gives rise to a local 1-cocycle [97, 98] in $\mathcal{F}_+$, i.e. a mapping $z : \Sigma_1 \to \mathcal{U}(\mathcal{F}_+)$ satisfying the cocycle identity $z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c)$, $c \in \Sigma_2$ and the locality condition $z(b) \in \mathcal{F}_+(|b|)$, $b \in \Sigma_1$. This cocycle can be used as in [98, 99] to extend π_+ to a representation π of $\mathcal{F}$ which has all desired properties. We omit the details. By this construction, the extensions of equivalent representations are equivalent, an intertwiner $T \in (\rho, \sigma)$ lifting to $\pi(T)$ on $\mathcal{H}$. ■

Theorem 2.2.13 *Let $\mathcal{F}$ be a complete normal field net associated with the pair $(\mathcal{A}, \Delta)$. Then $\mathcal{F}$ does not possess non-trivial bosonic even DHR representations with finite statistics. Equivalently, there are no non-trivial bosonic DHR representations of the even subalgebra $\mathcal{F}_+$ with finite statistics.*

Proof. Assume that $\mathcal{F}$ has non-trivial bosonic even DHR representations which by the lemma is equivalent to the existence of bosonic sectors of $\mathcal{F}_+$. For the latter the conventional DHR analysis goes through and gives rise to a semigroup Σ of endomorphisms of $\mathcal{F}_+$ with permutation symmetry etc. These morphisms lift to $\mathcal{F}$ and we can apply the DR construction to $(\mathcal{F}, \Sigma)$. Since all elements of Σ are bosonic, no bosonization in the sense of [47, (3.19)] is necessary. All this works irrespective of the fact that $\mathcal{F}$ is not a local net since the fermionic fields are mere spectators. That the resulting field net again satisfies normal commutation relations is more or less evident since the 'new' fields are purely bosonic. Furthermore, Lemma 2.2.3 is still true when the 'observable net' is $\mathbb{Z}_2$-graded. Now the rest of the argument works just as in Theorem 2.2.4. ■

Remarks. 1. In the fermionic case, the even subnet $\mathcal{F}_+$ has exactly one fermionic sector. This sector is simple and its square is equivalent to the identity, as follows from the fact that bosonic sectors of $\mathcal{F}_+$ do not exist.

2. At this point one might be suspicious that there exist relevant DHR-like representations of $\mathcal{F}$ which are not covered by this theorem. In particular the restriction to bosonic even DHR representations was made for reasons which may appear purely technical and physically weakly motivated. The next theorem shows that this is not the case.

Theorem 2.2.14 *Let $\Gamma \subset \Delta$ be a subsemigroup of DHR sectors containing not only bosonic sectors and let $\mathcal{F}_\Gamma$ be the incomplete $\mathbb{Z}_2$-graded field net associated with $(\mathcal{A}, \Gamma)$. Then an application of the DR construction with respect to the bosonic even sectors Σ of $\mathcal{F}_\Gamma$, as described above, leads to a field net $\mathcal{F}_{\Gamma,\Sigma}$ which is equivalent to the complete normal field net $\mathcal{F}_\Delta$.*

Proof. Since $\mathcal{F}$ is assumed to contain fermions, every $\mathcal{F}(\mathcal{O})$ contains unitaries which are odd under α_k, giving rise to fermionic automorphisms of $\mathcal{A}$. By composition with one of these, every irreducible endomorphism of $\mathcal{A}$ can be made bosonic. It is thus clear that it suffices to extend $\mathcal{F}$ by Bose fields which implement these bosonic sectors (more precisely, their extensions to $\mathcal{F}$). The rest of the argument goes as in the preceding subsection. ■

It is thus the existence of bosonic sectors of the even subnet which indicates that a fermionic field net is not complete, and only such sectors need to be considered when enlarging the field net in order to obtain the complete field net.

2.3 Degenerate Sectors in $1+1$ Dimensions

Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net of observables in $1+1$ dimensions satisfying Haag duality. As shown in [56] to each pair of localized endomorphisms there are associated two a priori different statistics operators $\varepsilon(\rho, \eta), \varepsilon(\eta, \rho)^* \in (\rho\eta, \eta\rho)$.

Definition 2.3.1 ([91]) *Two DHR sectors have trivial monodromy iff the corresponding morphisms satisfy $\varepsilon(\rho, \eta) = \varepsilon(\eta, \rho)^*$ or, equivalently, $\varepsilon_M(\rho, \eta) = \mathbf{1}$ (this is independent of the choice of ρ, η within their equivalence classes). A DHR sector is degenerate iff it has trivial monodromy with all sectors (it suffices to consider the irreducible ones).*

A convenient criterion for triviality of the monodromy of two morphisms is given by the following

Lemma 2.3.2 *Let ρ, η be irreducible localized endomorphisms. Furthermore, let η_L, η_R be equivalent to η localized to the spacelike left and right of ρ, respectively, with the (unique up to a constant) intertwiner $T \in (\eta_L, \eta_R)$. Then triviality of the monodromy $\varepsilon_M(\rho, \eta)$ is equivalent to $\rho(T) = T$.*

Proof. Using the intertwiners $T_{L/R} \in (\eta, \eta_{L/R})$ the statistics operators are given by $\varepsilon(\rho, \eta) = T_L^* \rho(T_L)$ and $\varepsilon(\eta, \rho) = T_R^* \rho(T_R)$. The monodromy operator is given by $\varepsilon_M(\rho, \eta) = T_L^* \rho(T_L T_R^*) T_R$. Thus, $\varepsilon_M(\rho, \eta) = \mathbf{1}$ is equivalent to $\rho(T_L T_R^*) = T_L T_R^*$. The proof is completed by the observation that $T_L T_R^*$ equals T^* up to a phase. ■

At first sight one might be tempted to erroneously conclude from this lemma that there is no nontrivial braid statistics as follows: The above charge transporting intertwiner T commutes with $\mathcal{A}(\mathcal{O})$ which, appealing to Haag duality, implies that it is contained in the algebra of the spacelike complement $\mathcal{O}'$. On this algebra every morphism localized in $\mathcal{O}$ acts trivially, so that the lemma implies permutation group statistics. The mistake in this argument is, of course, that T is contained in the weakly closed algebra $\mathcal{R}(\mathcal{O}') \equiv \mathcal{A}(\mathcal{O}')'' = \mathcal{A}(\mathcal{O})'$ but not necessarily in the C^*-subalgebra $\mathcal{A}(\mathcal{O}')$ of the quasilocal algebra $\mathcal{A}$. It is only the latter on which ρ is known to act trivially.

The results of this section depend on an additional axiom, the split property.

Definition 2.3.3 *An inclusion $A \subset B$ of von Neumann algebras is split [41], if there exists a type-I factor N such that $A \subset N \subset B$. A net of algebras satisfies the 'split property for double cones' if the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ is split whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$, i. e. the closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}_2$.*

The importance of this property derives from the fact [106, 4, 41] that it is equivalent to the following formulation: For each pair of double cones $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ the algebra $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)'$ is unitarily equivalent to the tensor product $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)'$. It is believed that this form of the split property is satisfied in all reasonable quantum field theories.

In the rest of this section we will give a sufficient criterion for the *absence* of degenerate sectors. The following result is independent of the number of space dimensions.

Proposition 2.3.4 *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net of observables fulfilling Haag duality and the split property for double cones. Let ρ be an endomorphism of the quasilocal algebra $\mathcal{A}$ which is localized in $\mathcal{O}$ and acts identically on the relative commutant $\mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$ whenever $\hat{\mathcal{O}} \supset\supset \mathcal{O}$. Then ρ is an inner endomorphism of $\mathcal{A}$, i. e. a direct sum of copies of the identity morphism.*

Proof. Choose double cones $\mathcal{O}_1, \mathcal{O}_2$ fulfilling $\mathcal{O} \subset\subset \mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset \hat{\mathcal{O}}$. Thanks to the split property there exist type I factors M_1, M_2 such that

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_1) \subset M_1 \subset \mathcal{A}(\mathcal{O}_2) \subset M_2 \subset \mathcal{A}(\hat{\mathcal{O}}). \tag{2.3.1}$$

We first show $\rho(M_1) \subset M_1$. If $A \in M_1$ we have $\rho(A) \in \mathcal{A}(\mathcal{O}_2)$. Due to $\mathcal{A}(\mathcal{O}) \subset M_1$ and the premises ρ acts trivially on $M_1' \cap \mathcal{A}(\hat{\mathcal{O}})$. Thus

$$\rho(A) \in (M_1' \cap \mathcal{A}(\hat{\mathcal{O}}))' \cap M_2 \subset (M_1' \cap M_2)' \cap M_2 = M_1. \tag{2.3.2}$$

The last identity follows from M_1, M_2 being type I factors. Thus ρ restricts to an endomorphism of M_1. Now every endomorphism of a type I factor is inner [80, Cor. 3.8], i. e. there is a (possibly infinite) family of isometries $V_i \in M_1,\ i \in I$ with $V_i^* V_j = \delta_{i,j}, \sum_{i\in I} V_i V_i^* = \mathbf{1}$ such that

$$\rho \restriction M_1 = \eta(A) \equiv \sum_{i\in I} V_i \ \cdot \ V_i^*. \tag{2.3.3}$$

(The sums over I are understood in the strong sense.) Now by (2.3.3) and the premises, $\eta \restriction \mathcal{A}(\mathcal{O})' \cap M_1 = id$, implying $V_i \in (\mathcal{A}(\mathcal{O})' \cap M_1)' \cap M_1 = \mathcal{A}(\mathcal{O})\ \forall i \in I$. Therefore $\rho = \eta$ on $\mathcal{A}(\mathcal{O}_1)$ and $\rho = \eta = id$ on $\mathcal{A}(\mathcal{O}')$. In order to prove $\rho = \eta$ on all of $\mathcal{A}$ it suffices to show $\rho(A) = \eta(A)\ \forall A \in \mathcal{A}(\mathcal{O}_2)$ where we may, of course, assume $\mathcal{O}_2 \supset \mathcal{O}_1$. For a second we take for granted that

$$\mathcal{A}(\mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \vee (\mathcal{A}(\mathcal{O}_2) \wedge \mathcal{A}(\mathcal{O})'). \tag{2.3.4}$$

Having just proved $\rho \restriction \mathcal{A}(\mathcal{O}_1) = \eta$ and remarking that $\rho \restriction \mathcal{A}(\mathcal{O}_2) \wedge \mathcal{A}(\mathcal{O})' = id = \eta$ is true by assumption we conclude by local normality that $\rho \restriction \mathcal{A}(\mathcal{O}_2) = \eta$. In order to prove (2.3.4) apply the split property to the inclusion $\mathcal{O} \subset\subset \mathcal{O}_1$. Under the split isomorphism we have

$$\begin{aligned} \mathcal{A}(\mathcal{O}) &\cong \mathcal{A}(\mathcal{O}) \otimes \mathbf{1}, \\ \mathcal{A}(\mathcal{O}_i) &\cong \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{A}(\mathcal{O}_i),\ i = 1, 2. \end{aligned} \tag{2.3.5}$$

Thus $\mathcal{A}(\mathcal{O})' \cong \mathcal{A}(\mathcal{O})' \otimes \mathcal{B}(\mathcal{H}_0)$ and $\mathcal{A}(\mathcal{O}_2) \wedge \mathcal{A}(\mathcal{O})' \cong \mathcal{A}(\mathcal{O})' \otimes \mathcal{A}(\mathcal{O}_2)$ from which (2.3.4) follows at once. ■

Remark. The first part of the proof is essentially identical to [40, Prop. 2.3]. There it was stated only for automorphisms but the possibility of the above extension was remarked. In [40] the C^*-version of the time-slice axiom was used to conclude $\rho = \eta$ on $\mathcal{A}$. We have dispensed with this assumption by requiring triviality of ρ on the relative commutant $\mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$ for all $\hat{\mathcal{O}} \supset\supset \mathcal{O}$. For our purposes this will be sufficient.

We are now in the position to state our criterion for the absence of degenerate sectors in $1+1$ dimensions:

Corollary 2.3.5 *Assume in addition to the conditions of the proposition that for each pair $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ the algebra $\mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$ is generated by the charge transporters from $\mathcal{O}_L$ to $\mathcal{O}_R$ (and vice versa). Here $\mathcal{O}_L, \mathcal{O}_R$ are the connected components of $\hat{\mathcal{O}} \cap \mathcal{O}'$, see the figure below. Then there are no degenerate sectors. More precisely, every degenerate endomorphism is inner in the above sense.*

Proof. Due to Lemma 2.3.2 a degenerate morphism localized in $\mathcal{O}$ acts trivially on the charge transporters between $\mathcal{O}_L$ to $\mathcal{O}_R$. As these are weakly dense in $\mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$ by assumption and due to local normality the morphism acts trivially on the relative commutant. This is true for every $\hat{\mathcal{O}} \supset\supset \mathcal{O}$. The statement now follows from Proposition 2.3.4. ■

Remarks. 1. In the next chapter (see also [88]) we will show that a much more far-reaching result can be proved if one requires the split property not only for double cones but also for wedge regions. This property can, however, hold only in massive quantum field theories.

2. Admittedly the condition on the relative commutant seems difficult to verify. One

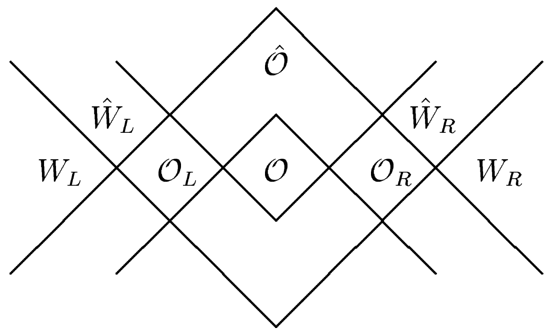


Figure 2.1: Relative commutants of double cones

may perhaps hope that something can be said in the case of rational theories, which have finitely many sectors.
3. It is likely that the condition on the relative commutant made in the corollary is also necessary. The argument goes as follows. If there are degenerate sectors then there is a field net $\mathcal{F}$ with group symmetry such that the net $\mathcal{A}$ is the restriction of the invariant subnet to the vacuum sector, cf. the next section. Assuming that the field net also satisfies the split property one can define localized implementers of the gauge group as in [20]. In particular, for every inclusion $\Lambda = (\mathcal{O} \subset\subset \hat{\mathcal{O}})$ and every $x \in U(G)'' \cap U(G)'$ one obtains an operator $U_\Lambda(x) \in \mathcal{A}(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O})'$. There is no reason to assume that $U_\Lambda(x)$ is contained in the algebra generated by $\mathcal{A}(\mathcal{O}_L), \mathcal{A}(\mathcal{O}_R)$ and the charge transporters.

2.4 On Degeneracies of Verlinde's Matrix S

2.4.1 Proof of a Conjecture by Rehren

We begin with two easy but crucial results on the set of degenerate DHR sectors.

Lemma 2.4.1 *A reducible DHR representation π is degenerate iff all irreducible subrepresentations are degenerate.*

Proof. Let ρ be equivalent to π and localized in $\mathcal{O}$, decomposing into irreducibles according to $\rho = \sum_{i \in I} V_i \, \rho_i(\cdot) \, V_i^*$. I.e., the ρ_i are localized in $\mathcal{O}$ and $V_i \in \mathcal{A}(\mathcal{O})$ with $V_i^* V_j = \delta_{i,j} \mathbf{1}$ and $\sum_i V_i V_i^* = \mathbf{1}$. By Lemma 2.3.2, π is degenerate iff $\rho(T) = T$ for every unitary intertwiner between (irreducible) morphisms σ, σ' which are localized in the two different connected components of $\mathcal{O}'$. Now, $\rho(T) = \sum_i V_i \, \rho_i(T) \, V_i^*$ equals T iff 'all matrix elements are equal', i.e., $V_j^* \, T \, V_k = \delta_{j,k} \, \rho_j(T) \; \forall j,k \in I$. But due to $T \in \mathcal{A}(\mathcal{O})'$ the left hand side equals $T V_j^* V_k = T \delta_{j,k}$ which leads to the necessary and sufficient condition $\rho_j(T) = T \; \forall j \in I$, which in turn is equivalent to all ρ_i being degenerate. ■

Lemma 2.4.2 *Let Δ_D be the set of all degenerate morphisms with finite statistical dimension. Then (Δ_D, ε) is a permutation symmetric, specially directed semigroup with* *subobjects and direct sums.*

Proof. Let ρ_1, ρ_2 be degenerate, i.e. $\varepsilon_M(\rho_i, \sigma) = \mathbf{1}\ \forall \sigma$. Due to the identities [57]

$$\begin{aligned} \varepsilon(\rho_1\rho_2, \sigma) &= \varepsilon(\rho_1, \sigma)\rho_1(\varepsilon(\rho_2, \sigma)), && (2.4.1)\\ \varepsilon(\sigma, \rho_1\rho_2) &= \rho_1(\varepsilon(\sigma, \rho_2))\varepsilon(\sigma, \rho_1) && (2.4.2)\end{aligned}$$

we have

$$\varepsilon_M(\rho_1\rho_2, \sigma) = \varepsilon(\rho_1\rho_2, \sigma)\varepsilon(\sigma, \rho_1\rho_2) = \varepsilon(\rho_1, \sigma)\rho_1(\varepsilon(\rho_2, \sigma)\varepsilon(\sigma, \rho_2))\varepsilon(\sigma, \rho_1) = \mathbf{1}. \qquad (2.4.3)$$

Thus Δ_D is closed under composition. By the preceding lemma the direct sum of degenerate morphisms is degenerate, and every irreducible morphism contained in a degenerate one is again degenerate. That (Δ_D, ε) is specially directed in the sense of [45, Sec. 5] follows as in [47, Lemma 3.7] from the fact that the degenerate sectors have permutation group statistics. ■

It is now clear that the spatial version [45, Cor. 6.2] of the construction of the crossed product can be applied to the quasilocal observable algebra and a semigroup Δ as above. As the proofs in [47, Sec. 3] were given for $\geq 2+1$ spacetime dimensions it seems advisable to reconsider them in order to be on the safe side, in particular as far as (twisted) duality for the field net is concerned.

Proposition 2.4.3 *Let $\mathcal{F}$ be the spatial crossed product [47, Cor. 6.2] of $\mathcal{A}$ by (Δ, ε) where Δ is as in Lemma 2.4.2 and let $\mathcal{F}(\mathcal{O})$ be defined as in the proof of [47, Thm. 3.5]. Then $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is a normal field system with gauge symmetry and satisfies twisted duality.*

Proof. The proof of existence in [47, Thm. 3.5] goes through unchanged as it relies only on algebraic arguments independent of the dimension. The same holds for [47, Thm. 3.6] with the possible exception of the argument leading to twisted duality on p. 73. The latter boils down algebraically to the identity $\mathcal{F}(\mathcal{O})' \cap G' = \pi(\mathcal{A}(\mathcal{O}'))^-$, $\mathcal{O} \in \mathcal{K}$. Finally, the proof of this formula in [47, Lemma 3.8] is easily verified to be correct in $1+1$ dimensions, too. ■

Remarks. 1. The reader who feels uneasy with these few remarks is encouraged to study the proofs of [47, Thms. 3.5, 3.6] himself, for it would make no sense to reproduce them here.

2. It may be confusing that in theories with group symmetry satisfying the split property for wedges (SPW), Haag duality for a field net $\mathcal{F}$ and the G-fixpoint net $\mathcal{A}$ (in the vacuum sector) are in fact incompatible [87]. The SPW has been verified for massive free scalar and Dirac fields and is probably true in all reasonable *massive* theories. On the other hand, a net of observables which satisfies Haag duality and the SPW does not admit DHR sectors anyway, cf. the next chapter. In view of this result, we implicitly assume in this section that the observables do *not* satisfy the SPW. The point is that one must be careful to distinguish between conformally covariant or at least massless theories, with which we are concerned here, and massive ones since the scenarios are quite different.

Theorem 2.4.4 *If the set Δ_D of all degenerate sectors with finite statistics is purely bosonic, the local field net $\mathcal{F}$ constructed from $\mathcal{A}$ and Δ_D does not have degenerate sectors with finite statistics. If Δ_D contains fermionic sectors, the normal field net $\mathcal{F}$ does not have degenerate bosonic even sectors with finite statistics. Equivalently, the even subnet has no degenerate bosonic sectors with finite statistics.*

Proof. The proofs of the Theorems 2.2.4, 2.2.13 are valid also in the 2-dimensional situation since neither the argument of Lemma 2.2.3 on the extendibility of local symmetries nor the uniqueness result of [47, Thm. 3.5] require any modification. ■

This result, which was conjectured by Rehren in [91], is quite interesting and potentially useful for the analysis of superselection structure in $1+1$ dimensions. It seems worthwhile to restate it in the following form.

Corollary 2.4.5 *Every degenerate quantum field theory in $1+1$ dimensions (in the sense that there are degenerate sectors, which in the rational case is equivalent to non-invertibility of S) arises as the fixpoint theory of a non-degenerate theory under the action of a compact group of inner symmetries. I.e., all degenerate theories are orbifold theories in the sense of [33].*

2.4.2 Relating the Superselection Structures of $\mathcal{A}$ and $\mathcal{F}$

In the preceding subsection we have seen, that whenever there are degenerate sectors one can construct an extended theory which is non-degenerate. The larger theory has a group symmetry such that the original theory is reobtained by retaining only the invariant operators. Equivalently, all degenerate theories are orbifold theories. By this result, a general analysis of the superselection structure in $1+1$ dimensions may begin by considering the nondegenerate case. It remains, however, to clarify the relation between the superselection structures of the degenerate and the extended theory. This will not be attempted here, but we will provide some results going in this direction.

Lemma 2.4.6 *All irreducible morphisms contained in the product of a degenerate and a non-degenerate morphism are non-degenerate.*

Proof. The fact that the composition of degenerate morphisms yields a sum of degenerate morphisms can be expressed in terms of the fusion coefficients N_{ij}^k as

$$i \text{ and } j \text{ degenerate, } k \text{ non-degenerate } \Rightarrow \; N_{ij}^k = 0. \tag{2.4.4}$$

By Frobenius reciprocity $N_{ij}^k = N_{i\bar{k}}^{\bar{j}}$ this implies

$$i \text{ and } j \text{ degenerate, } k \text{ non-degenerate } \Rightarrow \; N_{ik}^j = 0. \tag{2.4.5}$$

(We have used that the conjugate $\bar{\rho}$ is degenerate iff ρ is.) ■

Being able to apply the DR construction also in $1+1$ dimensions provided we consider only semigroups of degenerate endomorphisms, we are led to reconsider Lemma 2.2.6 concerning the extension of localized endomorphisms of the observable algebra to the field net. The construction given in Section 2 can not be used for the extension of nondegenerate morphisms η since we do not have a complete field net at our disposal. A prescription which does not rely on the existence of a complete field net was given by Rehren [92]. The claim of uniqueness made there has, however, to made more precise. Furthermore, it is not completely trivial to establish the existence part. Fortunately, both of these questions can be clarified in a relatively straightforward manner by generalizing results by Doplicher and Roberts. In [45, Sec. 8] they considered a similar extension problem, namely the extension of automorphisms of $\mathcal{A}$ to automorphisms of $\mathcal{F}$ commuting with the gauge group. The application these authors had in mind was the extension of spacetime symmetries to the field net [47, Sec. 6] under the provision that the endomorphisms implemented by the fields are covariant. For a morphism $\rho \in \Delta$ the inner endomorphism of $\mathcal{F}$ which extends ρ will also be denoted by ρ.

Lemma 2.4.7 *Let $\mathcal{B}$ be the crossed product [45] of the C^*-algebra $\mathcal{A}$ with center $\mathbb{C}\mathbf{1}$ by the permutation symmetric, specially directed semigroup (Δ, ε) of endomorphisms and let G be the corresponding gauge group. Let Γ be a semigroup of unital endomorphisms of $\mathcal{A}$. Then there is a one-to-one correspondence between actions of Γ on $\mathcal{B}$ by unital endomorphisms $\tilde{\eta}$ which extend $\eta \in \Gamma$ and commute elementwise with G and mappings $(\rho, \eta) \mapsto W_\rho(\eta)$ from $\Delta \times \Gamma$ to unitaries of $\mathcal{A}$ satisfying*

$$\begin{aligned} W_\rho(\eta) &\in (\rho\eta, \eta\rho), && (2.4.6)\\ W_{\rho'}(\eta) T W_\rho(\eta)^* &= \eta(T), \quad T \in (\rho, \rho'), && (2.4.7)\\ W_{\rho\rho'}(\eta) &= W_\rho(\eta)\, \rho(W_{\rho'}(\eta)), && (2.4.8)\\ W_\rho(\eta\eta') &= \eta(W_\rho(\eta')) W_\rho(\eta) && (2.4.9) \end{aligned}$$

for all $\eta, \eta' \in \Gamma$, $\rho, \rho' \in \Delta$. The correspondence is determined by

$$\tilde{\eta}(\psi) = W_\rho(\eta)\psi, \quad \psi \in H_\rho, \ \rho \in \Delta, \eta \in \Gamma. \qquad (2.4.10)$$

Furthermore, if a unitary $S \in (\eta, \eta')$, $\eta, \eta' \in \Gamma$ satisfies

$$S W_\rho(\eta) = W_\rho(\eta') \rho(S) \quad \forall \rho \in \Delta, \qquad (2.4.11)$$

then $S \in (\tilde{\eta}, \tilde{\eta}')$.

Proof. An inspection of the proofs of [45, Thm. 8.2, Cor. 8.3], where groups of automorphisms were considered, makes plain that they are valid also for the case of semigroups of true endomorphisms and we refrain from repeating them. Besides η not necessarily being onto, the only change occurred in (2.4.6) which replaces the property $W_\rho \in (\rho, \beta\rho\beta^{-1})$ which does not make sense for a proper endomorphism β. Given $\tilde{\eta}$ and setting

$$W_\rho = \sum_{i=1}^{d} \tilde{\eta}(\psi_i)\psi_i^*, \qquad (2.4.12)$$

where ψ_i, $i = 1, \ldots, d$ is a basis of H_ρ, it is clear that W_ρ satisfies (2.4.6). The other properties of the $W's$ are proved as in [45]. As to the converse direction, we are done provided we can show that [45, Thm. 8.1] concerning the extension of η to the cross product of $\mathcal{A}$ by a single endomorphism ρ generalizes to the case of η an endomorphism. We give only that part of the argument which differs from the one in [45].

Let thus $\mathcal{A}$ and ρ satisfy the assumptions of [45, Thm. 8.1], let η be an injective unital endomorphism of $\mathcal{A}$ and let $W \in (\rho\eta, \eta\rho)$ satisfy [45, (8.1), (8.2)]. As in [45, Thm. 8.1] we consider the monomorphisms of $\mathcal{A}$ and $\mathcal{O}_d$ into $\mathcal{A} \otimes_\mu \mathcal{O}_d$, defined by $\pi' : A \mapsto \eta(A) \otimes_\mu \mathbf{1}$ and $\zeta' : \psi \mapsto W \otimes_\mu \psi$, $\psi \in H$, respectively. The calculation leading to $\zeta'(\psi)\pi'(A) = \pi' \circ \rho(A)\zeta'(\psi)$ is correct also for η a true endomorphism. Furthermore, with $\zeta'_1 = \zeta' \restriction \mathcal{O}_{SU(d)}$ we have $\zeta'_1(\mathcal{O}_{SU(d)}) \in \eta(\mathcal{A})$ thanks to the conditions on W and the fact that $\mathcal{O}_{SU(d)}$ is generated by the elements S and θ, see [43]. Thus $\eta\rho\eta^{-1} \circ \zeta'_1$ is well defined and equals $\zeta'_1 \circ \sigma$, where σ is the canonical endomorphism of $\mathcal{O}_{SU(d)}$. As in [45] we conclude that $\zeta' \restriction \mathcal{O}_{SU(d)} = \eta \circ \mu$. Thus by the universality of $\mathcal{A} \otimes_\mu \mathcal{O}_d$ there is an isomorphism between $\mathcal{A} \otimes_\mu \mathcal{O}_d$ and the subalgebra generated by $\pi'(\mathcal{A})$ and $\zeta'(\psi)$. Equivalently, there is an endomorphism γ of $\mathcal{A} \otimes_\mu \mathcal{O}_d$ such that

$$\gamma(A \otimes_\mu \mathbf{1}) = \eta(A) \otimes_\mu \mathbf{1}, \quad A \in \mathcal{A}, \tag{2.4.13}$$

$$\gamma(\mathbf{1} \otimes_\mu \psi) = W \otimes_\mu \psi, \quad \psi \in H. \tag{2.4.14}$$

Now the rest of the proof goes exactly as in [45, Thm. 8.1], i.e. after factoring out the ideal J_ϕ we obtain an endomorphism $\tilde{\eta}$ of the crossed product $\mathcal{B} = (\mathcal{A} \otimes_\mu \mathcal{O}_d)/J_\phi$ which commutes with the action of the gauge group G.

Let now $S \in (\eta, \eta')$ satisfy (2.4.11). Then for $\psi \in H_\rho$ we have

$$S\tilde{\eta}(\psi) = SW_\rho(\eta)\psi = W_\rho(\eta')\rho(S)\psi = W_\rho(\eta')\psi S = \tilde{\eta}'(\psi)S. \tag{2.4.15}$$

Since $\tilde{\eta}, \tilde{\eta}'$ are determined by their action on the spaces H_ρ this implies $S \in (\tilde{\eta}, \tilde{\eta}')$. ■

We are now prepared to consider the wanted extensions of localized endomorphisms. Motivated by Lemma 2.2.6 where we had (in the case of bosonic ψ^ρ)

$$\hat{\eta}(\psi^\rho) = \sum_i \psi_i^\eta\, \psi^\rho\, \psi_i^{\eta *} = \left(\sum_{i,j} \psi_i^\eta \psi_j^\rho \psi_i^{\eta *} \psi_j^{\rho *} \right) \psi^\rho = \varepsilon(\rho, \eta)\, \psi^\rho, \tag{2.4.16}$$

we appeal to the preceding lemma with $W_\rho(\eta) = \varepsilon(\rho, \eta)$.

Proposition 2.4.8 *Let $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ be the field net obtained via the Doplicher-Roberts construction from the algebra $\mathcal{A}$ of observables and a semigroup of degenerate morphisms, closed under direct sums, subobjects and conjugates. Then every localized (unital) endomorphism η of $\mathcal{A}$ extends to a localized endomorphism $\tilde{\eta}$ of $\mathcal{F}$ commuting with the action of the gauge group. If η is localized in $\mathcal{O}$ the same holds for $\tilde{\eta}$. Every $S \in (\eta, \sigma)$ lifts to $S \in (\tilde{\eta}, \tilde{\sigma})$.*

Proof. We set $W_\rho(\eta) = \varepsilon(\rho, \eta)$ and verify the requirements (2.4.6-2.4.9). Obviously (2.4.6) is fulfilled by definition of the statistics operator. (2.4.8) and (2.4.9) follow from (2.4.1) and (2.4.2), respectively. Finally, (2.4.9) is just $\varepsilon(\rho', \eta)T = \eta(T)\varepsilon(\rho, \eta)$ which holds for $T \in (\rho, \rho')$. The statement on the localizations follows from the fact that $\varepsilon(\rho, \eta) = \mathbf{1}$ if ρ, η are spacelike localized. Finally, with $S \in (\eta, \sigma)$ one has $\varepsilon(\rho, \eta)S = \rho(S)\varepsilon(\rho, \sigma)$ such that the condition (2.4.11) is satisfied. Thus $S \in (\tilde{\eta}, \tilde{\sigma})$. ■

Remarks. 1. It is surprising that the result of [45, Sec. 8] in the guise of Lemma 2.4.7 finds an application quite different from the one in [47, Sec. 6] for which it was designed.
2. The above result is unaffected if the field net is fermionic. In this case the identitity $\tilde{\eta}\circ\alpha_k = \alpha_k\circ\tilde{\eta}$ where $k\in G$ is the grading element (which distinguishes bosonic and fermionic fields) shows that $\tilde{\eta}$ leaves the statistics of fields invariant. In fact, this observation provided the motivation for introducing the notion of even DHR sectors in Section 2.2.2.
3. In principle, the construction of the field algebra works for every family of sectors with permutation group statistics which is closed under direct sums and subobjects. As emphasized by Rehren [92], the extension $\tilde{\eta}$ is localized in a double cone only if the charged fields in $\mathcal{F}$ correspond to *degenerate* sectors, for otherwise $\varepsilon(\rho,\eta)=\mathbf{1}$ holds only if ρ is localized to the right of η (or left, if $\varepsilon(\eta,\rho)^*$ is used).

In the special case where η is an automorphism, the extension $\tilde{\eta}$ can be defined via [45, Thm. 8.2], using $W_\rho(\eta)$ as above. Clearly, $\tilde{\eta}$ is irreducible since it is an automorphism. In general, however, the extension $\tilde{\eta}$ will not be irreducible. Rehren's description [92] of the relative commutant can also be given a rigorous proof by adapting earlier results [44, Lemma 5.1].

Lemma 2.4.9 *The relative commutant $\mathcal{F}\cap\tilde{\eta}(\mathcal{F})'$ is generated as a closed linear space by sets of the form $(\rho\eta,\eta)H_\rho,\ \rho\in\Delta$.*

Proof. By twisted duality, an operator in $\tilde{\eta}(\mathcal{F})'$ is contained in $\mathcal{F}(\mathcal{O})^t$, where $\mathcal{O}$ is the localization region of η. Due to $\mathcal{F}\cap\mathcal{F}^t=\mathcal{F}_+$, the selfintertwiners of $\tilde{\eta}$ in $\mathcal{F}$ are bosonic. Obviously, $(\rho\eta,\eta)\psi,\ \psi\in H_\rho$ is in $\mathcal{F}\cap\eta(\mathcal{A})'$. Now, just as η, $\rho\eta$ can be extended to an endomorphism $\widetilde{\rho\eta}$ of $\mathcal{F}$ by the proposition. Furthermore, $T\in(\rho\eta,\eta)$ lifts to an intertwiner between $\tilde{\rho\eta}$ and $\tilde{\eta}$. Thus $(\rho\eta,\eta)\psi^\rho$ is in $\mathcal{F}\cap\tilde{\eta}(\mathcal{F})'$. As to the converse, $\tilde{\eta}(\mathcal{F})'\cap\mathcal{F}$ is globally stable under the action of G since $\tilde{\eta}$ commutes with G. Thus $\tilde{\eta}(\mathcal{F})'\cap\mathcal{F}$ is generated linearly by its irreducible tensors under G. If $T_1,\ldots,T_d$ is such a tensor from $\mathcal{F}\cap\tilde{\eta}(\mathcal{F})'$, then there is a multiplet $\psi_i, i=1,\ldots,d$ of isometries in $\mathcal{F}$ and transforming in the same way, since the field algebra has full Hilbert G-spectrum. With $X=\sum_{i=1}^d T_i\psi_i^*\in\mathcal{F}^G$ we have $T_i=X\psi_i$ and we must prove $X\in(\rho\eta,\eta)$. Now $T_iF=FT_i$ for $F\in\tilde{\eta}(\mathcal{F})$ implies

$$X\psi_i\,\tilde{\eta}(F)=X\rho(\tilde{\eta}(F))\psi_i=\tilde{\eta}(F)X\psi_i,\quad F\in\mathcal{F}, i=1,\ldots,d. \tag{2.4.17}$$

Multiplying the second identity with ψ_i^* and summing over i we obtain $X\rho(\tilde{\eta}(F)) = \tilde{\eta}(F)X,\ \ F\in\mathcal{F}$ since $\sum_i\psi_i\psi_i^*=\mathbf{1}$ by construction of the field algebra. Thus $X\in(\rho\eta,\eta)$. ■

Corollary 2.4.10 *$\tilde{\eta}$ is irreducible iff the endomorphism $\eta\bar{\eta}$ of $\mathcal{A}$ does not contain a non-trivial morphism $\rho\in\Delta$.*

Proof. By the lemma, the existence of a morphism $\rho\in\Delta$ with $(\eta\rho,\eta)\neq\{0\}$ is necessary and sufficient for the nontriviality of $\mathcal{F}\cap\tilde{\eta}(\mathcal{F})'$. But by Frobenius reciprocity, $\eta\rho\succ\eta$ is equivalent to $\eta\bar{\eta}\succ\bar{\rho}$. ■

Remark. The irreducible endomorphisms obtained by decomposing an extension $\tilde{\eta}$ are even, provided we use bosonic isometric intertwiners. This can always be done as the relative commutant is contained in $\mathcal{F}_+$ by Lemma 2.4.9.

We close this section with another remark. The considerations in this section were aimed primarily at conformally covariant theories in $1+1$ dimensions, since the superselection structure of massive models is quite different, see the next chapter. On the other hand, it is well known [84] that conformal theories live on a suitably compactified Minkowski space. This compactification renders the spacetime non-simply connected, which in turn implies the existence of a center in the algebra of observables [57]. Triviality of the center was however an essential requirement for the Doplicher-Roberts analysis, in particular [42, 45]. In a first approach one may circumvent this problem by working with the restriction of the net to Minkowski space. Since this 'removal of a point at infinity' may destroy Haag duality [27], an analysis on the compactified spacetime seems desirable. It should be obvious that in this case the DR construction may produce fields which live only on a covering space.

Chapter 3

Superselection Structure of Massive Quantum Field Theories in $1+1$ Dimensions

3.1 Introduction

There have long been indications that the DHR criterion might not be applicable to massive $2d$-theories as it stands. The first of these was the fact, known for some time, that the fixpoint nets of Haag-dual field nets with respect to the action of a global gauge group do *not* satisfy duality even in simple sectors, whereas this is true in $\geq 2+1$ dimensions. This phenomenon will be analyzed thoroughly in the next chapter (see also [87]) under the additional assumption that the fields satisfy the split property for wedges. This property plays an important role also in the present chapter which is devoted to deriving a number of new results on the superselection structure of massive quantum field theories in $1+1$ dimensions, not necessarily arising as fixpoint nets.

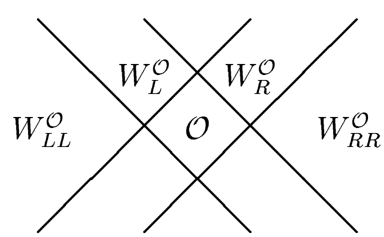
The split property for wedges (SPW), which will be defined precisely in the next section, is satisfied by free theories containing finitely many **massive** scalar and Dirac fields in $1+1$ dimensions, but is definitely violated by massless fields. Generalized free fields, even with mass-gap, are also incompatible with the split property if the mass spectrum is not discrete [41]. Whereas this shows the SPW to be more restrictive than the existence of a mass gap, massive theories without the SPW are considered pathological. In particular, the usual models like $P(\phi)_2, Y_2$, sine-Gordon, Gross-Neveu etc. are expected to satisfy this requirement, but this remains to be proved. As explained in Chapter 1, the split property (for double cones) can be derived from various criteria which limit the number of degrees of freedom in bounded regions of phase space. An analogous argument tailored to the SPW will be given in Appendix B.

In the next section we will prove some elementary consequences of Haag duality and the split property for wedges, in particular strong additivity and the time-slice property. The significance of our assumptions for superselection theory derives mainly from the fact that they preclude the existence of locally generated superselection sectors. More precisely, if the vacuum representation satisfies Haag duality and the SPW then every irreducible DHR representation is unitarily equivalent to the vacuum representation. This important and perhaps surprising result, to be proved in Section 4, indicates that the innocently looking assumptions of the DHR framework are quite restrictive when they are combined with the split property for wedges. While this may appear reasonable in view of the non-connectedness of $\mathcal{O}'$, our result also applies to the BF representations which are only localizable in wedges provided left and right handed wedges are admitted. In Section 5 we will prove the minimality of the relative commutant for an inclusion of double cone algebras which, via a result of Driessler, implies Haag duality in all locally normal irreducible representations. In Section 6 the facts gathered in the preceding sections will be applied to the theory of quantum solitons thereby concluding our discussion of the representation theory of Haag-dual nets. Summing up the results obtained so far, the representation theory of such nets is essentially trivial. On the other hand it is well known that charged representations, braid statistics and quantum symmetry exist in a host of more or less explicitly analyzed models. In order to accommodate these models, the only way seems to be to relax the duality requirement by postulating only wedge duality. In the final Section 7 Roberts' extension of localized representations to the dual net will be reconsidered. In this chapter we will not attempt to say anything concerning the quantum symmetry question.

3.2 Geometric Preliminaries and the Split Property for Wedges

Until further notice we fix a vacuum representation π_0 (which is always faithful) and omit the symbol π_0 identifying $\mathcal{A}(\mathcal{O}) \equiv \pi_0(\mathcal{A}(\mathcal{O}))$. Whereas we may assume the algebras $\mathcal{A}(\mathcal{O})$ to be weakly closed, for infinite regions like $W, \mathcal{O}'$ we carefully distinguish between the C^*-subalgebras $\mathcal{A}(W) \equiv \pi_0(\mathcal{A}(W))$ of $\pi_0(\mathcal{A})$ and their weak closures $\mathcal{R}(W) = \mathcal{A}(W)''$.

For any double cone $\mathcal{O} \in \mathcal{K}$ we designate the left and right spacelike complement by $W_{LL}^{\mathcal{O}}$ and $W_{RR}^{\mathcal{O}}$, respectively. Furthermore we write $W_L^{\mathcal{O}}$ and $W_R^{\mathcal{O}}$ for $W_{RR}^{\mathcal{O}\,\prime}$ and $W_{LL}^{\mathcal{O}\,\prime}$. These regions are wedge shaped, i.e. translates of the standard wedges $W_L = \{x \in \mathbb{R}^2 \mid x^1 < -|x^0|\}$ and $W_R = \{x \in \mathbb{R}^2 \mid x^1 > |x^0|\}$. We will not distinguish between open and closed regions, for definiteness one may consider $\mathcal{O}$ and all W-regions as open. With these definitions we have $\mathcal{O} = W_L^{\mathcal{O}} \cap W_R^{\mathcal{O}}$ and $\mathcal{O}' = W_{LL}^{\mathcal{O}} \cup W_{RR}^{\mathcal{O}}$ which graphically looks as follows.



(3.2.1)

Whereas we have already once made use of the split property in the preceding chapter, this property will play a more prominent role in this chapter and the following one.

Definition 3.2.1 *A net of algebras satisfies the split property for wedges if the inclusion* $\mathcal{R}(W_1) \subset \mathcal{R}(W_2)$ *is split whenever* $W_1 \subset\subset W_2$, *i. e. the closure of* W_1 *is contained in* *the interior of W_2 (which is equivalent to the existence of a double cone $\mathcal{O}$ such that $W_1 \cup W_2' = \mathcal{O}'$).*

We will now examine the implications of the split property for wedges (SPW). The power of this assumption in combination with Haag duality derives from the fact that one obtains strong results on the relation between the algebras of double cones and of wedges. In quantum field theories, where there are lots of cyclic and separating vectors for the local algebras by the Reeh-Schlieder theorem, the split property for wedges is equivalent [41] to the existence, for any double cone $\mathcal{O}$, of a unitary operator $Y^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ implementing an isomorphism between $\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_{RR}^{\mathcal{O}})$ and the tensor product $\mathcal{R}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{R}(W_{RR}^{\mathcal{O}})$ (in the sense of von Neumann algebras):

$$Y^{\mathcal{O}} A_1 A_2 Y^{\mathcal{O}*} = A_1 \otimes A_2 \quad \forall A_1 \in \mathcal{R}(W_{LL}^{\mathcal{O}}), A_2 \in \mathcal{R}(W_{RR}^{\mathcal{O}}). \tag{3.2.2}$$

Using the isomorphism implemented by $Y^{\mathcal{O}}$ we then have the following correspondences:

$$\begin{array}{ccccc} \mathcal{R}(W_{LL}^{\mathcal{O}}) & \cong & \mathcal{R}(W_{LL}^{\mathcal{O}}) & \otimes & \mathbf{1} \\ \mathcal{R}(W_{RR}^{\mathcal{O}}) & \cong & \mathbf{1} & \otimes & \mathcal{R}(W_{RR}^{\mathcal{O}}) \\ \mathcal{R}(W_{L}^{\mathcal{O}}) & \cong & \mathcal{B}(\mathcal{H}_0) & \otimes & \mathcal{R}(W_{L}^{\mathcal{O}}) \\ \mathcal{R}(W_{R}^{\mathcal{O}}) & \cong & \mathcal{R}(W_{R}^{\mathcal{O}}) & \otimes & \mathcal{B}(\mathcal{H}_0), \end{array} \tag{3.2.3}$$

whereas Haag duality for double cones yields

$$\mathcal{A}(\mathcal{O}) = \mathcal{R}(W_{L}^{\mathcal{O}}) \wedge \mathcal{R}(W_{R}^{\mathcal{O}}) \cong \mathcal{R}(W_{R}^{\mathcal{O}}) \otimes \mathcal{R}(W_{L}^{\mathcal{O}}). \tag{3.2.4}$$

In conjunction with the well known fact [48] that the algebras associated with wedge regions are factors of type III_1 we see that our assumptions imply that the algebras of double cones are type III_1 factors, too.
Remark. This computation of the intersection of tensor products is justified by [112, Cor. 5.10], which will be used quite frequently. Namely, for arbitrary von Neumann algebras M_1, M_2 on $\mathcal{H}_1$ and N_1, N_2 on $\mathcal{H}_2$ the following identities hold:

$$(M_1 \otimes N_1) \vee (M_2 \otimes N_2) = (M_1 \vee M_2) \otimes (N_1 \vee N_2), \tag{3.2.5}$$
$$(M_1 \otimes N_1) \wedge (M_2 \otimes N_2) = (M_1 \wedge M_2) \otimes (N_1 \wedge N_2). \tag{3.2.6}$$

3.3 Strong Additivity and the Time-Slice Axiom

Starting with the following easy lemma we will now show several further consequences of the pair of axioms Haag duality (HD) and SPW.

Lemma 3.3.1

$$\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{A}(\mathcal{O}) = \mathcal{R}(W_{L}^{\mathcal{O}}), \tag{3.3.1}$$
$$\mathcal{R}(W_{RR}^{\mathcal{O}}) \vee \mathcal{A}(\mathcal{O}) = \mathcal{R}(W_{R}^{\mathcal{O}}). \tag{3.3.2}$$

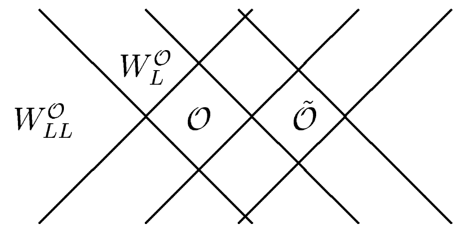


Figure 3.1: Double cones sharing one point

Remark. Equivalently, the inclusions $\mathcal{R}(W_{LL}^{\mathcal{O}}) \subset \mathcal{R}(W_L^{\mathcal{O}}), \mathcal{R}(W_{RR}^{\mathcal{O}}) \subset \mathcal{R}(W_R^{\mathcal{O}})$ are normal.
Proof. Under the unitary equivalence $\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_{RR}^{\mathcal{O}}) \cong \mathcal{R}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{R}(W_{RR}^{\mathcal{O}})$ we have $\mathcal{R}(W_{LL}^{\mathcal{O}}) \cong \mathcal{R}(W_{LL}^{\mathcal{O}}) \otimes \mathbf{1}$ and $\mathcal{A}(\mathcal{O}) = \mathcal{R}(W_L^{\mathcal{O}}) \cap \mathcal{R}(W_R^{\mathcal{O}}) \cong \mathcal{R}(W_R^{\mathcal{O}}) \otimes \mathcal{R}(W_L^{\mathcal{O}})$. Thus $\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{A}(\mathcal{O}) \cong (\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_R^{\mathcal{O}})) \otimes \mathcal{R}(W_L^{\mathcal{O}})$. Due to wedge duality and factoriality of the wedge algebras [48] this equals $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}(W_L^{\mathcal{O}}) \cong \mathcal{R}(W_L^{\mathcal{O}})$. The second equation is proved in the same way. ■

Consider now the situation depicted in Figure 3.1. In particular, $\mathcal{O}, \tilde{\mathcal{O}}$ are spacelike separated double cones the closures of which share one point. Such double cones will be called *adjacent*.

Lemma 3.3.2 *Let $\hat{\mathcal{O}} = \sup(\mathcal{O}, \tilde{\mathcal{O}})$ be the smallest double cone containing $\mathcal{O}, \tilde{\mathcal{O}}$. Then*

$$\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}}) = \mathcal{A}(\hat{\mathcal{O}}). \tag{3.3.3}$$

Proof. In the situation of Figure 3.1 we have $\hat{\mathcal{O}} = W_R^{\mathcal{O}} \cap W_L^{\tilde{\mathcal{O}}})$. Under the unitary equivalence considered above we have $\mathcal{A}(\tilde{\mathcal{O}}) \cong \mathbf{1} \otimes \mathcal{A}(\tilde{\mathcal{O}})$ as $\tilde{\mathcal{O}} \subset W_{RR}^{\mathcal{O}}$. Thus $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}}) \cong \mathcal{R}(W_R^{\mathcal{O}}) \otimes (\mathcal{R}(W_L^{\mathcal{O}}) \vee \mathcal{A}(\tilde{\mathcal{O}}))$. But now $W_L^{\mathcal{O}} = W_{LL}^{\tilde{\mathcal{O}}}$ leads to $\mathcal{R}(W_L^{\mathcal{O}}) \vee \mathcal{A}(\tilde{\mathcal{O}}) = \mathcal{R}(W_L^{\tilde{\mathcal{O}}})$ via the preceding lemma. Thus $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}}) \cong \mathcal{R}(W_R^{\mathcal{O}}) \otimes \mathcal{R}(W_L^{\tilde{\mathcal{O}}})$ which in turn is unitarily equivalent to $\mathcal{R}(W_R^{\mathcal{O}}) \wedge \mathcal{R}(W_L^{\tilde{\mathcal{O}}}) = \mathcal{A}(\hat{\mathcal{O}})$. ■
Remark. In analogy to chiral conformal field theory we denote this property *strong additivity.*

With these lemmas it is clear that the quantum field theories under consideration are *n-regular* in the sense of the following definition for all $n \geq 2$.

Definition 3.3.3 *A quantum field theory is n-regular if*

$$\mathcal{R}(W_1) \vee \mathcal{A}(\mathcal{O}_1) \vee \ldots \vee \mathcal{A}(\mathcal{O}_{n-2}) \vee \mathcal{R}(W_2) = \mathcal{B}(\mathcal{H}_0), \tag{3.3.4}$$

whenever $\mathcal{O}_i$, $i = 1, \ldots, n-2$ are mutually spacelike double cones such that the sets $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_{i+1}}$, $i = 1, \ldots, n-3$ each contain one point and where the wedges W_1, W_2 are such that

$$W_1 \cup W_2 = \left(\bigcup_{i=1}^{n-2} \mathcal{O}_i \right)'. \tag{3.3.5}$$

Corollary 3.3.4 *A quantum field theory in $1+1$ dimensions satisfying Haag duality and the SPW fulfills the (von Neumann version of the) time-slice axiom, i. e.*

$$\mathcal{R}(S) = \mathcal{B}(\mathcal{H}_0), \tag{3.3.6}$$

whenever $S = \{x \in \mathbb{R}^2 \mid x \cdot \eta \in (a,b)\}$ where $\eta \in \mathbb{R}^2$ is timelike and $a < b$.

Proof. The timeslice S contains an infinite string $\mathcal{O}_i$, $i \in \mathbb{Z}$ of mutually spacelike double cones as above. Thus the von Neumann algebra generated by all these double cones contains each $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ from which the claim follows by irreducibility. ■

Remarks. 1. We wish to emphasize that this statement on von Neumann algebras is weaker than the C^*-version of the timeslice axiom, which postulates that the C^*-algebra $\mathcal{A}(S)$ generated by the algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \subset S$ equals the quasilocal algebra $\mathcal{A}$. We follow the arguments in [71, Sec. III.3] to the effect that this stronger assumption should be avoided.

2. The above result fits well with the investigations [69, 86] concerning the time-slice property in the context of generalized free fields (in $3+1$ dimensions). In the cited works it was proved that generalized free fields possess the time-slice property iff (roughly) the spectral measure vanishes sufficiently fast at infinity. On the other hand, the split property imposes restrictions on the spectral measure [41, Thm. 10.2] which are considerably stronger. In particular, it is clear that the split property is by no means a necessary condition for the time-slice property.

3.4 Absence of Localized Charges

Whereas the results obtained so far are intuitively plausible, we will now prove a no-go theorem which shows that the combination of Haag duality and the SPW is extremely strong.

Theorem 3.4.1 *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net of observables satisfying Haag duality and the split property for wedges. Let π be a representation of the quasilocal algebra $\mathcal{A}$ which satisfies*

$$\pi \restriction \mathcal{A}(W) \cong \pi_0 \restriction \mathcal{A}(W) \quad \forall W \in \mathcal{W}, \tag{3.4.1}$$

where $\mathcal{W}$ is the set of all wedges (left and right handed). Then π is equivalent to an at most countable direct sum of representations which are unitarily equivalent to π_0:

$$\pi = \bigoplus_{i \in I} \pi_i, \qquad \pi_i \cong \pi_0. \tag{3.4.2}$$

In particular, if π is irreducible it is unitarily equivalent to π_0.

Remark. A fortiori, this applies to DHR representations (1.3.10).

Proof. Consider the geometry depicted in Figure 3.2. If π is a representation satisfying (3.4.1) then there is a unitary $V : \mathcal{H}_\pi \to \mathcal{H}_0$ such that, setting $\rho = V\pi(\cdot)V^*$, we have $\rho(A) = A$ if $A \in \mathcal{A}(W')$. Due to normality on wedges and wedge duality, ρ continues to

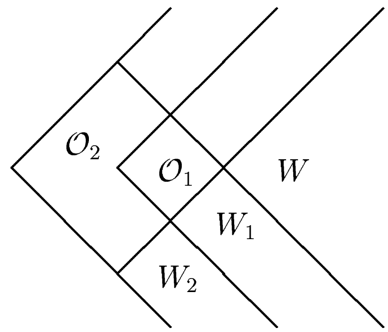


Figure 3.2: A split inclusion of wedges

normal endomorphisms of $\mathcal{R}(W), \mathcal{R}(W_1)$. By the split property there are type-I factors M_1, M_2 such that

$$\mathcal{R}(W) \subset M_1 \subset \mathcal{R}(W_1) \subset M_2 \subset \mathcal{R}(W_2). \tag{3.4.3}$$

Let $x \in M_1 \subset \mathcal{R}(W_1)$. Then $\rho(x) \in \mathcal{R}(W_1) \subset M_2$. Furthermore, ρ acts trivially on $M_1' \cap \mathcal{R}(W_2) \subset \mathcal{R}(W)' \cap \mathcal{R}(W_2) = \mathcal{A}(\mathcal{O}_2)$, where we have used Haag duality. Thus ρ maps M_1 into $M_2 \cap (M_1' \cap \mathcal{R}(W_2))' \subset M_2 \cap (M_1' \cap M_2)' = M_1$, the last identity following from M_1, M_2 being type-I factors. By [80, Cor. 3.8] every endomorphism of a type I factor is inner, i. e. there is a (possibly infinite) family of isometries $V_i \in M_1,\ i \in I$ with $V_i^* V_j = \delta_{i,j}, \sum_{i\in I} V_i V_i^* = \mathbf{1}$ such that

$$\rho(A) = \eta(A) \quad \forall A \in M_1, \tag{3.4.4}$$

where

$$\eta(A) \equiv \sum_{i\in I} V_i\, A\, V_i^*, \ \ A \in \mathcal{B}(\mathcal{H}_0). \tag{3.4.5}$$

(The sum over I is understood in the strong sense.) Now, ρ and thus η act trivially on $M_1 \cap \mathcal{R}(W)' \subset \mathcal{R}(W_1) \cap \mathcal{R}(W)' = \mathcal{A}(\mathcal{O}_1)$, which implies

$$V_i \in M_1 \cap (M_1 \cap \mathcal{R}(W)')' = \mathcal{R}(W). \tag{3.4.6}$$

Thanks to Lemma 3.3.1 we know that for every wedge $\hat{W} \supset\supset W$

$$\mathcal{R}(\hat{W}) = \mathcal{R}(W) \vee \mathcal{A}(\mathcal{O}), \tag{3.4.7}$$

where $\mathcal{O} = \hat{W} \cap W'$. From the fact that ρ acts trivially on $\mathcal{A}(W')$ it follows that (3.4.4) is true also for $A \in \mathcal{A}(\mathcal{O})$. By assumption, ρ is normal also on $\mathcal{A}(\hat{W})$ which leads to (3.4.4) on $\mathcal{A}(\hat{W})$. As this holds for every $\hat{W} \supset\supset W$, we conclude that

$$\pi(A) = \sum_{i\in I} V^* V_i\, A\, V_i^* V, \quad \forall A \in \mathcal{A}. \tag{3.4.8}$$

■

Remarks. 1. The main idea of the proof is taken from [40, Prop. 2.3].
2. The above result may seem inconvenient as it trivializes the DHR/FRS superselection theory [37, 39, 56] for a large class of massive quantum field theories in $1+1$ dimensions. It is not so clear what this means with respect to field theoretical models since there is little known about Haag duality in nontrivial models.
3. Conformal quantum field theories possessing no representations besides the vacuum representation, or 'holomorphic' theories, have been the starting point for an analysis of 'orbifold' theories in [33]. In [87], which was motivated by the desire to obtain a rigorous understanding of orbifold theories in the framework of massive two dimensional theories, the present author postulated the split property for wedges and claimed it to be weaker than the requirement of absence of nontrivial representations. Whereas this claim is disproved by Theorem 3.4.1, as far as localized (DHR or BF) representations of Haag dual theories are concerned, none of the results of [87] is invalidated or rendered obsolete.

3.5 Haag Duality in Locally Normal Representations

A further crucial consequence of the split property for wedges is observed in the following

Proposition 3.5.1 *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net satisfying Haag duality (for double cones) and the split property for wedges. Then for every pair $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ we have*

$$\mathcal{A}(\hat{\mathcal{O}}) \wedge \mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R). \tag{3.5.1}$$

Proof. By the split property for wedges there is [20, 87] a unitary operator $Y^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ such that $\mathcal{R}(\hat{W}_L) \vee \mathcal{R}(\hat{W}_R) = Y^{\mathcal{O}*}(\mathcal{R}(\hat{W}_L) \otimes \mathcal{R}(\hat{W}_R))Y^{\mathcal{O}}$. More concretely,

$$Y^{\mathcal{O}}\, xy\, Y^{\mathcal{O}*} = x \otimes y \quad \forall x \in \mathcal{R}(\hat{W}_L),\ y \in \mathcal{R}(\hat{W}_R). \tag{3.5.2}$$

By Haag duality $\mathcal{A}(\mathcal{O})' = \mathcal{R}(\hat{W}_L) \vee \mathcal{R}(\hat{W}_R) \cong \mathcal{R}(\hat{W}_L) \otimes \mathcal{R}(\hat{W}_R)$ and $\mathcal{A}(\hat{\mathcal{O}})' = \mathcal{R}(W_L) \vee \mathcal{R}(W_R)$. Now $\mathcal{R}(W_{L/R}) \subset \mathcal{R}(\hat{W}_{L/R})$ implies $\mathcal{A}(\hat{\mathcal{O}})' \cong \mathcal{R}(W_L) \otimes \mathcal{R}(W_R)$ and thus

$$\mathcal{A}(\hat{\mathcal{O}}) \cong (\mathcal{R}(W_L) \otimes \mathcal{R}(W_R))' = \mathcal{R}(W_L') \otimes \mathcal{R}(W_R'), \tag{3.5.3}$$

where we have used wedge duality and the commutation theorem for tensor products. Now we can compute the relative commutant as follows:

$$\begin{aligned} \mathcal{A}(\hat{\mathcal{O}}) \wedge \mathcal{A}(\mathcal{O})' &\cong (\mathcal{R}(W_L') \otimes \mathcal{R}(W_R')) \wedge (\mathcal{R}(\hat{W}_L) \otimes \mathcal{R}(\hat{W}_R)) \\ &= (\mathcal{R}(\hat{W}_L) \wedge \mathcal{R}(W_L')) \otimes (\mathcal{R}(\hat{W}_R) \wedge \mathcal{R}(W_R')) \\ &= \mathcal{A}(\mathcal{O}_L) \otimes \mathcal{A}(\mathcal{O}_R) \cong \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R). \end{aligned} \tag{3.5.4}$$

We have used Haag duality in the form $\mathcal{R}(\hat{W}_L) \wedge \mathcal{R}(W_L') = \mathcal{A}(\mathcal{O}_L)$ and similarly for $\mathcal{A}(\mathcal{O}_R)$. All unitary equivalences $\cong$ are, of course, implemented by the same operator $Y^{\mathcal{O}}$.
■

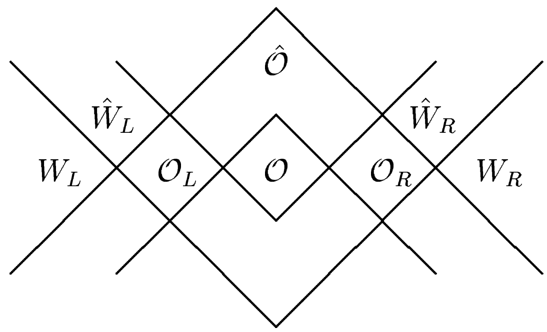


Figure 3.3: Relative commutant of double cones

This result should contribute to the understanding of Theorem 3.4.1 as far as DHR representations are concerned. In fact, it already implies the absence of DHR sectors as can be shown by an application of the triviality criterion for local 1-cohomologies [97] given in [98], see also [99].

Sketch of proof. Let $z \in Z^1(\mathcal{A})$ be the local 1-cocycle associated according to [97, 98] with a representation π satisfying the DHR criterion. Due to Proposition 3.5.1 it satisfies $z(b) \in \mathcal{A}(|\partial_0 b|) \vee \mathcal{A}(|\partial_1 b|)$ for every $b \in \Sigma_1$ such that $|\partial_0 b| \subset\subset |\partial_1 b|'$. Thus the arguments in the proof of [98, Thm. 3.5] are applicable despite the fact that we are working in $1+1$ dimensions. We thereby see that there are unique Hilbert spaces $H(\mathcal{O}) \subset \mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \Sigma_0 \equiv \mathcal{K}$ of support I such that $z(b)H(\partial_1 b) = H(\partial_0 b)\ \forall b \in \Sigma_1$. Each of these Hilbert spaces implements an endomorphism $\rho_{\mathcal{O}}$ of $\mathcal{A}$ such that $\rho_{\mathcal{O}} \cong \pi$. This implies that ρ is either reducible or an inner automorphism. ■

Remarks. 1. A third proof of the absence of DHR sectors results by combining Proposition 3.5.1 with Lemma 2.3.4.

2. The above argument needs the split property for double cones. It is not completely trivial that this follows from the split property for wedges. It is clear that the latter implies unitary equivalence of $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ and $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1, \mathcal{O}_2$ are double cones separated by a finite spacelike distance. The split property for double cones requires more, namely unitary equivalence of $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\hat{\mathcal{O}})'$ and $\mathcal{A}(\mathcal{O}) \otimes \mathcal{A}(\hat{\mathcal{O}})'$ whenever $\mathcal{O} \subset\subset \hat{\mathcal{O}}$, which is equivalent to the existence of a type I factor N such that $\mathcal{A}(\mathcal{O}) \subset N \subset \mathcal{A}(\hat{\mathcal{O}})$.

Lemma 3.5.2 *Let $\mathcal{A}$ be a local net satisfying Haag duality and the split property for wedges. Then the split property for double cones holds.*

Proof. Using the notation of the preceding proof we have

$$\mathcal{A}(\mathcal{O}) \cong \mathcal{R}(\hat{W}_L') \otimes \mathcal{R}(\hat{W}_R'), \tag{3.5.5}$$

$$\mathcal{A}(\hat{\mathcal{O}}) \cong \mathcal{R}(W_L') \otimes \mathcal{R}(W_R'). \tag{3.5.6}$$

By the SPW there are type I factors N_L, N_R such that $\mathcal{R}(\hat{W}_L') \subset N_L \subset \mathcal{R}(W_L')$ and $\mathcal{R}(\hat{W}_R') \subset N_R \subset \mathcal{R}(W_R')$. Thus $Y^{\mathcal{O}*}(N_L \otimes N_R)Y^{\mathcal{O}}$ is a type I factor sitting between $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\hat{\mathcal{O}})$. ■

Having disproved the existence of nontrivial representations localized in double cones or wedges, we will now prove a result which concerns a considerably larger class of representations.

Theorem 3.5.3 *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net of observables satisfying Haag duality and the SPW. Then every irreducible, locally normal representation of the quasilocal algebra $\mathcal{A}$ fulfills Haag duality.*

Proof. We will show that our assumptions imply those of [49, Thm. 1]. $\mathcal{A}$ satisfies the split property for double cones (called 'funnel property' in [49, 105]) by Lemma 3.5.2, whereas we also assume condition (1) of [49, Theor. 1] (Haag duality and irreducibility). Condition (3), which concerns relative commutants $\mathcal{A}(\mathcal{O}_2) \cap \mathcal{A}(\mathcal{O}_1)'$, $\mathcal{O}_2 \supset\supset \mathcal{O}_1$ in the vacuum representation, is an immediate consequence of Proposition 3.5.1 (we may even take $\mathcal{O} = \mathcal{O}_1, \mathcal{O}_2 = \mathcal{O}_3$). Finally, Lemma 3.3.1 implies

$$\mathcal{A}(\mathcal{O})' = \mathcal{A}(\hat{\mathcal{O}})' \vee \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R), \tag{3.5.7}$$

where we again use the notation of Figure 3.3. This is more than required by Driessler's condition (2). Now [49, Theor. 1] applies and we are done. ■

Remarks. 1. In [105] a slightly simplified version of [49, Theor. 1] is given which dispenses with condition (2) at the price of a stronger form of condition (3). This condition is still (more than) fulfilled by our class of theories.

2. Observing that soliton representations are locally normal with respect to both asymptotic vacua [58, 101], we conclude at once that Haag duality holds for every irreducible soliton sector where at least one of the vacua satisfies Haag duality and the SPW. Consequences of this fact will be explored in the next section. We remark without going into details that our results are also of relevance for the construction of soliton sectors with prescribed asymptotic vacua in [101].

3.6 Applications to the Theory of Quantum Solitons

In [18] it has been shown that every factorial massive one-particle representation (MOPR) in $\geq 2+1$ dimensions is a multiple of an irreducible representation which is localizable in every spacelike cone. (Here, MOPR means that the lower bound of the energy-momentum spectrum consists of a hyperboloid of mass $m > 0$ which is separated from the rest of the spectrum by a mass gap.) In $1+1$ dimensions one is led to irreducible soliton sectors [58] which we will now reconsider in the light of Theorems 3.4.1, 3.5.3. In this section, where we are concerned with inequivalent vacuum representations, we will consider a QFT to be defined by a net of abstract C^*-algebras instead of the algebras in a concrete representation. Given two vacuum representations π_0^L, π_0^R, a representation π is said to be a soliton representation of type (π_0^L, π_0^R) if it is translation covariant and

$$\pi \restriction \mathcal{A}(W_{L/R}) \cong \pi_0^{L/R} \restriction \mathcal{A}(W_{L/R}), \tag{3.6.1}$$

where W_L, W_R are arbitrary left and right handed wedges, respectively. An obvious consequence of (3.6.1) is local normality of π_0^L, π_0^R with respect to each other. In order to formulate a useful theory of soliton representations [58, 59] one must assume $\pi_0^{L/R}$ to satisfy wedge duality. After giving a short review of the formalism in [58, 59], we will show in this section that considerably more can be said under the stronger assumption that one of the vacuum representations satisfies duality for double cones and the SPW. (Then the other vacuum is automatically Haag dual, too.)

Let π_0 be a vacuum representation and $W \in \mathcal{W}$ a wedge. Then by $\mathcal{A}(W)_{\pi_0}$ we denote the W^*-completion of the C^*-algebra $\mathcal{A}(W)$ with respect to the family of seminorms given by

$$\|A\|_T = |\mathrm{tr}\, T\pi_0(A)|, \tag{3.6.2}$$

where T runs through the set of all trace class operators in $\mathcal{B}(\mathcal{H}_{\pi_0})$. Furthermore, we define extensions $\mathcal{A}_{\pi_0}^L$, $\mathcal{A}_{\pi_0}^R$ of the quasilocal algebra $\mathcal{A}$ by

$$\mathcal{A}_{\pi_0}^{L/R} = \overline{\bigcup_{W \in \mathcal{W}_{L/R}} \mathcal{A}(W)_{\pi_0}}^{\|\cdot\|}, \tag{3.6.3}$$

where $\mathcal{W}_L, \mathcal{W}_R$ are the sets of left and right wedges, respectively. Now, it has been demonstrated in [59] that, given a (π_0^L, π_0^R)-soliton representation π, there are homomorphisms ρ from $\mathcal{A}_{\pi_0^R}^R$ to $\mathcal{A}_{\pi_0^L}^R$ such that

$$\pi \cong \pi_0^L \circ \rho \tag{3.6.4}$$

(strictly speaking, π_0^L must be extended to $\mathcal{A}_{\pi_0^L}^R$, which is trivial since $\mathcal{A}(W)_{\pi_0}$ is isomorphic to $\pi_0(\mathcal{A}(W))''$). The morphism ρ is localized in some right wedge W in the sense that

$$\rho \restriction \mathcal{A}(W') = \mathrm{id} \restriction \mathcal{A}(W'). \tag{3.6.5}$$

Provided that the vacua of two soliton representations π, π' 'fit together' $\pi_0^R \cong \pi_0'^L$ one can define a soliton representation $\pi \times \pi'$ of type $\pi_0^L, \pi_0'^R$ via composition of the corresponding morphisms:

$$\pi \times \pi' \cong \pi_0^L \circ \rho\rho' \restriction \mathcal{A}. \tag{3.6.6}$$

Alternatively, the entire analysis may be done in terms of left localized morphisms η from $\mathcal{A}_{\pi_0^L}^L$ to $\mathcal{A}_{\pi_0^R}^L$. As proved in [59], the unitary equivalence class of the composed representation depends neither on the use of left or right localization nor on the concrete choice of the morphisms.

Whereas for soliton representations there is no analog to the theory of statistics [37, 39, 56], one can still *define* a 'dimension' d_ρ via

$$d_\rho^2 \equiv [\mathcal{A}(W)_{\pi_0^L} : \rho(\mathcal{A}(W)_{\pi_0^R})], \tag{3.6.7}$$

where ρ is localized in the right wedge W and $[M : N]$ is the Jones index of the inclusion $N \subset M$.

Proposition 3.6.1 *Let π be an irreducible soliton representation such that at least one of the asymptotic vacua π_0^L, π_0^R satisfies Haag duality and the SPW. Then the associated morphism satisfies ind$(\rho) = 1$.*

Proof. Since the representation π satisfies Haag duality by Theorem 3.5.3 we have in particular $\pi(\mathcal{A}(W))^- = \pi(\mathcal{A}(W'))'$. Thus

$$\pi_0^L \circ \rho(\mathcal{A}(W))^- = \pi_0^L \circ \rho(\mathcal{A}(W'))' = \pi_0^L(\mathcal{A}(W'))' = \pi_0^L(\mathcal{A}(W))^-. \tag{3.6.8}$$

By ultraweak continuity on $\mathcal{A}(W)$ of π_0^L and of ρ this implies

$$\rho(\mathcal{A}(W)_{\pi_0^R}) = \mathcal{A}(W)_{\pi_0^L} \tag{3.6.9}$$

whence the claim. ■

This result rules out soliton sectors with infinite index so that [100, Thm. 3.2] applies and yields equivalence of the various possibilities of constructing antisoliton sectors considered in [100]. In particular the antisoliton sector is uniquely defined up to unitary equivalence. Now we can formulate our main result concerning soliton representations.

Theorem 3.6.2 *Let π_0^L, π_0^R be vacuum representations, at least one of which satisfies Haag duality and the SPW. Then all soliton representations of type (π_0^L, π_0^R) are unitarily equivalent.*

Remark. Equivalently, up to unitary equivalence, a soliton representation is completely characterized by the pair of asymptotic vacua.
Proof. Let π, π' be irreducible soliton representations of types (π_0, π_0') and (π_0', π_0), respectively. They may be composed, giving rise to a soliton representation of type (π_0, π_0) (or (π_0', π_0')). This representation is irreducible since the morphisms ρ, ρ' must be isomorphisms by the proposition. Now, $\pi \times \pi'$ is unitarily equivalent to π_0 on left *and* right handed wedges, which by Theorem 3.4.1 and irreducibility implies $\pi \times \pi' \cong \pi_0$. We conclude that every (π_0', π_0)-soliton is an antisoliton of every (π_0, π_0')-soliton. This implies the statement of the theorem since for every soliton representation with finite index there is a corresponding antisoliton which is unique up to unitary equivalence. ■
Remark. The above proof relies on the absence of nontrivial representations which are localizable in wedges. Knowing just that DHR sectors do not exist, as follows already from Proposition 3.5.1, is not enough.

3.7 DHR Representations of Nets without Haag Duality

We have observed that the theory of localized representations of Haag-dual nets of observables which satisfy the SPW is trivial. There are, however, quantum field theories in $1+1$ dimensions where the net of algebras which is most naturally considered as the net of observables does not fulfill Haag duality in the strong form (1.2.9). As mentioned in Chapter 1, this is the case if the observables are defined as the fixpoints under a global symmetry group of a field net which satisfies (twisted) duality and the SPW. The weaker property of wedge duality, namely

$$\mathcal{R}(W)' = \mathcal{R}(W') \quad \forall W \in \mathcal{W}, \tag{3.7.1}$$

remains, however. This property is also known to hold whenever the local algebras arise from a Wightman field theory [12]. However, for the analysis in [37, 39, 57] as well as Section 4 above one needs full Haag duality. Therefore it is of relevance that, starting from a net of observables satisfying only (3.7.1), one can define a larger but still local net

$$\mathcal{A}^d(\mathcal{O}) \equiv \mathcal{R}(W_L^{\mathcal{O}}) \wedge \mathcal{R}(W_R^{\mathcal{O}}) \tag{3.7.2}$$

which satisfies Haag duality, whence the name *dual net*. Here $W_L^{\mathcal{O}}, W_R^{\mathcal{O}}$ are wedges such that $W_L^{\mathcal{O}} \cap W_R^{\mathcal{O}} = \mathcal{O}$ and duality is seen to follow from the fact that the wedge algebras $\mathcal{R}(W), W \in \mathcal{W}$ are the same for the nets $\mathcal{A}, \mathcal{A}^d$. (For observables arising as group fixpoints the dual net has been computed explicitly in [87].) Now, *for* $\geq 2+1$ *dimensions* it is known [98, 99] that representations π satisfying the DHR criterion (1.3.10) extend uniquely to DHR representations $\hat{\pi}$ of the dual net. Here the original net is required to satisfy essential duality. Furthermore, the categories of DHR representations of $\mathcal{A}$ and $\mathcal{A}^d$, respectively, and their intertwiners are isomorphic. Thus, instead of $\mathcal{A}$ one may as well study $\mathcal{A}^d$ to which the usual methods are applicable. In $1+1$ dimensions things are more complicated. As shown in [97] there are in general two different extensions $\hat{\pi}^L, \hat{\pi}^R$. They coincide iff one (thus both) of them is a DHR representation. Even before defining precisely these extensions we can state the following consequence of Theorem 3.4.1.

Proposition 3.7.1 *Let $\mathcal{A}$ be a net of observables satisfying wedge duality and the SPW. Let π be an irreducible DHR or BF representation of $\mathcal{A}$ which is not unitarily equivalent to the defining (vacuum) representation. Then there is no extension $\hat{\pi}$ to the dual net $\mathcal{A}^d$ which is still localized in the DHR or BF sense.*

Proof. Assume π to be the restriction to $\mathcal{A}$ of a wedge-localized representation $\hat{\pi}$ of $\mathcal{A}^d$. As the latter is known to be either reducible or unitarily equivalent to π_0, the same holds for π. This is a contradiction. ∎

The fact that the extension of a localized representation of $\mathcal{A}$ to the dual net $\mathcal{A}^d$ cannot be localized, too, partially undermines the original motivation for considering these extensions. Nevertheless, one may entertain the hope that there is something to be learnt which is useful for a model-independent analysis of the phenomena observed in exactly soluble models. Therefore, we now turn to the examination of the extensions $\hat{\pi}^L, \hat{\pi}^R$, assuming first that π is localizable only in wedges.

Let $\mathcal{O}$ be a double cone and let W_L, W_R be left and right handed wedges, respectively, containing $\mathcal{O}$. By assumption the restriction of π to $\mathcal{A}(W_L), \mathcal{A}(W_R)$ is unitarily equivalent to π_0. Choose unitary implementers U_L, U_R such that

$$\begin{aligned} Ad\,U_L \restriction \mathcal{A}(W_L) &= \pi \restriction \mathcal{A}(W_L), \\ Ad\,U_R \restriction \mathcal{A}(W_R) &= \pi \restriction \mathcal{A}(W_R). \end{aligned} \tag{3.7.3}$$

Then $\hat{\pi}^L, \hat{\pi}^R$ are defined for $A \in \mathcal{A}^d(\mathcal{O})$ by

$$\begin{aligned} \hat{\pi}^L(A) &= U_L\,A\,U_L^*, \\ \hat{\pi}^R(A) &= U_R\,A\,U_R^*. \end{aligned} \tag{3.7.4}$$

Independence of these definitions of the choice of W_L, W_R and the implementers U_L, U_R follows straightforwardly from wedge duality. We state some immediate consequences of this definition.

Proposition 3.7.2 *$\hat{\pi}^L, \hat{\pi}^R$ are irreducible, locally normal representations of $\mathcal{A}^d$ and satisfy Haag duality. $\hat{\pi}^L, \hat{\pi}^R$ are normal on left and right handed wedges, respectively.*

Proof. Irreducibility is a trivial consequence of the assumed irreducibility of π whereas local normality is obvious from the definition (3.7.4). Thus, Theorem 3.5.3 applies and yields Haag duality in both representations. Normality of, say, $\hat{\pi}^L$ on left handed wedges W follows from the fact that we may use the same auxiliary wedge $W_L \supset W$ and implementer U_L for all double cones $\mathcal{O} \subset W$. ■

Clearly, the extensions $\hat{\pi}^L, \hat{\pi}^R$ cannot be normal to π_0 on right and left wedges, respectively, for otherwise Theorem 3.4.1 would imply unitary equivalence to π_0. In general, we can only conclude localizability in the following weak sense. Given an arbitrary left handed wedge W, $\hat{\pi}^L$ is equivalent to a representation ρ on $\mathcal{H}_0$ such that $\rho(A) = A \;\forall A \in \mathcal{A}(W)$. Furthermore, by duality ρ is an isomorphism of $\mathcal{A}(W')$ onto a weakly dense subalgebra of $\mathcal{R}(W')$ which is only continuous in the norm. In favorable cases this is a local symmetry, acting as an automorphism of $\mathcal{A}(W')$. But we will see shortly that there are perfectly non-pathological situations where the extensions are not of this particularly nice type. In complete generality, the best one can hope for is normality with respect to another vacuum representation π_0'. In particular, this is automatically the case if π is a massive one-particle representation [18] which we did not assume so far.

If the representation π satisfies the DHR criterion, i.e. is localizable in double cones, we can obtain stronger results concerning the localization properties of the extended representations $\hat{\pi}_L, \hat{\pi}_R$. By the criterion, there are unitary operators $X^{\mathcal{O}}: \mathcal{H}_0 \to \mathcal{H}_\pi$ such that

$$\pi^{\mathcal{O}}(A) \equiv X^{\mathcal{O}*}\,\pi(A)\,X^{\mathcal{O}} = A, \quad \forall A \in \mathcal{A}(\mathcal{O}'). \tag{3.7.5}$$

(By wedge duality, $X^{\mathcal{O}}$ is unique up to right multiplication by operators in $\mathcal{A}^d(\mathcal{O})$.) Considering the representations

$$\hat{\pi}^{\mathcal{O}}_{L/R} = X^{\mathcal{O}*}\hat{\pi}_{L/R}X^{\mathcal{O}} \tag{3.7.6}$$

on the vacuum Hilbert space $\mathcal{H}_0$, it is easy to verify that

$$\hat{\pi}^{\mathcal{O}}_L \restriction \mathcal{A}^d(W^{\mathcal{O}}_{LL}) \;=\; \mathrm{id} \restriction \mathcal{A}^d(W^{\mathcal{O}}_{LL}), \tag{3.7.7}$$
$$\hat{\pi}^{\mathcal{O}}_R \restriction \mathcal{A}^d(W^{\mathcal{O}}_{RR}) \;=\; \mathrm{id} \restriction \mathcal{A}^d(W^{\mathcal{O}}_{RR}). \tag{3.7.8}$$

We restrict our attention to $\hat{\pi}^{\mathcal{O}}_L$, the other extension behaving similarly. If $A \in \mathcal{A}(\tilde{\mathcal{O}})$ then $\hat{\pi}_L(A) = X^{\mathcal{O}_r}\,A\,X^{\mathcal{O}_r *}$ whenever $\mathcal{O}_r > \tilde{\mathcal{O}}$. Therefore

$$\hat{\pi}^{\mathcal{O}}_L(A) = X^{\mathcal{O}*}X^{\mathcal{O}_r}\,A\,X^{\mathcal{O}_r *}X^{\mathcal{O}}, \tag{3.7.9}$$

where the unitary $X^{\mathcal{O}*}X^{\mathcal{O}_r}$ intertwines $\pi^{\mathcal{O}}$ and $\pi^{\mathcal{O}_r}$. Associating with every pair $(\mathcal{O}_1, \mathcal{O}_2)$ two other double cones by

$$\hat{\mathcal{O}} \;=\; \sup(\mathcal{O}_1, \mathcal{O}_2), \tag{3.7.10}$$
$$\mathcal{O}_0 \;=\; \hat{\mathcal{O}} \cap \mathcal{O}_1' \cap \mathcal{O}_2' \tag{3.7.11}$$

($\mathcal{O}_0$ may be empty) and defining

$$\mathcal{C}(\mathcal{O}_1, \mathcal{O}_2) = \mathcal{A}^d(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O}_0)', \tag{3.7.12}$$

we can conclude by wedge duality that

$$X^{\mathcal{O}*}X^{\mathcal{O}_r} \in \mathcal{C}(\mathcal{O}, \mathcal{O}_r). \tag{3.7.13}$$

Thus $\hat{\pi}_L^{\mathcal{O}}(A)$ as given by (3.7.9) is contained in $\mathcal{A}^d(\sup(\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{O}_r))$ which already shows that $\hat{\pi}_L^{\mathcal{O}}$ maps the quasilocal algebra $\mathcal{A}^d$ into itself (this does not follow if π is only localizable in wedges). Since the double cone $\mathcal{O}_r > \mathcal{O}$ may be chosen arbitrarily small and appealing to outer regularity of the dual net $\mathcal{A}^d$ we even have $\hat{\pi}_L^{\mathcal{O}}(A) \in \mathcal{A}^d(\sup(\mathcal{O}, \tilde{\mathcal{O}}))$ and thus finally

$$\hat{\pi}_L^{\mathcal{O}}(\mathcal{A}^d(\tilde{\mathcal{O}})) \subset \mathcal{C}(\mathcal{O}, \tilde{\mathcal{O}}). \tag{3.7.14}$$

This result shows the representation $\hat{\pi}_L^{\mathcal{O}}$ (and $\hat{\pi}_R^{\mathcal{O}}$) deteriorates the localization but still maps local algebras into local algebras. We will see shortly that this phenomenon is not just a theoretical possibility but really occurs. The above considerations are similar to Roberts' local 1-cohomology [97, 98, 99], but (3.7.14) seems to be new. In Section 4.4.6 we will return to the considerations of this chapter and apply them to the fixpoint nets of a field net under the action of a symmetry group, which are known to violate Haag duality. We will see that the weak localization property (3.7.14) is in fact realized if the group G is non-abelian, whereas in the abelian case one obtains soliton representations of a more familiar type.

We conclude this section by giving an example of a representation which is equivalent to a vacuum representation only on one side. Consider, e.g., two free massive scalar fields ϕ_1, ϕ_2 of different masses. In [65] the automorphism Λ_θ of the quasilocal algebra $\mathcal{A}$ which arises from the transformation

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}(x,0) = \Theta(x)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(x,0), \quad \begin{pmatrix} \dot{\phi}_1' \\ \dot{\phi}_2' \end{pmatrix}(x,0) = \Theta(x)\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}(x,0) \tag{3.7.15}$$

of the time-zero fields, where

$$\Theta(x) = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \tag{3.7.16}$$

and $\theta' \in \mathcal{D}(\mathbb{R})$, $\theta(-\infty) = 0$, $\theta(\infty) \neq n\pi$, was shown not to be unitarily implemented in the vacuum representation and was conjectured to lead to a superselection sector via $\pi_\theta = \pi_0 \circ \Lambda_\theta$. But, by Theorem 3.4.1 there are no representations of DHR or BF type since the vacuum representation of massive free (scalar and Dirac) fields is known to satisfy Haag duality and the SPW. On the other hand, Λ_θ is easily seen to be spatial in restriction to left handed wedges. (Thus, by Theorem 3.5.3 Haag duality holds in the representation π_θ due to irreducibility and local normality.) Finally, π cannot be a soliton state since

$$\omega_r \equiv \lim_{x\to+\infty} \omega_0 \circ \Lambda_\theta \circ \alpha_x = \omega_0 \circ \Lambda_{\theta(\infty)}, \tag{3.7.17}$$

where the global $SO(2)$ rotation $\Lambda_{\theta(\infty)}$ of ϕ_1, ϕ_2 commutes only with the space translations but not with the time evolution. It seems doubtful that representations of this type are physically meaningful.

Chapter 4

On Massive Orbifold Theories

4.1 Introduction

Since the notion of the 'quantum double' was coined by Drinfel'd in his famous ICM lecture [50] there have been several attempts aimed at a clarification of its relevance to two dimensional quantum field theory. The quantum double appears implicitly in the work [33] on orbifold constructions in conformal field theory, where conformal quantum field theories (CQFTs) are considered whose operators are fixpoints under the action of a symmetry group on another CQFT. Whereas the authors emphasize that 'the fusion algebra of the holomorphic G-orbifold theory naturally combines both the representation and class algebra of the group G' the relevance of the double is fully recognized only in [34]. There the construction is also generalized by allowing for an arbitrary 3-cocycle in $H^3(G, U(1))$ leading only, however, to a quasi quantum group in the sense of [51]. The quantum double also appears in the context of integrable quantum field theories, e.g. [10], as well as in certain lattice models (e.g. [108]). Common to these works is the role of disorder operators or 'twist fields' which are 'local with respect to $\mathcal{A}$ up to the action of an element $g \in G$' [33]. Finally, it should be mentioned that the quantum double and its twisted generalization also play a role in spontaneously broken gauge theories in $2+1$ dimensions (for a review and further references see [8]).

Regrettably most of these works (with the exception of [108]) are not very precise in stating the premises and the results in mathematically unambiguous terms. For example it is usually unclear whether the 'twist fields' have to be constructed or are already present in some sense in the theory one starts with. The aim of this chapter is to improve on this state of affairs by using the methods of algebraic quantum field theory. We will demonstrate the role of the quantum double as a hidden symmetry in *every* quantum field theory with group symmetry in $1+1$ dimensions fulfilling (besides the usual assumptions like locality) only two technical assumptions (Haag duality and split property for wedges) but independent of conformal covariance or exact integrability.

As explained in Chapter 1, the category of DHR representations of a Haag dual net of observables in $\geq 2+1$ dimensions is isomorphic to the representation category of a compact group G and, furthermore, there exists a field net with symmetry group G such that the reducible representation of the fixpoint net contains all equivalence classes of DHR representations. This result is definitely not true in $1+1$ dimensions due to the role of braid group statistics and quantum symmetries. Nevertheless one may, of course, study nets of local algebras which arise as fixpoint nets or 'orbifold theories'. Thus, as in [35] our starting point will be a net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ of von Neumann algebras on the common Hilbert space $\mathcal{H}$, satisfying the properties of a field algebra as described in Section 3 of Chapter 1. I.e. we assume irreducibility of the quasilocal algebra, Bose/Fermi commutation relations with twisted duality, and Poincaré covariance with spectrum condition. Covariance under the conformal group, however, is *not* required. Finally, there shall be a compact group G, represented in a strongly continuous fashion by unitary operators on $\mathcal{H}$ leaving invariant the vacuum vector Ω such that the automorphisms $\alpha_g(F) = Ad\, U(g)(F)$ of $\mathcal{B}(\mathcal{H})$ respect the local structure: $\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O})$. The action shall be faithful, i.e. $Aut\, \mathcal{F} \ni \alpha_g \neq \mathrm{id}\ \forall g \in G$. This is no real restriction, for the kernel of the homomorphism $G \rightarrow \mathrm{Aut}(\mathcal{F})$ can be divided out. (Compactness of G need in fact not be postulated, as it follows [41, Theorem 3.1] from the split property which we will also require.)

One remark on the commutativity of inner and spacetime symmetries is in order. Assuming the (distal) split property one can show, for quantum field theories in $\geq 2+1$ dimensions, that inner symmetries automatically commute with the action of the Poincaré group [41]. Since this result hinges on the non-existence of finite dimensional unitary representations of the latter, it is not necessarily true in 1+1 dimensions. In Appendix A we will, however, prove that in theories satisfying the distal split property the translations commute with the inner symmetries whereas the boosts act by automorphisms on the group G_{max} of *all* inner symmetries. As we will postulate a stronger version of the split property in the next section the cited result applies to the situation at hand. What we still have to assume is that the one parameter group of Lorentz boosts maps the subgroup $G \subset G_{\mathrm{max}}$ which we consider as the inner symmetries into itself and commutes with $V = U(k)$. This assumption is indispensable for the covariance of the fixpoint net $\mathcal{A}$ as well as of another net to be constructed later.

We will now comment on another two dimensional peculiarity which is less known and at first sight of interest only within the algebraic approach to quantum field theory. We refer to the fact that the step from (1.3.6) to (1.2.9) fails in 1+1 dimensions. This means that one cannot conclude from twisted duality of the fields that duality holds for the observables in simple sectors, which in fact is violated. The origin of this phenomenon is easily understood. Let $\mathcal{O} \in \mathcal{K}$ be a double cone. One can then construct gauge invariant operators in $\mathcal{F}(\mathcal{O}')$ which are obviously contained in $\mathcal{A}(\mathcal{O})'$ but not in $\mathcal{A}(\mathcal{O}')$. This is seen remarking that the latter algebra, belonging to a disconnected region, is defined to be generated by the observable algebras associated to the left and right spacelike complements of $\mathcal{O}$, respectively. This algebra does not contain gauge invariant operators constructed using fields localized in both components.

This is reminiscent of Roberts' analysis [95] of spontaneously broken symmetry symmetries, which was mentioned in Chapter 1. There nets of observables (in at least 2+1 dimensions) which arise as fixpoints under a group of inner symmetries from a field theory were shown to violate Haag duality whenever the symmetry is spontaneously broken in the sense that the vacuum is not invariant under the whole group. In this case the observables fulfill only essential duality, i.e. $\mathcal{A}^{dd}(\mathcal{O}) = \mathcal{A}^d(\mathcal{O})$ with (1.2.10). In the present situation, however, Haag duality is violated even though the group symmetry is unbroken!

We now come to the plan of this chapter. The aim will be to explore the relation between a quantum field theory with symmetry group G in 1+1 dimensions and the fixpoint theory. In addition to the general properties of such a theory stated above, twisted duality (1.3.6) is assumed to hold for the large theory. As explained above, in this situation duality of the fixpoint theory fails even in the case of unbroken group symmetry. As we will show in the next two sections essential duality holds, however, and we will compute the dual net quite explicitly. To this end we will need as an additional postulate the split property for wedges, which has already played an important role in the preceding chapter. In Section 4 we will uncover a hidden symmetry under the action of the quantum double $D(G)$ whose spontaneous breakdown one might interpret as the actual reason for the failure of Haag duality. The construction given there allows in particular the mathematically rigorous construction of quantum field theories with $D(G)$-symmetry for any finite group G. The final Section 5 contains a partial extension of the results of Section 4 to infinite compact groups, a treatment of Jordan-Wigner transformations and bosonization with our methods, and the first steps of an analysis of chiral orbifold nets on the circle. We refer to Appendix C of this thesis for a summary of the needed facts on quantum groups and quantum doubles.

4.2 Disorder Variables and the Split Property

4.2.1 Disorder Operators: Definition

Whereas, as we have remarked in Chapter 1, Haag duality for double cones is violated in the fixpoint theory $\mathcal{A}$, one obtains the following weaker form of duality. (In this chapter all algebras associated with infinite regions $\mathcal{O}', W$ are meant to be weakly closed, i.e. $\mathcal{F}(W)$ stands for $\pi_0(\mathcal{F}(W))''$.)

Proposition 4.2.1 *The representation of the fixpoint net $\mathcal{A}$ fulfills duality for wedges*

$$\mathcal{A}(W)' = \mathcal{A}(W') \tag{4.2.1}$$

and essential duality in all simple sectors.

Proof. The spacelike complement of a wedge region is itself a wedge, thus connected, whereby the proof of [35, Theorem 4.1] applies, yielding the first statement. The second follows from wedge duality via

$$\mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' = (\mathcal{A}(W_{LL}^{\mathcal{O}}) \vee \mathcal{A}(W_{RR}^{\mathcal{O}}))' = \mathcal{A}(W_R^{\mathcal{O}}) \wedge \mathcal{A}(W_L^{\mathcal{O}}), \tag{4.2.2}$$

as locality of the dual net is equivalent to essential duality of $\mathcal{A}$. ■

We will now introduce the central notion for this paper.

Definition 4.2.2 *A family of disorder operators consists, for any $\mathcal{O} \in \mathcal{K}$ and any $g \in G$, of two unitary operators $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g)$ verifying*

$$\begin{array}{rcl} Ad\,U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) = Ad\,U_R^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) & = & \alpha_g, \\ Ad\,U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) = Ad\,U_R^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) & = & id. \end{array} \tag{4.2.3}$$

In words: the adjoint action of $U^{\mathcal{O}}_{L/R}(g)$ on fields localized in the left and right spacelike complements of $\mathcal{O}$, respectively, equals the global group action on one side and is trivial on the other. As a consequence of (twisted) wedge duality we have at once

$$U^{\mathcal{O}}_L(g) \in \mathcal{F}(W^{\mathcal{O}}_L)^t,\ U^{\mathcal{O}}_R(g) \in \mathcal{F}(W^{\mathcal{O}}_R)^t. \tag{4.2.4}$$

On the other hand it is clear that disorder operators cannot be contained in the local algebras $\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O})^t$ nor in the quasilocal algebra $\mathcal{F}$, for in this case locality would not allow their adjoint action to be as stated on operators localized arbitrarily far to the left or right. Heuristically, assuming $U(g)$ arises from a conserved local current via $U(g) = e^{i\int j^0(t=0,x)dx}$, one may think of $U^{\mathcal{O}}_L(g)$ as given by

$$U^{\mathcal{O}}_L(g) = e^{i\int_{-\infty}^{x_0} j^0(x)dx}, \tag{4.2.5}$$

where integration takes place over a spacelike curve from left spacelike infinity to a point x_0 in $\mathcal{O}$. The need for a finitely extended interpolation region $\mathcal{O}$ arises from the distributional character of the current which necessitates a smooth cutoff. We refrain from discussing these matters further as they play no role in the sequel. In massive free field theories disorder operators can be constructed rigorously (e.g. [65, 1]) using the CCR/CAR structure and the criteria due to Shale.

Lemma 4.2.3 *Let $U^{\mathcal{O}}_{L,1}(g), U^{\mathcal{O}}_{L,2}(g)$ be disorder operators associated with the same double cone and the same group element. Then $U^{\mathcal{O}}_{L,1}(g) = F\, U^{\mathcal{O}}_{L,2}(g)$ with $F \in \mathcal{F}(\mathcal{O})^t$ unitary. An analogous statement holds for the right handed disorder operators.*

Proof. Consider $F = U^{\mathcal{O}}_{L,1}(g)\, U^{\mathcal{O}*}_{L,2}(g)$. By construction $F \in \mathcal{F}(W^{\mathcal{O}}_L)^t$. On the other hand $Ad\,F \restriction \mathcal{F}(W^{\mathcal{O}}_{LL}) = id$ holds as $U^{\mathcal{O}}_{L,1}(g)$ and $U^{\mathcal{O}}_{L,2}(g)$ implement the same automorphism of $\mathcal{F}(W^{\mathcal{O}}_{LL})$. By (twisted) wedge duality we have $F \in \mathcal{F}(W^{\mathcal{O}}_R)^t$ and (twisted) duality for double cones implies $F \in \mathcal{F}(\mathcal{O})^t$. ■

Remark. This result shows that disorder operators are unique up to unitary elements of $\mathcal{F}(\mathcal{O})^t$ where $\mathcal{O}$ is the interpolation region. The obvious fact that $U^{\mathcal{O}}_L(g)\, U^{\mathcal{O}}_L(h)$ and $U(g)\, U^{\mathcal{O}}_L(h)\, U(g)^*$ are disorder operators for the group elements gh and ghg^{-1}, respectively, implies that a family of disorder operators constitutes a projective representation of G with the cocycle taking values in $\mathcal{F}(\mathcal{O})^t$.

For the purposes of the present investigation the mere existence of disorder operators is not enough, for we need them to obey certain further restrictions. Our first aim will be to obtain such operators by a construction which is model independent to the largest possible extent, making use only of properties valid in any reasonable model. To this effect we reconsider an idea due to Doplicher [40] and developed further in, e.g., [41, 20]. It consists of using the split property [19] to obtain, for any $g \in G$ and any pair of double cones $\Lambda = (\mathcal{O}_1, \mathcal{O}_2)$ such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$, an operator $U_\Lambda(g) \in \mathcal{F}(\mathcal{O}_2)$ such that

$$U_\Lambda(g) F U_\Lambda(g)^* = U(g) F U(g)^*\ \forall F \in \mathcal{F}(\mathcal{O}_1). \tag{4.2.6}$$

In order to be able to do the same thing with wedges we complete our list of postulates by requiring the net $\mathcal{F}$ to satisfy the split property for wedges (SPW) as defined in the preceding chapter (Definition 3.2.1). (In our case, where wedge duality holds, the split property for inclusions of left handed wedges entails the same for the right handed ones). This property is known to be fulfilled for the free massive scalar and Dirac fields and should be true in all reasonable massive theories in $1+1$ dimensions.

Before we proceed one remark is in order. As shown in the preceding chapter (Theorem 3.4.1) (bosonic) nets of algebras satisfying Haag duality and the SPW do not possess non-trivial representations corresponding to charges which are localizable in the DHR or BF sense. This fact, which was discovered after completing [87], shows that the analysis in this chapter (and the latter reference) is related to the one in [33] even stronger than originally expected. These authors explicitly required the conformal theory they started with to be 'holomorphic' in the sense that the chiral algebra has no representations besides the vacuum representation. Keeping in mind that here we are concerned with *massive* models, the combination of Haag duality and the SPW is just a very convenient way of characterizing theories with trivial representation theory.

In analogy to the preceding chapter one has, for any double cone $\mathcal{O}$, a unitary operator $Y^{\mathcal{O}}: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing an isomorphism between $\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}})^t$ and the tensor product $\mathcal{F}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})^t$:

$$Y^{\mathcal{O}} F_1 F_2^t Y^{\mathcal{O}*} = F_1 \otimes F_2^t \quad \forall F_1 \in \mathcal{F}(W_{LL}^{\mathcal{O}}), F_2 \in \mathcal{F}(W_{RR}^{\mathcal{O}}). \tag{4.2.7}$$

That one of the algebras $\mathcal{F}(W_{LL}^{\mathcal{O}})$ and $\mathcal{F}(W_{RR}^{\mathcal{O}})$, which are associated to spacelike separated regions, has to be twisted in order for an isomorphism as above to exist is clear as in general these algebras do not commute while the factors of a tensor product do commute. Analogously, there is a spatial isomorphism between $\mathcal{F}(W_{LL}^{\mathcal{O}})^t \vee \mathcal{F}(W_{RR}^{\mathcal{O}})$ and $\mathcal{F}(W_{LL}^{\mathcal{O}})^t \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})$ implemented by $\tilde{Y}^{\mathcal{O}}$. We will stick to the use of $Y^{\mathcal{O}}$ throughout. In order not to obscure the basic simplicity of the argument we assume for a moment that the theory $\mathcal{F}$ is purely bosonic, i.e. fulfills locality and duality without twisting. Then the equations (3.2.3), (3.2.4) hold when $\mathcal{F}$ is replaced by $\mathcal{A}$ everywhere. Furthermore, the following property of the maps $Y^{\mathcal{O}}$ will be pivotal for the considerations below. Given any unitary U implementing a local symmetry (i.e. $U\mathcal{F}(\mathcal{O})U^* = \mathcal{F}(\mathcal{O})\ \forall \mathcal{O}$) and leaving invariant the vacuum ($U\Omega = \Omega$) the following identity holds:

$$Y^{\mathcal{O}} U = (U \otimes U)\, Y^{\mathcal{O}}. \tag{4.2.8}$$

For the construction of $Y^{\mathcal{O}}$ as well as for the proof of (4.2.8) we refer to [41, 20], the difference that those authors work with double cones being unimportant.

4.2.2 Disorder Operators: Construction

The operators $Y^{\mathcal{O}}$ will now be used to obtain disorder operators. To this purpose we give the following

Definition 4.2.4 *For any double cone $\mathcal{O} \in \mathcal{K}$ and any $g \in G$ we set*

$$\begin{aligned} U_L^{\mathcal{O}}(g) &= Y^{\mathcal{O}*}\,(U(g) \otimes \mathbf{1})\, Y^{\mathcal{O}}, \\ U_R^{\mathcal{O}}(g) &= Y^{\mathcal{O}*}\,(\mathbf{1} \otimes U(g))\, Y^{\mathcal{O}}. \end{aligned} \tag{4.2.9}$$

As an immediate consequence of this definition we have the following

Proposition 4.2.5 *The disorder operators defined above satisfy*

$$\begin{aligned} \left[U_L^{\mathcal{O}}(g), U_R^{\mathcal{O}}(h)\right] &= 0, && (4.2.10) \\ U_L^{\mathcal{O}}(g)\; U_R^{\mathcal{O}}(g) &= U(g), && (4.2.11) \end{aligned}$$

$$U(g)\, U_{L/R}^{\mathcal{O}}(h)\, U(g)^* = U_{L/R}^{\mathcal{O}}(ghg^{-1}). \tag{4.2.12}$$

Proof. The first statement is trivial and the second follows from (4.2.8). The covariance property (4.2.12) is another consequence of (4.2.8). ■

Remark. We have thus obtained some kind of factorization of the global action of the group G into two commuting *true* (i.e. no cocycles) representations of G such that the original action is recovered as the diagonal. Furthermore, these operators transform covariantly under global gauge transformations. In particular they are bosonic since $k \in Z(G)$.

It remains to be shown that the $U_{L/R}^{\mathcal{O}}$ indeed fulfill the requirements of Definition 4.2.2. The second requirement follows from Definition 4.2.4, which with (3.2.3) obviously yields

$$U_L^{\mathcal{O}}(g) \in \mathcal{F}(W_L^{\mathcal{O}}),\; U_R^{\mathcal{O}}(g) \in \mathcal{F}(W_R^{\mathcal{O}}). \tag{4.2.13}$$

The first one is seen by the following computation valid for $F \in \mathcal{F}(W_{LL}^{\mathcal{O}})$

$$\begin{aligned} U_L^{\mathcal{O}}(g) F U_L^{\mathcal{O}*}(g) &\cong (U(g) \otimes \mathbf{1})(F \otimes \mathbf{1})(U(g) \otimes \mathbf{1})^* \\ &= (U(g) F U(g)^* \otimes \mathbf{1}) \cong U(g) F U(g)^* \end{aligned} \tag{4.2.14}$$

appealing to the isomorphism $\cong$ implemented by $Y^{\mathcal{O}}$.

Returning now to the more general case including fermions we have to consider the apparent problem that there are now two ways to define the operators $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g)$, depending upon whether we choose $Y^{\mathcal{O}}$ or $\tilde{Y}^{\mathcal{O}}$. (By contrast, the tensor product factorization (3.2.4) of the local algebras is of a purely technical nature, rendering it irrelevant whether we use $Y^{\mathcal{O}}$ or $\tilde{Y}^{\mathcal{O}}$.) This ambiguity is resolved by remarking that the element $k \in G$ giving rise to V by $V = U(k)$ is central, implying that the operators $U(g)$, $g \in G$, are bosonic (even). For even operators $F_1 \in \mathcal{F}(W_{LL}^{\mathcal{O}}), F_2 \in \mathcal{F}(W_{RR}^{\mathcal{O}})$ we have $F_1 = F_1^t, F_2 = F_2^t$ and thus

$$Y^{\mathcal{O}}\, F_1 F_2\, Y^{\mathcal{O}*} = \tilde{Y}^{\mathcal{O}}\, F_1 F_2\, \tilde{Y}^{\mathcal{O}*} = F_1 \otimes F_2 \tag{4.2.15}$$

so that the disorder variables are uniquely defined even operators.

The first two equations of (3.2.3) are replaced by

$$\begin{array}{rcccc} \mathcal{F}(W_{LL}^{\mathcal{O}}) & \cong & \mathcal{F}(W_{LL}^{\mathcal{O}}) & \otimes & \mathbf{1}, \\ \mathcal{F}(W_{RR}^{\mathcal{O}})^t & \cong & \mathbf{1} & \otimes & \mathcal{F}(W_{RR}^{\mathcal{O}})^t. \end{array} \tag{4.2.16}$$

By taking commutants we obtain

$$\begin{array}{rcccc} \mathcal{F}(W_{L}^{\mathcal{O}}) & \cong & \mathcal{B}(\mathcal{H}) & \otimes & \mathcal{F}(W_{L}^{\mathcal{O}}), \\ \mathcal{F}(W_{R}^{\mathcal{O}})^t & \cong & \mathcal{F}(W_{R}^{\mathcal{O}})^t & \otimes & \mathcal{B}(\mathcal{H}) \end{array} \tag{4.2.17}$$

and an application of the twist operation to the second equations of (4.2.16) and (4.2.17) yields

$$\begin{array}{rcccccccc} \mathcal{F}(W_{RR}^{\mathcal{O}}) & \cong & \mathbf{1} & \otimes & \mathcal{F}(W_{RR}^{\mathcal{O}})_+ & + & V & \otimes & \mathcal{F}(W_{RR}^{\mathcal{O}})_-, \\ \mathcal{F}(W_{R}^{\mathcal{O}}) & \cong & \mathcal{F}(W_{R}^{\mathcal{O}}) & \otimes & \mathcal{B}(\mathcal{H})_+ & + & \mathcal{F}(W_{R}^{\mathcal{O}})\,V & \otimes & \mathcal{B}(\mathcal{H})_-. \end{array} \tag{4.2.18}$$

The identity $\mathcal{F}(\mathcal{O}) = \mathcal{F}(W_L^{\mathcal{O}}) \wedge \mathcal{F}(W_R^{\mathcal{O}})$ which is valid in the fermionic case, too, finally leads to

$$\mathcal{F}(\mathcal{O}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ \;+\; \mathcal{F}(W_R^{\mathcal{O}})\,V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-. \tag{4.2.19}$$

While this is not as nice as (3.2.4) it is still sufficient for the considerations in the sequel. That $\mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ is a factor is, however, less obvious than in the pure Bose case and will be proved only in Subsection 4.3.3.

The following easy result will be of considerable importance later on.

Lemma 4.2.6 *The disorder operators $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g)$ associated with the double cone $\mathcal{O}$ implement automorphisms of the local algebra $\mathcal{F}(\mathcal{O})$.*

Proof. In the pure Bose case this is obvious from Definition 4.2.4, equation (3.2.4) and the fact that $Ad\,U(g)$ acts as an automorphism on all wedge algebras. In the Bose-Fermi case (4.2.19) the same is true since $U(g)$ commutes with $V = U(k)$. ■

Definition 4.2.7 $\alpha_g^{\mathcal{O}} = Ad\,U_L^{\mathcal{O}}(g), \quad g \in G, \mathcal{O} \in \mathcal{K}.$

We close this section with one remark. We have seen that the split property for wedges implies the existence of disorder operators which constitute true representations of the symmetry group and which transform covariantly under the global symmetry. Conversely, one can show that the existence of disorder operators, possibly with group cocycle, in conjunction with the split property for wedges for the fixpoint net $\mathcal{A}$ implies the split property for the field net $\mathcal{F}$. This in turn allows to remove the cocyle using the above construction. We refrain from giving the argument which is similar to those in [40, pp. 79, 85].

4.3 Field Extensions and Haag Duality

4.3.1 The Extended Field Net

Having defined the disorder variables we now take the next step, which at first sight may seem unmotivated. Its relevance will become clear in the sequel. We define two new nets of algebras $\mathcal{O} \mapsto \hat{\mathcal{F}}_{L/R}(\mathcal{O})$ by adding the disorder variables associated with the double cone $\mathcal{O}$ to the fields localized in this region.

Definition 4.3.1

$$\hat{\mathcal{F}}(\mathcal{O})_{L/R} = \mathcal{F}(\mathcal{O}) \vee U_{L/R}^{\mathcal{O}}(G)''. \tag{4.3.1}$$

Remarks. 1. In accordance with the common terminology in statistical mechanics and conformal field theory the operators which are composed of fields (order variables) and disorder variables might be called *parafermion operators*.

2. As there is a complete symmetry between left and right there is no fundamental difference between the extensions $\hat{\mathcal{F}}_L$ and $\hat{\mathcal{F}}_R$. With the exception of Subsection 4.4.6, we will therefore stick to $\hat{\mathcal{F}}_L$ throughout this paper, writing $\hat{\mathcal{F}} \equiv \hat{\mathcal{F}}_L$ for simplicity. Including both the left and right handed disorder operators would have the unpleasant consequence that there would be translation invariant operators (namely the $U(g)$'s) in the local algebras.

3. The local algebra $\hat{\mathcal{F}}(\mathcal{O})$ of the above definition resembles the crossed product of $\mathcal{F}(\mathcal{O})$ by the automorphism group $\alpha_g^{\mathcal{O}}$, the interesting aspect being that the automorphism group depends on the region $\mathcal{O}$. These two constructions differ, however, with respect to the Hilbert space on which they are defined. Whereas the crossed product $\mathcal{F}(\mathcal{O}) \rtimes_{\alpha^{\mathcal{O}}} G$ lives on the Hilbert space $L^2(G, \mathcal{H})$, our algebras $\hat{\mathcal{F}}(\mathcal{O})$ are defined on the original space $\mathcal{H}$. For later purposes it will be necessary to know whether these algebras are isomorphic, but we prefer first to discuss those aspects which are independent of this question.

The first thing to check is, of course, that the Definition 4.3.1 specifies a net of von Neumann algebras.

Proposition 4.3.2 *The assignment $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ satisfies isotony.*

Proof. Let $\mathcal{O} \subset \hat{\mathcal{O}}$ be an inclusion of double cones. Obviously we have $\mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\hat{\mathcal{O}})$. In order to prove $U_L^{\mathcal{O}}(g) \in \hat{\mathcal{F}}(\hat{\mathcal{O}})$ we observe that $U_L^{\mathcal{O}}(g)$ is a disorder operator for the larger region $\hat{\mathcal{O}}$, too. Thus, by Lemma 4.2.3 we have $U_L^{\mathcal{O}}(g) = F\, U_L^{\hat{\mathcal{O}}}(g)$ with $F \in \mathcal{F}(\hat{\mathcal{O}})^t$. Since the disorder operators are bosonic by construction we even have $F \in \mathcal{F}(\hat{\mathcal{O}})$. Now it is clear that $U_L^{\mathcal{O}}(g) \in \hat{\mathcal{F}}(\hat{\mathcal{O}})$. ■

Remark. From this we can conclude that the net $\hat{\mathcal{F}}(\mathcal{O})$ is uniquely defined in the sense that any family of bosonic disorder operators gives rise to the same net $\hat{\mathcal{F}}(\mathcal{O})$ provided such operators exist at all. For most of the arguments in this paper we will, however, need the detailed properties proved above which follow from the construction via the split property.

It is obvious that the net $\hat{\mathcal{F}}$ is nonlocal. While the spacelike commutation relations of fields and disorder operators are known by construction we will have more to say on this subject later. On the other hand it should be clear that the nets $\hat{\mathcal{F}}$ and $\mathcal{A}$ are local relative to each other. This is simply the fact that the disorder operators commute with the fixpoints of α_g in both spacelike complements.

Proposition 4.3.3 *The net $\hat{\mathcal{F}}$ is Poincaré covariant with the original representation of $\mathcal{P}$. In particular $\alpha_a(U_L^{\mathcal{O}}(g)) = U_L^{\mathcal{O}+a}(g)$ whereas for the boosts we have*

$$\alpha_\Lambda(U_L^{\mathcal{O}}(g)) = U_L^{\Lambda\mathcal{O}}(h), \tag{4.3.2}$$

if $U(\Lambda)\, U(g)\, U(\Lambda)^* = U(h)$.

Proof. The family $Y^{\mathcal{O}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ of unitaries provided by the split property fulfills the identity

$$Y^{\Lambda\mathcal{O}+a} = (U(\Lambda, a) \otimes U(\Lambda, a))\, Y^{\mathcal{O}}\, U(\Lambda, a)^*, \tag{4.3.3}$$

as is easily seen to follow from the construction in [41, 20]. This implies

$$\begin{aligned}\alpha_{\Lambda,a}(U_L^{\mathcal{O}}(g)) &= U(\Lambda,a)\,Y^{\mathcal{O}*}\,(U(g)\otimes\mathbf{1})\,Y^{\mathcal{O}}\,U(\Lambda,a)^* \\ &= Y^{\Lambda\mathcal{O}+a*}\,(U(\Lambda,a)\,U(g)\,U(\Lambda,a)^*\otimes\mathbf{1})\,Y^{\Lambda\mathcal{O}+a} \\ &= U_L^{\Lambda\mathcal{O}+a}(h),\end{aligned} \tag{4.3.4}$$

where $U(\Lambda)\,U(g)\,U(\Lambda)^* = U(h)$. ■

Proposition 4.3.4 *The vacuum vector Ω is cyclic and separating for $\hat{\mathcal{F}}(\mathcal{O})$.*

Proof. Follows from

$$\mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O}) \subset \mathcal{F}(W_L^{\mathcal{O}}) \tag{4.3.5}$$

since Ω is cyclic and separating for $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(W_L^{\mathcal{O}})$. ■

Proposition 4.3.5 *The wedge algebras for the net $\hat{\mathcal{F}}$ take the form*

$$\hat{\mathcal{F}}(W_L^{\mathcal{O}}) = \mathcal{F}(W_L^{\mathcal{O}}), \quad \hat{\mathcal{F}}(W_R^{\mathcal{O}}) = \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' = \mathcal{A}(W_{LL}^{\mathcal{O}})'. \tag{4.3.6}$$

As a consequence Ω is not separating for $\hat{\mathcal{F}}(W_R^{\mathcal{O}})$!

Proof. The first identity is obvious, while the second follows from $\mathcal{F}(W_R^{\mathcal{O}}) \ni U_R^{\hat{\mathcal{O}}}(g)\ \forall \hat{\mathcal{O}} \in W_R^{\mathcal{O}}$ and the factorization property (4.2.11). The last statement is equivalent to Ω not being cyclic for $\mathcal{A}(W_{LL}^{\mathcal{O}})$. ■

Proposition 4.3.6 *Let $\hat{F} \in \mathcal{F}(\mathcal{O})U_L^{\mathcal{O}}(g)$. Then the following cluster properties hold.*

$$w - \lim_{x\to-\infty} \alpha_x(\hat{F}) = \langle\Omega, \hat{F}\Omega\rangle \cdot \mathbf{1}, \tag{4.3.7}$$

$$w - \lim_{x\to+\infty} \alpha_x(\hat{F}) = \langle\Omega, \hat{F}\Omega\rangle \cdot U(g). \tag{4.3.8}$$

Proof. The first identity follows from $\hat{F} \in \hat{\mathcal{F}}(W_L^{\mathcal{O}})$ and the usual cluster property. The second is seen by writing $\hat{F} = F\,U_R^{\mathcal{O}}(g^{-1})\,U(g)$ and applying the weak convergence of $U_R^{\mathcal{O}}$ as above, the translation invariance of $U(g)$ and the invariance of the vacuum under $U(g)$. ■

4.3.2 Haag Duality

Observing by (4.2.12) that the adjoint action of the global symmetry group leaves the 'localization' (in the sense of Definition 4.2.2) of the disorder operators invariant it is clear that the automorphisms $\alpha_g = Ad\,U(g)$ extend to local symmetries of the enlarged net $\hat{\mathcal{F}}$. We are thus in a position to define yet another net, the fixpoint net of $\hat{\mathcal{F}}$

Definition 4.3.7

$$\hat{\mathcal{A}}(\mathcal{O}) = \hat{\mathcal{F}}(\mathcal{O}) \wedge U(G)', \tag{4.3.9}$$

whereby we have the following square of local inclusions

$$\begin{array}{ccc} \hat{\mathcal{A}}(\mathcal{O}) & \subset & \hat{\mathcal{F}}(\mathcal{O}) \\ \cup & & \cup \\ \mathcal{A}(\mathcal{O}) & \subset & \mathcal{F}(\mathcal{O}). \end{array} \tag{4.3.10}$$

Remark. The conditional expectation $m(\cdot) = \int dg\, \alpha_g(\cdot)$ from $\hat{\mathcal{F}}(\mathcal{O})$ to $\hat{\mathcal{A}}(\mathcal{O})$ clearly restricts to a conditional expectation from $\mathcal{F}(\mathcal{O})$ to $\mathcal{A}(\mathcal{O})$. In Section 4 we will see that there is also a conditional expectation γ_e from $\hat{\mathcal{F}}(\mathcal{O})$ to $\mathcal{F}(\mathcal{O})$ which restricts to a conditional expectation from $\hat{\mathcal{A}}(\mathcal{O})$ to $\mathcal{A}(\mathcal{O})$, provided the group G is finite. Since γ_e commutes with m the square (4.3.10) then constitutes a commuting square in the sense of Popa.

Proposition 4.3.8 *The net $\mathcal{O} \mapsto \hat{\mathcal{A}}(\mathcal{O})$ is local.*

Proof. Let $\mathcal{O} < \tilde{\mathcal{O}}$ be two regions spacelike to each other, $\tilde{\mathcal{O}}$ being located to the right of $\mathcal{O}$. From $\hat{\mathcal{A}}(\mathcal{O}) \subset \mathcal{A}(W_L^{\mathcal{O}})$ and the relative locality of observables and fields we conclude that $\hat{\mathcal{A}}(\mathcal{O})$ commutes with $\mathcal{F}(\tilde{\mathcal{O}})$. On the other hand the operators $U_L^{\tilde{\mathcal{O}}}(g)$ commute with $\hat{\mathcal{A}}(\mathcal{O}) \subset \hat{\mathcal{F}}(W_L^{\mathcal{O}}) = \mathcal{F}(W_L^{\mathcal{O}})$ since $Ad\, U_L^{\tilde{\mathcal{O}}}(g) \restriction \mathcal{F}(W_L^{\mathcal{O}}) = \alpha_g$ and $\hat{\mathcal{A}}(\mathcal{O})$ is pointwise gauge invariant. ■

We have just proved that the net $\hat{\mathcal{A}}$ constitutes a local extension of the observable net $\mathcal{A}$, thereby confirming our initial observation that $\mathcal{A}$ does not satisfy Haag duality. Using ideas from the proof of [35, Thm. 4.1], we demonstrated in [87] that $\hat{\mathcal{A}}$ satisfies Haag duality in all simple sectors, which provides justification for the Definitions 4.3.1 and 4.3.7. Now, since $\hat{\mathcal{A}} \restriction \mathcal{H}_0$ satisfies the SPW, see below, our abstract result in Theorem 3.5.3 to the effect that Haag duality obtains in *all* locally normal irreducible representations of the dual net applies to the situation at hand. We can thus conclude that Haag duality also holds for the non-simple sectors of $\hat{\mathcal{A}}$ which by necessity occur for non-abelian groups G. Since this result is somewhat counterintuitive (which explains why it was overlooked in [87]) we verify it by the following direct calculation, which replaces Lemma 3.9 and Theorem 3.10 of [87].

Lemma 4.3.9 *The commutants of the algebras $\hat{\mathcal{A}}_L(\mathcal{O})$ are given by*

$$\hat{\mathcal{A}}_L(\mathcal{O})' = \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}}) \quad \forall \mathcal{O} \in \mathcal{K}. \tag{4.3.11}$$

Proof. For simplicity we assume $\mathcal{F}$ to be a local net for a moment. Then

$$\begin{aligned} \hat{\mathcal{A}}_L(\mathcal{O})' &= (\hat{\mathcal{F}}_L(\mathcal{O}) \wedge U(G)')' = \hat{\mathcal{F}}_L(\mathcal{O})' \vee U(G)'' \\ &= (\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'')' \vee U(G)'' = (\mathcal{F}(\mathcal{O})' \wedge U_L^{\mathcal{O}}(G)') \vee U(G)'' \\ &= ((\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}})) \wedge U_L^{\mathcal{O}}(G)') \vee U(G)'' \\ &= (\mathcal{F}(W_{LL}^{\mathcal{O}}) \wedge U_L^{\mathcal{O}}(G)') \vee \mathcal{F}(W_{RR}^{\mathcal{O}}) \vee U(G)'' \\ &= \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}}). \end{aligned} \tag{4.3.12}$$

The fourth line follows from the third using the split property. In the last step we have used the identities $\hat{\mathcal{A}}_L(W_L) = \mathcal{A}_L(W_L)$ and $\mathcal{F}_L(W_R) \vee U(G)'' = \hat{\mathcal{F}}_L(W_R)$ which hold for all left (right) handed wedges W_L (W_R), cf. Proposition 4.3.5. Now, if $\mathcal{F}$ satisfies twisted duality, (4.2.19) leads to $\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \cong \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' \otimes \mathcal{F}(W_L^{\mathcal{O}})$ and $(\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'')' \cong \mathcal{A}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})^t$. Using this it is easy to verify that (4.3.11) is still true. ■

Theorem 4.3.10 *The net $\hat{\mathcal{A}}_L$ satisfies Haag duality in restriction to every invariant subspace of $\mathcal{H}$ on which $\hat{\mathcal{A}}_L$ acts irreducibly (e.g., in particular the vacuum sector).*

Proof. We recall that the representation π of $\hat{\mathcal{A}}_{L/R}$ on $\mathcal{H}$ is of the form $\pi = \oplus_{\xi \in \hat{G}} d_\xi \pi_\xi$. Let thus P be an orthogonal projection onto a subspace $\mathcal{H}_\xi \subset \mathcal{H}$ on which $\hat{\mathcal{A}}_L$ acts as the irreducible representation π_ξ. Since P commutes with $\mathcal{A}_L(\mathcal{O})$ and $\mathcal{A}_L(W_{LL}^{\mathcal{O}})$ we have

$$\begin{aligned} P\,\hat{\mathcal{A}}_L(\mathcal{O})'\,P &= P\,\hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}})\,P \\ &= \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee (P\,\hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}})\,P) \\ &= P\,\hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}})\,P, \end{aligned} \tag{4.3.13}$$

which implies

$$(\hat{\mathcal{A}}_L(\mathcal{O}) \restriction \mathcal{H}_\xi)' = \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}}) \restriction \mathcal{H}_\xi. \tag{4.3.14}$$

■

This provides a concrete verification of Theorem 3.5.3 in a special, albeit important situation. The above arguments make it clear that Haag duality cannot hold for the net $\mathcal{A}(\mathcal{O})$ even in simple sectors. This is not necessarily so if the split property for wedges does not hold. In conformally invariant theories gauge invariant combinations of field operators in the left and the right spacelike complements of a double cone $\mathcal{O}$ may well be contained in $\mathcal{A}(\mathcal{O}')$ due to spacetime compactification. One would think, however, that this is impossible in massive theories, even those without the split property.

4.3.3 Computation of $\hat{\mathcal{A}}(\mathcal{O})$

While Theorem 4.3.10 allows us in principle to construct the dual net $\hat{\mathcal{A}}$ one would like to know more explicitly how the elements of $\hat{\mathcal{A}}$ look in terms of the fields in $\mathcal{F}$ and the disorder operators. In the case of an abelian group G this is easy to see. As a consequence of the covariance property (4.2.12) we then have

$$U(g)\,U_{L/R}^{\mathcal{O}}(h)\,U(g)^* = U_{L/R}^{\mathcal{O}}(ghg^{-1}) = U_{L/R}^{\mathcal{O}}(h), \tag{4.3.15}$$

that is the disorder operators are gauge invariant and thus contained in $\hat{\mathcal{A}}(\mathcal{O})$. It is then obvious that

$$\hat{\mathcal{A}}(\mathcal{O}) = \mathcal{A}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'', \quad (G \text{ abelian!}) \tag{4.3.16}$$

as $\hat{\mathcal{A}}(\mathcal{O})$ is spanned by operators of the form $FU_L^{\mathcal{O}}(g)$, $F \in \mathcal{F}(\mathcal{O})$ which are invariant iff $F \in \mathcal{A}(\mathcal{O})$.

The case of the group G being non-abelian is more complicated and we limit ourselves to finite groups leading already to structures which are quite interesting. In order to proceed we would like to know that every operator $\hat{F} \in \hat{\mathcal{F}}(\mathcal{O})$ has a unique representation of the form

$$\hat{F} = \sum_{g \in G} F(g)\, U_L^{\mathcal{O}}(g), \quad F(g) \in \mathcal{F}(\mathcal{O}). \tag{4.3.17}$$

While this true for the crossed product $\mathcal{M} \rtimes G$ on $L^2(G, \mathcal{H})$ (only for finite groups!) it is not obvious for the algebra $\mathcal{M} \vee U(G)''$ on $\mathcal{H}$. The latter may be considered as the image of the former under a homomorphism which might have a nontrivial kernel. In this case there would be equations of the type

$$\sum_{g \in G} F(g)\, U_L^{\mathcal{O}}(g) = 0, \tag{4.3.18}$$

where not all $F(g)$ vanish. Fortunately at least for finite groups (infinite, thus noncompact, discrete groups are ruled out by the split property) this undesirable phenomenon can be excluded without imposing further assumptions using the following result due to Buchholz [30].

Proposition 4.3.11 *The automorphisms $\alpha_g = Ad\, U(g)$ act outerly on the wedge algebras.*

Proof. Let W be the standard wedge $W = \{x \in \mathbb{R}^2 \mid x^1 > |x^0|\}$ and assume there is a unitary $V_g \in \mathcal{F}(W)$ such that $Ad\, V_g \restriction \mathcal{F}(W) = \alpha_g$. Define $V_{g,x} = \alpha_x(V_g)$ for all $x \in W$. Obviously $V_{g,x} \in \mathcal{F}(W_x)$. By the commutativity $\alpha_x \circ \alpha_g = \alpha_g \circ \alpha_x$ of translations and gauge transformations we have $Ad\, V_{g,x} \restriction \mathcal{F}(W_x) = \alpha_g$. By the computation (for $x \in W$)

$$\begin{array}{rclclcl} V_g\, V_{g,x}\, V_g^* & = & \alpha_g(V_{g,x}) & = & \alpha_g \circ \alpha_x(V_g) & = & \alpha_x \circ \alpha_g(V_g) \\ & = & \alpha_x(V_g\, V_g\, V_g^*) & = & \alpha_x(V_g) & = & V_{g,x} \end{array} \tag{4.3.19}$$

we obtain

$$V_g\, V_{g,x} = V_{g,x}\, V_g \quad \forall x \in W. \tag{4.3.20}$$

The von Neumann algebra

$$\mathcal{V} = \{V_{g,x},\ x \in W\}'' \tag{4.3.21}$$

is mapped into itself by translations α_x where $x \in W$ and the vacuum vector Ω it is separating for $\mathcal{V}$ as we have $\mathcal{V} \subset \mathcal{F}(W)$. This allows us to apply the arguments in [48] to conclude that $\mathcal{V}$ is either trivial (i.e. $\mathcal{V} = \mathbb{C}\mathbf{1}$) or a factor of type III_1. The assumed existence of V_g, which cannot be proportional to the identity due to the postulate $\alpha_g \neq \mathrm{id}$, excludes the first alternative whereas the second is incompatible with (4.3.20) according to which V_g is central. Contradiction! ∎

Remark. This result may be interpreted as a manifestation of an ultraviolet problem. The automorphism α_g being inner on a wedge W, wedge duality would imply it to be inner on the complementary wedge W', too, giving rise to a factorization $U(g) = V_L(g)\, V_R(g)$, $V_L(g) \in \mathcal{F}(W)$, $V_R(g) \in \mathcal{F}(W')$. This would be incompatible with the distributional character of the local current from which $U(g)$ derives.

We cite the following well known result on automorphism groups of factors.

Proposition 4.3.12 *Let $\mathcal{M}$ be a factor and α an outer action of the finite group G. Then the inlusions $\mathcal{M}^G \subset \mathcal{M}$, $\pi(\mathcal{M}) \subset \mathcal{M} \rtimes G$ are irreducible, i.e. $\mathcal{M} \rtimes G \cap \pi(\mathcal{M})' = \mathcal{M} \cap \mathcal{M}^{G'} = \mathbb{C}\mathbf{1}$. In particular $\mathcal{M} \rtimes G$ and $\mathcal{M}^G$ are factors. If the action α is unitarily implemented $\alpha_g = Ad\, U(g)$ then $\mathcal{M} \rtimes G$ and $\mathcal{M} \vee U(G)''$ are isomorphic.*

Proof. The irreducibility statements $\mathcal{M} \rtimes G \cap \pi(\mathcal{M})' = \mathcal{M} \cap \mathcal{M}^{G'} = \mathbb{C}\mathbf{1}$ are standard consequences of the relative commutant theorem [103, §22] for crossed products. Remarking that finite groups are discrete and compact the proof is completed by an application of [73, Corollary 2.3] which states that $\mathcal{M} \rtimes G$ and $\mathcal{M} \vee U(G)''$ are isomorphic if the former algebra is factorial and G is compact. ■

We are now in a position to prove several important corollaries to Proposition 4.3.11.

Corollary 4.3.13 *The algebras $\mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ are factors also in the Bose-Fermi case.*

Proof. Since $Ad\, V$ acts outerly on the factor $\mathcal{F}(W_R^{\mathcal{O}})$ by Proposition 4.3.11 $M_1 = \mathcal{F}(W_R^{\mathcal{O}}) \vee \{V\}$ is a factor and there is an automorphism β of M_1 leaving $\mathcal{F}(W_R^{\mathcal{O}})$ pointwise invariant such that $\beta(V) = -V$. The automorphism $\beta \otimes \alpha_k$ of $M_1 \otimes \mathcal{F}(W_L^{\mathcal{O}})$ clearly has $Y^{\mathcal{O}}\, \mathcal{F}(\mathcal{O})\, Y^{\mathcal{O}*}$ as fixpoint algebra, cf. (4.2.19). Since α_k is outer the same holds [103, Prop. 17.6] for $\beta \otimes \alpha_k$. Thus the fixpoint algebra is factorial by another application of Proposition 4.3.12. ■

Corollary 4.3.14 *Let $\mathcal{O} \in \mathcal{K}$. The automorphisms $\alpha_g = Ad\, U(g)$ and $\alpha_g^{\mathcal{O}} = Ad\, U_L^{\mathcal{O}}(g)$ act outerly on the algebra $\mathcal{F}(\mathcal{O})$.*

Proof. The pure Bose case is easy. $\mathcal{F}(\mathcal{O})$, $\alpha_g^{\mathcal{O}}$ and α_g are unitarily equivalent to $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})$, $\alpha_g \otimes id$, and $\alpha_g \otimes \alpha_g$, respectively. Since $\alpha_g = Ad\, U(g)$ is outer on $\mathcal{F}(W_R^{\mathcal{O}})$ the same holds by [103, Prop. 17.6] for the automorphisms $\alpha_g \otimes id$ and $\alpha_g \otimes \alpha_g$ of the above tensor product.

Turning to the Bose-Fermi case let $X_g \in \mathcal{F}(\mathcal{O})$ be an implementer of α_g or $\alpha_g^{\mathcal{O}}$ and define $\hat{X}_g = Y^{\mathcal{O}}\, X_g\, Y^{\mathcal{O}*}$. Then $(\mathbf{1} \otimes V)\hat{X}_g(\mathbf{1} \otimes V)$ also implements $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$, respectively, since k is central. $\mathcal{F}(\mathcal{O})$ being a factor this implies $(\mathbf{1} \otimes V)\hat{X}_g(\mathbf{1} \otimes V) = c_g\, \hat{X}_g$ with $c_g^2 = \pm 1$ due to $k^2 = e$. $\hat{X}_g$ is thus contained either in $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+$ or in $\mathcal{F}(W_R^{\mathcal{O}})\, V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-$. In the first case the restriction of $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$ to $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+$ is inner which can not be true by the same argument as for the Bose case. (Observe that $\mathcal{F}(W_L^{\mathcal{O}})_+$ is factorial.) On the other hand, no $\hat{X}_g \in \mathcal{F}(W_R^{\mathcal{O}})\, V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-$ can implement $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$ since both automorphisms are trivial on the subalgebra $\mathbf{1} \otimes \mathcal{F}(W_L^{\mathcal{O}}) \cap U(G)'$ which requires $\hat{X}_g \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{F}(W_L^{\mathcal{O}})^{G'}$. This, however, is impossible: $\mathcal{F}(W_L^{\mathcal{O}})_- \cap \mathcal{F}(W_L^{\mathcal{O}})^{G'} = [\mathcal{F}(W_L^{\mathcal{O}}) \cap \mathcal{F}(W_L^{\mathcal{O}})^{G'}]_- = [\mathbb{C}\mathbf{1}]_- = \emptyset$, where we have used the irreducibility of $\mathcal{F}(W_L^{\mathcal{O}})^G \subset \mathcal{F}(W_L^{\mathcal{O}})$. ■

Corollary 4.3.15 *Let the symmetry group G be finite. Then the enlarged algebra $\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)''$ is isomorphic to the crossed product $\mathcal{F}(\mathcal{O}) \rtimes_{\alpha^{\mathcal{O}}} G$ and the inclusions $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O})$, $\mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ are irreducible (i.e. the relative commutants are trivial).*

Proof. Obvious from Proposition 4.3.12 and Corollaries 4.3.13, 4.3.14. ■

Remark. If G is a compact continuous group outerness of the action does not allow us to draw these conclusions. In this case an additional postulate is needed. It would be sufficient to assume irreducibility of the inclusion $\mathcal{A}(W) \subset \mathcal{F}(W)$, for, as shown by Longo, this property in conjunction with proper infiniteness of $\mathcal{A}(W)$ implies dominance of the action and factoriality of the crossed product. Irreducibility of $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ follows as above.

We are now able to give an explicit description of the dual net $\hat{\mathcal{A}}$.

Theorem 4.3.16 *Every operator $\hat{A} \in \hat{\mathcal{A}}(\mathcal{O})$ can be uniquely written in the form*

$$\hat{A} = \sum_{g \in G} A(g)\, U_L^{\mathcal{O}}(g), \tag{4.3.22}$$

where the $A(g) \in \mathcal{F}(\mathcal{O})$ satisfy

$$A(kgk^{-1}) = \alpha_k(A(g)) \quad \forall g, k \in G. \tag{4.3.23}$$

Conversely, every choice of $A(g)$ complying with this constraint gives rise to an element of $\hat{\mathcal{A}}(\mathcal{O})$. An analogous representation for the algebras $\hat{\mathcal{A}}(W_R^{\mathcal{O}})$ is obtained by replacing $U_L^{\mathcal{O}}(g)$ by $U(g)$.

Remark. Condition (4.3.23) implies

$$A(g) \in \mathcal{F}(\mathcal{O}) \cap U(N_g)', \tag{4.3.24}$$

where $N_g = \{h \in G \mid gh = hg\}$ is the normalizer of g in G.
Proof. By Proposition 4.3.11 any $\hat{A} \in \hat{\mathcal{A}}(\mathcal{O})$ can be represented uniquely according to (4.3.22). Since $\alpha_k(\hat{A})$ is given by $\sum_g \alpha_k(A(g))\, U_L^{\mathcal{O}}(kgk^{-1}) = \sum_g \alpha_k(A(k^{-1}gk))\, U_L^{\mathcal{O}}(g)$ equation (4.3.23) follows by comparing coefficients. It is obvious that the arguments can be reversed. The statement on the wedge algebras $\hat{\mathcal{A}}(W_R^{\mathcal{O}})$ follows from the fact that $\hat{\mathcal{F}}(W_R^{\mathcal{O}})$ is the crossed product of $\mathcal{F}(W_R^{\mathcal{O}})$ by the global automorphism group, cf. Proposition 4.3.5. ■

4.3.4 The Split Property

The prominent role played by the split property in our investigations so far gives rise to the question whether it extends to the enlarged nets $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$. As to the net $\hat{\mathcal{F}}$ it is clear that a twist operation is needed in order to achieve commutativity of the algebras of two spacelike separated regions. Let $\mathcal{O}_1 < \mathcal{O}_2$ be double cones. Then one has $\hat{\mathcal{F}}(\mathcal{O}_2)^T \subset \hat{\mathcal{F}}(\mathcal{O}_1)'$ where

$$\left(\sum_g F(g)\, U_L^{\mathcal{O}}(g) \right)^T := \sum_g F(g)^t\, U_L^{\mathcal{O}}(g)\; U(g^{-1}) = \sum_g F(g)^t\, U_R^{\mathcal{O}}(g)^*, \tag{4.3.25}$$

and the ^{t} on $F(g)$ denotes the Bose-Fermi twist of Chapter 1. (By the crossed product nature of the algebras $\hat{\mathcal{F}}(\mathcal{O})$ it is clear that this map is well defined and invertible.) That commutativity holds as claimed follows easily from $\hat{\mathcal{F}}(\mathcal{O}_1) \subset \mathcal{F}(W_L^{\mathcal{O}_1})$ and $\hat{\mathcal{F}}(\mathcal{O}_2)^T \subset \mathcal{F}(W_R^{\mathcal{O}_2})^t$. It is interesting to observe that the twist has to be applied to the algebra located to the right for this construction to work. This twist operation lacks, however, several indispensable features. Firstly, there is no unitary operator S implementing the twist as in the Bose-Fermi case. The second, more important objection refers to the fact that the map (4.3.25) becomes noninvertible when extended to right-handed wedge regions, for the operators $U_R^{\mathcal{O}}(g)$ are contained in $\mathcal{F}(W_R^{\mathcal{O}})$.

Concerning the net $\hat{\mathcal{A}}$ which, in contrast, is local there is no conceptual obstruction to proving the split property. We start by observing that $\hat{\mathcal{A}}(W_{LL}^{\mathcal{O}}) = \mathcal{A}(W_{LL}^{\mathcal{O}})$. Furthermore, in restriction to a simple sector $\mathcal{H}_1$ wedge duality (Proposition 4.2.1) implies $\hat{\mathcal{A}}(W_{RR}^{\mathcal{O}}) \upharpoonright \mathcal{H}_1 = \mathcal{A}(W_{RR}^{\mathcal{O}}) \upharpoonright \mathcal{H}_1$. As the split property for the fields carries over [40] to the observables in the vacuum sector there is nothing to do if we restrict ourselves to the latter. We intend to prove now that the net $\hat{\mathcal{A}}$ fulfills the split property on the big Hilbert space $\mathcal{H}$. To this purpose we draw upon the pioneering work [40] where it was shown that the split property (for double cones) of a field net with group symmetry and twisted locality follows from the corresponding property of the fixpoint net provided the group G is finite abelian. (The case of general groups constitutes an open problem, but given nuclearity for the observables and some restriction on the masses in the charged sectors nuclearity and thus the split property for the fields can be proved.)

Proposition 4.3.17 *The net $\mathcal{O} \mapsto \hat{\mathcal{A}}(\mathcal{O})$ satisfies the split property for wedge regions, provided the group G is finite.*

Proof. The split property for wedges is equivalent [17] to the existence, for every double cone $\mathcal{O}$, of a product state $\phi^{\mathcal{O}}$ satisfying $\phi^{\mathcal{O}}(AB) = \phi^{\mathcal{O}}(A) \cdot \phi^{\mathcal{O}}(B)\ \forall A \in \hat{\mathcal{A}}(W_{LL}^{\mathcal{O}}), B \in \hat{\mathcal{A}}(W_{RR}^{\mathcal{O}})$. For the rest of the proof we fix one double cone $\mathcal{O}$ and omit it in the formulae. We have already remarked that for the net $\mathcal{A}$ product states ϕ_0 are known to exist. In order to construct a product state for $\hat{\mathcal{A}}$ we suppose γ_e is a conditional expectation from $\mathcal{A}(W_{LL}) \vee \hat{\mathcal{A}}(W_{RR})$ to $\mathcal{A}(W_{LL}) \vee \mathcal{A}(W_{RR})$ such that $\gamma_e(\hat{\mathcal{A}}(W_{RR})) = \mathcal{A}(W_{RR})$. Then $\gamma_e(AB) = \gamma_e(A)\,\gamma_e(B)$ where A, B are as above, implying that $\phi = \phi_0 \circ \gamma_e$ is a product state. It remains to find the conditional expectation γ_e. To make plain the basic idea we consider abelian groups G first. In this case γ_e is given by

$$\gamma_e(\hat{A}) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \psi_\chi^* \hat{A} \psi_\chi, \tag{4.3.26}$$

where $\psi_\chi \in \mathcal{F}(\mathcal{O})$ is a unitary field operator transforming according to $\alpha_g(\psi_\chi) = \chi(g) \cdot \psi_\chi$ under the group G. This map has all the desired properties. The pointwise invariance of $\hat{\mathcal{A}}(W_{LL})$ follows from the fact that this algebra commutes with the unitaries ψ_χ. On the other hand

$$\psi_\chi^*\, U_L^{\tilde{\mathcal{O}}}(g)\, \psi_\chi = \chi(g) \cdot U_L^{\tilde{\mathcal{O}}}(g), \quad \tilde{\mathcal{O}} \subset W_{RR}^{\mathcal{O}} \tag{4.3.27}$$

in conjunction with the identity $\sum_{\chi \in \hat{G}} \chi(g) = |G|\, \delta_{g,e}$ (valid also for non-abelian groups) implies that the operators $U_L^{\tilde{\mathcal{O}}}(g) \in \hat{\mathcal{A}}(W_{RR}),\ g \neq e$ are annihilated by γ_e. Finally, the existence of $\psi_\chi \in \mathcal{F}(\mathcal{O})$ for all χ (i.e. the dominance of the group action α on $\mathcal{F}(\mathcal{O})$) is well known to follow from the outerness of the group action α. The generalization to non-abelian groups is straightforward. The unitaries ψ_χ are replaced by multiplets $\psi_{r,i}$ of isometries for all irreducible representations r of G. They fulfill the following relations of orthogonality and completeness:

$$\psi_{r,i}^* \, \psi_{r,j} \;=\; \delta_{i,j}\mathbf{1}, \tag{4.3.28}$$

$$\sum_{i=1}^{d_r} \psi_{r,i} \, \psi_{r,i}^* \;=\; \mathbf{1} \tag{4.3.29}$$

and transform according to

$$\alpha_g(\psi_{r,i}) = \sum_{i'} D^r_{i',i}(g) \, \psi_{r,i'} \tag{4.3.30}$$

under the group. That the conditional expectation γ_e given by

$$\gamma_e(\hat{A}) = \frac{1}{|G|} \sum_{r \in \hat{G}} \sum_{i=1}^{d_r} \psi_{r,i}^* \hat{A} \psi_{r,i}, \tag{4.3.31}$$

does the job follows from

$$\sum_{i=1}^{d_r} \psi_{r,i}^* \, U_L^{\tilde{\mathcal{O}}}(g) \, \psi_{r,i} = tr\, D^r(g) \cdot U_L^{\tilde{\mathcal{O}}}(g) = \chi_r(g) \cdot U_L^{\tilde{\mathcal{O}}}(g). \tag{4.3.32}$$

Again the existence of such multiplets is guaranteed by our assumptions. ■

Remark. Tensor multiplets satisfying (4.3.28, 4.3.29) were first considered in [45] where the relation between the charged fields in a net of field algebras and the inequivalent representations of the observables was studied in the framework of [35]. Multiplets of this type will play a role in our subsequent investigations, too.

4.3.5 Irreducibility of $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$

The inclusions $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ are of the form

$$\mathcal{N} = \mathcal{P}^K \subset \mathcal{P} \subset \mathcal{P} \rtimes L = \mathcal{M}, \tag{4.3.33}$$

where K and L are finite subgroups of Aut$\mathcal{P}$, as studied in [11] (albeit for type II_1 factors). There $\mathcal{P}^K \subset \mathcal{P} \rtimes L$ was shown to be irreducible iff $K \cap L = \{e\}$ in Out$\mathcal{P}$ and to be of finite depth if and only if the subgroup Q of Out$\mathcal{P}$ generated by K and L is finite. Furthermore, the inclusion has depth two (i.e. $\mathcal{N}' \wedge \mathcal{M}_2$ is a factor where $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ is the Jones tower corresponding to the subfactor $\mathcal{N} \subset \mathcal{M}$) in the special case when $Q = K \cdot L$ (i.e. every $q \in Q$ can be written as $q = kl$, $k \in K, l \in L$).

In our situation, where $K = \mathrm{Diag}(G \times G)$ and $L = G \times \mathbf{1}$, all these conditions are fulfilled, as we have $Q = G \times G$ and $g \times h = (h \times h) \cdot (h^{-1}g \times e)$. The interest of this observation for our purposes derives from the following result, discovered by Ocneanu and proved, e.g., in [110, 82]. It states that an irreducible inclusion $\mathcal{N} \subset \mathcal{M}$ arises via $\mathcal{N} = \mathcal{M}^H = \{x \in \mathcal{M} \mid \gamma_a(x) = \varepsilon(x)\mathbf{1} \; \forall a \in H\}$ from the action of a Hopf algebra H on $\mathcal{M}$ iff the inclusion has depth two. In the next section this Hopf algebra will be identified and related to our quantum field theoretic setup.

For the irreducibility of $\mathcal{A}(\mathcal{O})$ in $\hat{\mathcal{F}}(\mathcal{O})$ we now give a proof independent of any sophisticated inclusion theoretic machinery.

Proposition 4.3.18 *For any $\mathcal{O} \in \mathcal{K}$ we have*

$$\hat{\mathcal{F}}(\mathcal{O}) \wedge \mathcal{A}(\mathcal{O})' = \mathbb{C}\mathbf{1}. \tag{4.3.34}$$

Proof. All unitary equivalences in this proof are implemented by $Y^{\mathcal{O}}$. With the abbreviations $\mathcal{M}_1 = \mathcal{F}(W_R^{\mathcal{O}})^t$ and $\mathcal{M}_2 = \mathcal{F}(W_L^{\mathcal{O}})$ we have $\mathcal{M}_1' \vee \mathcal{M}_2' \cong \mathcal{M}_1' \otimes \mathcal{M}_2'$. By (3.2.4) if $\mathcal{F}$ is bosonic or (4.2.19) in the Bose-Fermi case we have

$$\hat{\mathcal{F}}(\mathcal{O}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' \otimes \mathcal{F}(W_L^{\mathcal{O}}) = \mathcal{M}_1 \vee U(G)'' \otimes \mathcal{M}_2, \tag{4.3.35}$$

where we have used $\mathcal{M}^t \vee U(G)'' = \mathcal{M} \vee U(G)''$ (which is true for every von Neumann algebra $\mathcal{M}$). Furthermore,

$$\begin{aligned}\mathcal{A}(\mathcal{O})' &= \mathcal{F}(\mathcal{O})' \vee U(G)'' = (\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}}))^t \vee U(G)'' \\ &= \mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}}) \vee U(G)'' = \mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}})^t \vee U(G)'' \\ &= \mathcal{M}_1' \vee \mathcal{M}_2' \vee U(G)'' \cong (\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\, g \in G\}''.\end{aligned} \tag{4.3.36}$$

The relative commutant $\hat{\mathcal{F}}(\mathcal{O}) \wedge \mathcal{A}(\mathcal{O})'$ is thus equivalent to

$$(\mathcal{M}_1 \vee U(G)'' \otimes \mathcal{M}_2) \wedge [(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\, g \in G\}'']. \tag{4.3.37}$$

The obvious inclusion $(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\, g \in G\}'' \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_2' \vee U(G)''$ in conjunction with the irreducibility property $\mathcal{M}_2 \wedge (\mathcal{M}_2' \vee U(G)'') = \mathbb{C}\mathbf{1}$ (Corollary 4.3.15) yields

$$[(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\, g \in G\}''] \wedge (\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_2) \subset \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}. \tag{4.3.38}$$

Now let X be an element of the algebra given by eq. (4.3.37). By the same arguments as used earlier, every operator $X \in (\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\, g \in G\}''$ has a unique representation of the form $X = \sum_g F_g\, (U(g) \otimes U(g))$ where $F_g \in \mathcal{M}_1' \otimes \mathcal{M}_2'$. The condition $X \in \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$ implies $F_g = 0$ for all $g \neq e$ and thereby $X \in \mathcal{M}_1' \otimes \mathbf{1}$. We thus have $X \in (\mathcal{M}_1' \wedge (\mathcal{M}_1 \vee U(G)'')) \otimes \mathbf{1}$ and, once again using the irreducibility of the group inclusions, $X \propto \mathbf{1} \otimes \mathbf{1}$. ■

4.3.6 The Operators $Y^{\mathcal{O}}$ and Jones Theory

As witnessed by this chapter and the preceding one, the unitary operators $Y^{\mathcal{O}}$ which map $\mathcal{H}$ onto $\mathcal{H} \otimes \mathcal{H}$ and implement the isomorphisms $\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}}) \cong \mathcal{F}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})$ play an important role in the analysis of quantum field theories satisfying the SPW. Since the restriction of the fixpoint net $\mathcal{A}$ to $\mathcal{H}_0$ also satisfies wedge duality and the SPW, it makes sense to compare $Y_{\mathcal{A}}^{\mathcal{O}}$ and $Y_{\mathcal{F}}^{\mathcal{O}} \restriction \mathcal{H}_0$, where the suffixes $\mathcal{A}, \mathcal{F}$ indicate which net was used in the construction of $Y^{\mathcal{O}}$. In this subsection we assume $\mathcal{F}$ to be a local net for simplicity.

The tempting conjecture $Y_{\mathcal{F}}^{\mathcal{O}} \restriction \mathcal{H}_0 = Y_{\mathcal{A}}^{\mathcal{O}}$ is easily seen to be false. If U is a local symmetry of $\mathcal{F}$ (i.e. $Ad\, U(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O})\ \forall \mathcal{O} \in \mathcal{K}, U\Omega = \Omega$) then $Y_{\mathcal{F}}^{\mathcal{O}} U = (U \otimes U) Y_{\mathcal{F}}^{\mathcal{O}}$. With the projector on $\mathcal{H}_0$

$$P_0 = \int dg\; U(g), \tag{4.3.39}$$

where the integration takes place over a compact group G of inner symmetries, we have $Y^{\mathcal{O}}_{\mathcal{F}} P_0 = P_D Y^{\mathcal{O}}_{\mathcal{F}}$ with $P_D \equiv \int dg\ U(g) \otimes U(g)$. But P_D projects onto the subspace $\mathcal{H} \otimes \mathcal{H}^{\mathrm{Diag}(G)} \subset \mathcal{H} \otimes \mathcal{H}$ which is strictly larger than $\mathcal{H}_0 \otimes \mathcal{H}_0 = \mathcal{H} \otimes \mathcal{H}^{G \times G}$. Thus, $Y^{\mathcal{O}}_{\mathcal{F}}$ does not map $\mathcal{H}_0$ into $\mathcal{H}_0 \otimes \mathcal{H}_0$.

Considering the operator $I^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_0$ defined by

$$I^{\mathcal{O}} = (Y^{\mathcal{O}*}_{\mathcal{F}} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0)\, Y^{\mathcal{O}}_{\mathcal{A}} \tag{4.3.40}$$

it is easy to verify that $I^{\mathcal{O}}$ is an isometry. The range projection $E^{\mathcal{O}} = I^{\mathcal{O}} I^{\mathcal{O}*}$ can be computed as follows. Even if G is non-abelian, $P_0 \otimes P_0$ commutes with P_D, such that $P^{\mathcal{O}} \equiv Y^{\mathcal{O}*}_{\mathcal{F}} (P_0 \otimes P_0) Y^{\mathcal{O}}_{\mathcal{F}}$ restricts to a projection on $\mathcal{H}_0$, and $P^{\mathcal{O}} \upharpoonright \mathcal{H}_0 = E^{\mathcal{O}}$. Now, $U(g) \upharpoonright \mathcal{H}_0 = id$ implies

$$E^{\mathcal{O}} = \left(\int dg\, U^{\mathcal{O}}_L(g) \right) \upharpoonright \mathcal{H}_0, \tag{4.3.41}$$

where $U^{\mathcal{O}}_L(g) = Y^{\mathcal{O}*}_{\mathcal{F}}(U(g) \otimes \mathbf{1}) Y^{\mathcal{O}}_{\mathcal{F}}$ is the disorder operator introduced in Definition 4.2.4, and in particular $E^{\mathcal{O}} \in \mathcal{A}^d(\mathcal{O})$. That $I^{\mathcal{O}} \in \mathcal{A}^d(\mathcal{O})$ is also true is seen as follows. Let $A \in \mathcal{A}(W^{\mathcal{O}}_{LL}), B \in \mathcal{A}(W^{\mathcal{O}}_{RR})$. Then

$$\begin{aligned} (Y^{\mathcal{O}*}_{\mathcal{F}} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0)\, Y^{\mathcal{O}}_{\mathcal{A}}\ AB &= (Y^{\mathcal{O}*}_{\mathcal{F}} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0)\ (A \otimes B)\ Y^{\mathcal{O}}_{\mathcal{A}} \\ &= AB\ (Y^{\mathcal{O}*}_{\mathcal{F}} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0)\, Y^{\mathcal{O}}_{\mathcal{A}}, \end{aligned} \tag{4.3.42}$$

where the second identity follows from $Y^{\mathcal{O}}_{\mathcal{F}}\, \pi(AB) = \pi(A) \otimes \pi(B)\, Y^{\mathcal{O}}_{\mathcal{F}}$ and $[\pi(A) \otimes \pi(B), P_0 \otimes P_0] = 0$. Thus, $I^{\mathcal{O}} \in (\mathcal{A}(W^{\mathcal{O}}_{LL}) \vee \mathcal{A}(W^{\mathcal{O}}_{RR}))' = \mathcal{A}^d(\mathcal{O})$ as claimed.

In order to understand better the significance of $I^{\mathcal{O}}$ and $E^{\mathcal{O}}$, we recall the construction of the operators $Y^{\mathcal{O}}_{\mathcal{F}}$ [20], for simplicity assuming $\mathcal{F}$ to be local. By the isomorphism between $\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR})$ and $\mathcal{F}(W^{\mathcal{O}}_{LL}) \otimes \mathcal{F}(W^{\mathcal{O}}_{RR})$ there is a normal state ω_η on $\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR})$ such that

$$\omega_\eta(AB) = (\Omega, A\Omega) \cdot (\Omega, B\Omega), \quad A \in \mathcal{F}(W^{\mathcal{O}}_{LL}), B \in \mathcal{F}(W^{\mathcal{O}}_{RR}). \tag{4.3.43}$$

Since $\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR})$ admits a separating vector (e.g., Ω) and ω_η is faithful, there is a unique cyclic and separating vector $\eta \in \mathcal{P}^\natural(\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR}), \Omega)$ such that $\omega_\eta = (\eta, \cdot\, \eta)$. (Here $\mathcal{P}^\natural(\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR}), \Omega) = \overline{\Delta^{1/4}(\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR}))_+\Omega}$ is the natural cone, see, e.g., the appendix of [41] for a review and references.) As a consequence of the obvious gauge invariance of ω_η we have $\eta \in \mathcal{H}_0$ (and in fact, $\eta \in \mathcal{P}^\natural(M, \Omega) \subset \mathcal{H}_0$). Making an analogous construction with the net $\mathcal{A} = \mathcal{F}^G$ in the vacuum sector we obtain a vector $\eta_0 \in \mathcal{P}^\natural(N, \Omega) \subset \mathcal{H}_0$, where $N \equiv \mathcal{A}(W^{\mathcal{O}}_{LL}) \vee \mathcal{A}(W^{\mathcal{O}}_{RR})$. With these vectors the definition (4.3.40) of the isometry $I^{\mathcal{O}}$ takes the form

$$I^{\mathcal{O}}\, AB\, \eta_0 = AB\, \eta, \quad A \in \mathcal{F}(W^{\mathcal{O}}_{LL}), B \in \mathcal{F}(W^{\mathcal{O}}_{RR}). \tag{4.3.44}$$

We thus have $E^{\mathcal{O}} = [N\eta]$ which provides another proof of $E^{\mathcal{O}} \in N' = \mathcal{A}^d(\mathcal{O})$. We will now prove that $E^{\mathcal{O}}$ is just the Jones projection $e_N \in M_1$ of the inclusion $N \subset M \equiv (\mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR})) \cap U(G)'$. (See, e.g., [83, Sec. 2] for a review of inclusion theory for infinite factors.) Firstly, the conditional expectation $\mu : \mathcal{F}(W^{\mathcal{O}}_{LL}) \vee \mathcal{F}(W^{\mathcal{O}}_{RR}) \to \mathcal{A}(W^{\mathcal{O}}_{LL}) \vee \mathcal{A}(W^{\mathcal{O}}_{RR})$ given by

$$\mu(x) = \int dg\, dh\ U^{\mathcal{O}}_L(g) U^{\mathcal{O}}_R(h)\, x\, U^{\mathcal{O}}_R(h)^* U^{\mathcal{O}}_L(g)^* \tag{4.3.45}$$

restricts to a conditional expectation $\mu : M \to N$. Clearly, $\mu \restriction M$ is 'implemented' by $E^{\mathcal{O}}$:

$$E^{\mathcal{O}} x E^{\mathcal{O}} = \mu(x)\, E^{\mathcal{O}}, \quad x \in M. \tag{4.3.46}$$

Furthermore, $\omega_\eta \circ \mu = \omega_\eta$ follows from the invariance of $\Omega \otimes \Omega$ under $U(g) \otimes U(h)$. But now it is clear that $E^{\mathcal{O}} = e_N$, since the latter operator is defined through

$$e_N = [N\xi], \tag{4.3.47}$$

where ξ is a cyclic and separating vector for M such that ω_ξ is a μ-invariant faithful normal state on M. For the dual inclusion $M' = \mathcal{A}(\mathcal{O}) \subset \mathcal{A}^d(\mathcal{O}) = N'$ there is a normal conditional expectation $\mu' : N' \to M'$ (given by γ_e in the notation of the next section) iff the group G is finite. In this case one can easily check explicitly that $\mu'(E^{\mathcal{O}}) = 1/|G| \cdot \mathbf{1}$ as follows also from the general inclusion theory. Furthermore, the general theory tells us that $E^{\mathcal{O}} = e_N \in \mathcal{A}^d(\mathcal{O})$ is also the Jones projection for the inclusion $M_1' \subset M' = \mathcal{A}(\mathcal{O})$.

4.4 Quantum Double Symmetry

4.4.1 Abelian Groups

As we have shown above the algebras $\hat{\mathcal{F}}(\mathcal{O})$ may be considered as crossed products of $\mathcal{F}(\mathcal{O})$ with the actions of the respective automorphism groups $\alpha^{\mathcal{O}}$. In the case of abelian (locally compact) groups there is a canonical action [111] of the dual (character-) group $\hat{G}$ on $\mathcal{M} \rtimes G$ given by

$$\begin{array}{rcl} \hat{\alpha}_\chi(\pi(x)) & = & \pi(x) \\ \hat{\alpha}_\chi(U_g) & = & \overline{\chi(g)} \cdot U_g \end{array} \quad , \chi \in \hat{G}. \tag{4.4.1}$$

Making use of $U^{\mathcal{O}_1}(g)\, U^{\mathcal{O}_2}(g)^* \in \mathcal{F}$, $\forall \mathcal{O}_i$ one can consistently define an action of $\hat{G}$ on the net $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$, respecting the local structure and thus extending to the quasilocal algebra $\hat{\mathcal{F}}$. The action of $\hat{G}$ commutes with the original action of G as extended to $\hat{\mathcal{F}}$, implying that the locally compact group $G \times \hat{G}$ is a group of local symmetries of the extended theory $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$. The square structure (4.3.10) can now easily be interpreted in terms of the larger symmetry:

$$\hat{\mathcal{A}} = \hat{\mathcal{F}}^G, \quad \mathcal{F} = \hat{\mathcal{F}}^{\hat{G}}, \quad \mathcal{A} = \hat{\mathcal{F}}^{G \times \hat{G}}. \tag{4.4.2}$$

The symmetry between the subgroups G and $\hat{G}$ of $G \times \hat{G}$ is, however, not perfect, as only the automorphisms α_g, $g \in G$ are unitarily implemented on the Hilbert space $\mathcal{H}$. That there can be no unitary implementer $U(\chi)$ for $\hat{\alpha}_\chi$, $\chi \in \hat{G}$ leaving invariant the vacuum Ω is shown by the following computation which would be valid for all $A \in \mathcal{A}(\mathcal{O})$

$$\begin{aligned} \langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle &= \langle \Omega, U(\chi)\, AU_L^{\mathcal{O}}(g)\, U(\chi)^*\Omega \rangle \\ &= \langle \Omega, A\hat{\alpha}_\chi(U_L^{\mathcal{O}}(g))\Omega \rangle = \overline{\chi(g)} \cdot \langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle. \end{aligned} \tag{4.4.3}$$

This can only be true if $\chi(g) = 1$ or $\langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle = 0\ \forall A \in \mathcal{A}(\mathcal{O})$. The latter, however, can be ruled out, since the density of $\mathcal{A}(\mathcal{O})\Omega$ in $\mathcal{H}_0$ would imply $U_L^{\mathcal{O}}(g)\Omega \perp \mathcal{H}_0$ which is impossible, Ω being unitary and gauge invariant. This argument shows that the vacuum state $\omega = \langle\Omega, \cdot\,\Omega\rangle$ is not invariant under the automorphisms $\hat{\alpha}(\chi),\ \chi \in \hat{G}$, in other words, the symmetry under $\hat{G}$ is spontaneously broken.

The preceding argument is just a special case of the much more general analysis in [95], where non-abelian groups were considered, too. There, to be sure, the field net acted upon by the group was supposed to fulfill Bose-Fermi commutation relations, whereas in our case the field net is nonlocal. Furthermore, whereas the net $\mathcal{F}(\mathcal{O})$, the point of departure for our analysis, fulfills (twisted) duality, the extended net $\hat{\mathcal{F}}(\mathcal{O})$ enjoys no obvious duality properties. Nevertheless the analogy to [95] goes beyond the above argument. Indeed, as shown by Roberts, spontaneous breakdown of group symmetries is accompanied by a violation of Haag duality for the observables, restricted to the vacuum sector $\mathcal{H}_0$. Defining the net $\mathcal{B}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{G_0}$, the fixpoint net under the action of the unbroken part $G_0 = \{g \in G \,|\, \omega_0 \circ \alpha_g = \omega_0\}$ of the symmetry group, a combination of the arguments in [35] and [95] leads to the conclusion that (in the vacuum sector $\mathcal{H}_0$) $\mathcal{B}(\mathcal{O})$ is just the dual net $\mathcal{A}^d(\mathcal{O})$ which verifies Haag duality. Our analysis in Section 2, leading to the identification of the dual net as $\mathcal{A}^d = \hat{\mathcal{A}} = \hat{\mathcal{F}}^G$, is obviously in accord with the general theory as we have shown above that G is the unbroken part, corresponding to G_0, of the full symmetry group $G \times \hat{G}$.

In the case of spontaneously broken group symmetries it is known [28] that, irrespective of the nonexistence of global unitary implementers leaving invariant the vacuum, one can find local implementers for the whole symmetry group. This means that for each double cone $\mathcal{O}$ there exists a unitary representation $G \ni g \mapsto V_{\mathcal{O}}(g)$ satisfying $Ad\, V_{\mathcal{O}}(g) \restriction \mathcal{F}(\mathcal{O}) = \alpha_g$, the important point being the dependence on the region $\mathcal{O}$. (Due to the large commutant of $\mathcal{F}(\mathcal{O})$ such operators are far from unique.) A particularly nice construction, which applied to an unbroken symmetry g automatically yields the global implementer ($V_{\mathcal{O}}(g) = U(g)\, \forall \mathcal{O}$), was given in [29]. The construction given there applies without change to the situation at hand where the action of the dual group $\hat{G}$ on $\hat{\mathcal{F}}(\mathcal{O})$ is spontaneously broken.

An immediate consequence of the above and Theorem 4.3.10 is that the dual net $\mathcal{A}^d$ corresponding to the non-dual fixpoint net $\mathcal{A} = \mathcal{F}^G$ has a $\hat{G}$ symmetry. We will return to this in Subsection 4.4.6. An interesting example is provided by the free massive Dirac field which as already mentioned fulfills our postulates, including twisted duality and the split property. Its symmetry group $U(1)$ being compact and abelian, the extended net $\hat{\mathcal{F}}$ and the action of the dual group $\mathbb{Z}$ can be constructed as described above. By restriction of the net $\hat{\mathcal{A}}$ to the vacuum sector $\mathcal{H}_0$ one obtains a local net fulfilling Haag duality with symmetry group $\mathbb{Z}$. Wondering to which quantum field theory this net might correspond, it appears quite natural to think of the sine-Gordon theory at the free fermion point $\beta^2 = 4\pi$ as discussed, e.g., in [78].

4.4.2 Non-abelian Groups

We refrain from a further discussion of the abelian case and turn over to the more interesting case of G being non-abelian and finite. (Infinite compact groups will be treated in Subsection 4.5.1.) For non-abelian groups the dual object is not a group but either some Hopf algebraic structure or a category of representations. Correspondingly, the action of the dual group in [111] has to be replaced by a coaction of the group or the action of a group dual in the sense of [96]. For our present purposes these high-brow approaches will not be necessary. Instead we choose to generalize (4.4.1) in the following straightforward way. We observe that the characters of a compact abelian group constitute an orthogonal basis of the function space $L^2(G)$, whereas in the non-abelian case they span only the subspace of class functions. This motivates us to define an action of $\mathbb{C}(G)$, the $|G|$-dimensional space of *all* complex valued functions on G, on $\hat{\mathcal{F}}(\mathcal{O})$ in the following way:

$$\gamma_F\left(\sum_{g\in G} x(g)\, U_L^{\mathcal{O}}(g)\right) = \sum_{g\in G} F(g)\, x(g)\, U_L^{\mathcal{O}}(g), \quad x(g)\in\mathcal{F}(\mathcal{O}), F\in\mathbb{C}(G). \tag{4.4.4}$$

Again this action of $\mathbb{C}(G)$ is consistent with the local structure of the net $\mathcal{O}\mapsto\hat{\mathcal{F}}(\mathcal{O})$ and extends to the quasilocal C^*-algebra $\hat{\mathcal{F}}$. In general, of course, γ_F is no homomorphism but only a linear map. (That the maps γ_F are well defined for every $F\in\mathbb{C}(G)$ should be obvious, see also the next section.) Introducing the 'deltafunctions' $\delta_g(h)=\delta_{g,h}$ any function can be written as $F=\sum_g F(g)\,\delta_g$, and γ_{δ_g} will be abbreviated by γ_g. The latter are projections, i.e. they satisfy $\gamma_g^2=\gamma_g$. The images of $\hat{\mathcal{F}}(\mathcal{O})$ and $\hat{\mathcal{F}}$ under these will be designated $\hat{\mathcal{F}}_g(\mathcal{O})$ and $\hat{\mathcal{F}}_g$, respectively. Obviously we have $\hat{\mathcal{F}}_g(\mathcal{O})=\mathcal{F}(\mathcal{O})\,U_L^{\mathcal{O}}(g)$ and $\hat{\mathcal{F}}_g=\mathcal{F}\,U_L^{\mathcal{O}}(g)$ with $\mathcal{O}\in\mathcal{K}$ arbitrary. It should be clear that the decomposition

$$\hat{\mathcal{F}}=\bigoplus_{g\in G}\hat{\mathcal{F}}_g \tag{4.4.5}$$

represents a grading of $\hat{\mathcal{F}}$ by the group, i.e.

$$\hat{\mathcal{F}}_g\,\hat{\mathcal{F}}_h\subset\hat{\mathcal{F}}_{gh}\;\forall g,h\in G. \tag{4.4.6}$$

(In fact we have equality, but this will play no role in the sequel.) This group grading which is, of course, not surprising as it holds for every crossed product by a finite group allows us to state the behavior of γ_g under products:

$$\gamma_g(AB)=\sum_h \gamma_h(A)\,\gamma_{h^{-1}g}(B). \tag{4.4.7}$$

The novel aspect, however, is that $\hat{\mathcal{F}}$ is at the same time acted upon by the group G, these two structures being coupled by

$$\alpha_g(\hat{\mathcal{F}}_h)=\hat{\mathcal{F}}_{ghg^{-1}} \tag{4.4.8}$$

as a consequence of (4.2.12). This is equivalent to the relation

$$\alpha_g\circ\gamma_h=\gamma_{ghg^{-1}}\circ\alpha_g. \tag{4.4.9}$$

In this context it is of interest to remark that several years ago algebraists studied (see [32] and references given there) analogies between group graded algebras and algebras acted upon by a finite group. Similar studies have been undertaken in the context of inclusions of von Neumann algebras. As it turns out the situation at hand, which is rather more interesting, can be neatly described in terms of the action, as defined, e.g., in [109], of a Hopf algebra (in our case finite dimensional) on $\hat{\mathcal{F}}$. The relations fulfilled by the α_g and γ_h, in particular (4.4.9), motivate us to cite the following well known

Definition 4.4.1 *Let $\mathbb{C}(G)$ be the algebra of (complex valued) functions on the finite group G and consider the adjoint action of G on $\mathbb{C}(G)$ according to $\alpha_g : f \mapsto f \circ Ad(g^{-1})$. The quantum double $D(G)$ is defined as the crossed product $D(G) = \mathbb{C}(G) \rtimes_\alpha G$ of $\mathbb{C}(G)$ by this action. In terms of generators $D(G)$ is the algebra generated by elements U_g and V_h, $g, h \in G$ with the relations*

$$U_g U_h = U_{gh}, \tag{4.4.10}$$
$$V_g V_h = \delta_{g,h} V_g, \tag{4.4.11}$$
$$U_g V_h = V_{ghg^{-1}} U_g \tag{4.4.12}$$

and the identification $U_e = \sum_g V_g = \mathbf{1}$.

It is easy to see that $D(G)$ is of the finite dimension $|G|^2$, where as a convenient basis one may choose $V(g)U(h)$, $g, h \in G$, multiplying according to $V(g_1)U(h_1)\,V(g_2)U(h_2) = \delta_{g_1, h_1 g_2 h_1^{-1}} \cdot V(g_1)U(h_1 h_2)$. This is just a special case of a construction given by Drinfel'd [50] in greater generality which we do not bother to retain. For the purposes of this work it suffices to state the following well known properties of $D(G)$, referring to [50, 93, 34] for further discussion, see also Appendix C.

In order to define an action of a Hopf algebra on von Neumann algebras we further need a star structure on the former which in our case is provided by the following

Proposition 4.4.2 *With the definition $U_g^* = U_{g^{-1}}$, $V_h^* = V_h$ and the appropriate extension, $D(G)$ is a *-algebra. $D(G)$ is semisimple.*

Proof. Trivial calculation. Finite dimensional *-algebras are automatically semisimple. ■

Before stating how the quantum double $D(G)$ acts on $\hat{\mathcal{F}}$ we define precisely the properties of a Hopf algebra action.

Definition 4.4.3 *A bilinear map $\gamma : H \times \mathcal{M} \to \mathcal{M}$ is an action of the Hopf *-algebra H on the *-algebra $\mathcal{M}$ iff the following hold for any $a, b \in H$, $x, y \in \mathcal{M}$:*

$$\gamma_{\mathbf{1}}(x) = x, \tag{4.4.13}$$
$$\gamma_a(\mathbf{1}) = \varepsilon(a)\mathbf{1}, \tag{4.4.14}$$
$$\gamma_{ab}(x) = \gamma_a \circ \gamma_b(x), \tag{4.4.15}$$
$$\gamma_a(xy) = \gamma_{a^{(1)}}(x)\gamma_{a^{(2)}}(y), \tag{4.4.16}$$
$$(\gamma_a(x))^* = \gamma_{S(a^*)}(x^*). \tag{4.4.17}$$

We have used the standard notation $\Delta(a) = a^{(1)} \otimes a^{(2)}$ for the coproduct where on the right side there is an implicit summation. The map γ is assumed to be weakly continuous with respect to $\mathcal{M}$ and continuous with respect to some C^*-norm on H (which is unique in the case of finite dimensionality).

After these lengthy preparations it is clear how to define the action of $D(G)$ on $\hat{\mathcal{F}}$.

Theorem 4.4.4 *Defining* $\gamma_a(\hat{F}),\ \hat{F} \in \hat{\mathcal{F}}$ *for* $a \in \{U(g), V(h) | g, h \in G\}$ *by*

$$\begin{aligned} \gamma_{U_g}(\hat{F}) &= \alpha_g(\hat{F}) && (4.4.18)\\ \gamma_{V_h}(\hat{F}) &= \gamma_h(\hat{F}), && (4.4.19) \end{aligned}$$

using (4.4.15) to define γ *on the basis* $V(g)U(h)$ *and extending linearly to* $D(G)$ *one obtains an action in the sense of Definition 4.4.3.*

Proof. (4.4.13) follows from $\mathbf{1}_{D(G)} = \sum_g V_g$, (4.4.14) from $\mathbf{1}_{\hat{\mathcal{F}}} \in \hat{\mathcal{F}}_e$ and (C.1.4), whereas (4.4.15) is an obvious consequence of the definition. Furthermore, (4.4.16) is a consequence of α_g being a homomorphism, the coproduct property (4.4.7) and the definition (C.1.5). The statement (4.4.17) on the *-operation finally follows from $(\alpha_g(x))^* = \alpha_g(x^*)$ and $S(U_g^*) = U_g$ on the one hand and $(\hat{\mathcal{F}}_g)^* = \hat{\mathcal{F}}_{g^{-1}}$ and $S(V_g^*) = V_{g^{-1}}$ on the other. ∎

Remarks. 1. It should be obvious that the action of $D(G)$ on $\hat{\mathcal{F}}$ commutes with the translations and that it commutes with the boosts iff the group G does. Otherwise, $U(\Lambda)\, U(g)\, U(\Lambda)^* = U_h$ implies $\alpha_\Lambda \circ \gamma_g = \gamma_h \circ \alpha_\Lambda$.

2. In the case of G being abelian $U_\chi = \sum_{g \in G} \overline{\chi(g)} \cdot V_g,\ \chi \in \hat{G}$ constitutes an alternative basis for the subalgebra $\mathbb{C}(G) \subset D(G)$. The resulting formulae $U_\chi\, U_\rho = U_{\chi\rho},\ \Delta(U_\chi) = U_\chi \otimes U_\chi$ and $\gamma_{U_\chi}(\cdot) = \hat{\alpha}_\chi(\cdot)$ establish the equivalence of the quantum double with the group $G \times \hat{G}$. The abelian case is special insofar as $D(G)$ is spanned by its grouplike elements, which is not true for G non-abelian.

4.4.3 Spontaneously Broken Quantum Symmetry

Having shown in the abelian case that the symmetry under the dual group $\hat{G}$ is spontaneously broken it should not come as a surprise that the same holds for non-abelian groups G where, of course, the notion of unitary implementation has to be generalized.

Definition 4.4.5 *An action* γ *of the Hopf algebra* H *on the *-algebra* $\mathcal{M}$ *is said to be implemented by the (homomorphic) representation* $U:\ H \to \mathcal{B}(\mathcal{H})$ *if for all* $a \in H, x \in \mathcal{M}$

$$U(a)\, x = \gamma_{a^{(1)}}(x)\, U(a^{(2)}) \tag{4.4.20}$$

or equivalently

$$\gamma_a(x) = U(a^{(1)})\, x U(S(a^{(2)})). \tag{4.4.21}$$

The representation is said to be unitary if the map U *is a *-homomorphism.*

In complete analogy to the abelian case we see that only a subalgebra of $D(G)$, namely the group algebra $\mathbb{C}G$ is implemented in the above sense. A similar phenomenon has already been observed to occur in the Coulomb gas representation of the minimal models [67] and in [10] where two dimensional theories without conformal covariance were considered. It would be interesting to know whether there exists, in some sense, a 'quantum version' of Goldstone's theorem for spontaneously broken Hopf algebra symmetries.

In an earlier section we defined a twist operation (4.3.25) which bijectively maps $\hat{\mathcal{F}}(\mathcal{O})$ into an algebra $\hat{\mathcal{F}}(\mathcal{O})^T$ which commutes with all field operators localized in the left spacelike complement $W_{LL}^{\mathcal{O}}$ of $\mathcal{O}$. With the notation introduced in this chapter this operation can be written as $F^T = \sum_g \gamma_g(F)^t\, U(g^{-1})$. One might wonder whether there is a map $\bar{T}$ which achieves the same thing for the right spacelike complement $W^{\mathcal{O}}_{RR}$. If the quantum symmetry were not spontaneously broken, such a map would be given by

$$F^{\bar{T}} = \sum_g \alpha_g(F)^t\, V(g), \tag{4.4.22}$$

where the $V(g)$ are the projectors implementing the dual $\mathbb{C}(G)$ of the group G. Using the spacelike commutation relations and the property $U^{\mathcal{O}}(g)\, V(h) = V(gh)\, U^{\mathcal{O}}(g)$ this claim is easily verified.

In the discussion of the abelian case we have mentioned that one can construct, e.g. by the method given in [29], local implementers of the dual group $\hat{G}$. For the quantum double $D(G)$ of a non-abelian group G, however, which is not spanned by its grouplike elements, another approach is needed. What we are looking for is, for every double cone $\mathcal{O}$, a family of orthogonal projections $V_{\mathcal{O}}(g)$ fulfilling

$$V_{\mathcal{O}}(g)\, V_{\mathcal{O}}(h) = \delta_{g,h}\, V_{\mathcal{O}}(g)\ , \quad \sum_g V_{\mathcal{O}}(g) = \mathbf{1}, \tag{4.4.23}$$

$$\gamma_g \restriction \hat{\mathcal{F}}(\mathcal{O}) = \sum_h V_{\mathcal{O}}(gh)\, \cdot\, V_{\mathcal{O}}(h) \tag{4.4.24}$$

and transforming correctly under the (unbroken) group G

$$U(g)\, V_{\mathcal{O}}(h)\, U(g)^* = V_{\mathcal{O}}(ghg^{-1}). \tag{4.4.25}$$

In order to obtain operators with these properties we make use of the isomorphism, for every wedge W, between $\mathcal{F}(W) \vee U(G)''$ and $\mathcal{F}(W) \rtimes_\alpha G$. We shortly remind the construction of the crossed product $\mathcal{M} \rtimes G$. It is represented on the Hilbert space $\bar{\mathcal{H}} = L^2(G, \mathcal{H})$ of square integrable functions from G to $\mathcal{H}$. The algebra $\mathcal{M}$ acts according to $(\pi(x)f)(g) = \alpha_{g^{-1}}(x)\, f(g)$ whereas the group G is unitarily represented by $(\bar{U}(k)f)(g) = f(k^{-1}g)$. With these definitions one can easily verify the equation $\bar{U}(k)\,\pi(x)\,\bar{U}(k)^* = \pi \circ \alpha_k(x)$. If the group G is finite one can furthermore define the projections $(\bar{E}(k)f)(g) = \delta_{g,k}\, f(g)$ for which one obviously has $\bar{U}(g)\, \bar{E}(k) = \bar{E}(gk)\, \bar{U}(g)$. As already discussed above there is, as a consequence of the outerness of the action of the group, an isomorphism between the algebras $\mathcal{M} \vee U(G)''$ and $\mathcal{M} \rtimes_\alpha G$ sending $\sum_g x_g\, U(g)$ to $\sum_g \pi(x_g)\, \bar{U}(g)$. As both algebras are of type III and live on separable Hilbert spaces this isomorphism is unitarily implemented and can be used to pull back the projections $\bar{E}(k)$ to the Hilbert space $\mathcal{H}$ where we denote them by $E(k)$. ($E(e)$ is nothing but the Jones projection in the extension $\mathcal{M}_2$ of the inclusion $\mathcal{M} \subset \mathcal{M} \vee U(G)''$.) Applying these considerations to the algebras of the wedges $W^{\mathcal{O}}_L$ and $W^{\mathcal{O}}_R$ we obtain the families of projections $E^{\mathcal{O}}_{L/R}(k)$, satisfying

$$U(g)\, E^{\mathcal{O}}_{L/R}(k)\, U(g)^* = E^{\mathcal{O}}_{L/R}(gk), \tag{4.4.26}$$

which we use to define

$$V_{\mathcal{O}}(g) = Y^{\mathcal{O}*}\, (\sum_h E^{\mathcal{O}}_R(gh) \otimes E^{\mathcal{O}}_L(h))\, Y^{\mathcal{O}}. \tag{4.4.27}$$

The properties (4.4.23) of orthogonality and completeness are obvious whereas covariance (4.4.25) follows from (4.4.26) and $U(k) = Y^{\mathcal{O}*}\, U(k) \otimes U(k)\, Y^{\mathcal{O}}$ as follows

$$\begin{aligned} Ad\, U(k)(V_{\mathcal{O}}(g)) &= Y^{\mathcal{O}*}\,(\sum_h E_R^{\mathcal{O}}(kgh) \otimes E_L^{\mathcal{O}}(kh))\, Y^{\mathcal{O}} \\ &= Y^{\mathcal{O}*}\,(\sum_h E_R^{\mathcal{O}}(kgk^{-1}h) \otimes E_L^{\mathcal{O}}(h))\, Y^{\mathcal{O}} \\ &= V_{\mathcal{O}}(kgk^{-1}). \end{aligned} \tag{4.4.28}$$

It remains to show the implementation property (4.4.24). Using $E_L^{\mathcal{O}}(g)\, \mathcal{F}(W_L^{\mathcal{O}})\, E_L^{\mathcal{O}}(h) = \{0\}$ if $g \neq h$ and $\hat{\mathcal{F}}(\mathcal{O}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' \otimes \mathcal{F}(W_L^{\mathcal{O}})$ we obtain

$$\begin{aligned} Y^{\mathcal{O}} \sum_h V_{\mathcal{O}}(gh)\, \hat{F}\, V_{\mathcal{O}}(h)\, Y^{\mathcal{O}*} &= \sum_{h,k,l} E_R^{\mathcal{O}}(ghk) \otimes E_L^{\mathcal{O}}(k)\, F_1 \otimes F_2\, E_R^{\mathcal{O}}(hl) \otimes E_L^{\mathcal{O}}(l) \\ &= \sum_{h,k} E_R^{\mathcal{O}}(ghk) \otimes E_L^{\mathcal{O}}(k)\, F_1 \otimes F_2\, E_R^{\mathcal{O}}(hk) \otimes E_L^{\mathcal{O}}(k) \\ &= (\sum_h E_R^{\mathcal{O}}(gh)\, F_1\, E_R^{\mathcal{O}}(h)) \otimes (\sum_k E_L^{\mathcal{O}}(k)\, F_2\, E_L^{\mathcal{O}}(k)) \\ &= \sum_h E_R^{\mathcal{O}}(gh)\, F_1\, E_R^{\mathcal{O}}(h) \otimes F_2 \end{aligned} \tag{4.4.29}$$

where we have written (abusively) $F_1 \otimes F_2$ for $Y^{\mathcal{O}}\, \hat{F}\, Y^{\mathcal{O}*}$. With $\sum_h E_R^{\mathcal{O}}(gh)\, U(k)\, E_R^{\mathcal{O}}(h) = \delta_{g,k} U(k)$ it is clear that the above map projects $\hat{\mathcal{F}}(\mathcal{O})$ onto $\mathcal{F}(\mathcal{O}) U_L^{\mathcal{O}}(g)$, thus implementing the restriction of γ_g to $\hat{\mathcal{F}}(\mathcal{O})$. It should be remarked that the apparently simpler definition $\tilde{V}_{\mathcal{O}}(g) = Y^{\mathcal{O}*}\, E_R^{\mathcal{O}}(g) \otimes \mathbf{1}\, Y^{\mathcal{O}}$, which also satisfies (4.4.24), does not lead to a representation of $D(G)$ as these $V_{\mathcal{O}}$'s do not transform according to the adjoint representation (4.4.25).

4.4.4 Spectral Properties

The above discussion was to a large extent independent of the quantum field theoretic application insofar as the action of the quantum double on a certain class of *-algebras was concerned. As we have seen, any *-algebra which is at the same time acted upon by a finite group G and graded by G supports an action of the double provided the relation (4.4.8) holds. The converse is also true. Let $\mathcal{M}$ be a *-algebra on which the double acts. Then $\mathcal{M}_g = \gamma_g(\mathcal{M})$ induces a G-grading satisfying (4.4.8). It may however happen that $\mathcal{M}_g = \{0\}$ for g in a normal subgroup. This possibility can be eliminated by demanding the existence of a unitary representation of G in $\mathcal{M}$: $G \ni g \mapsto U(g) \in \mathcal{M}_g$. In the situation at hand this condition is fulfilled by construction.

We now turn to the spectral properties of the action of the double. To this purpose we introduce the following notion [96], already encountered implicitly in the proof of Proposition 4.3.17.

Definition 4.4.6 *A normclosed linear subspace $\mathcal{T}$ of a von Neumann algebra $\mathcal{M}$ is called a Hilbert space in $\mathcal{M}$ if $x^*x \in \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{T}$ and $x \in \mathcal{M}$ and $xa = 0\ \forall a \in \mathcal{T}$ implies $x = 0$.*

The name is justified as $\langle x, y\rangle \mathbf{1} = x^* y$ defines a scalar product in $\mathcal{T}$. One can thus choose a basis $\psi_i,\ i = 1 \dots d_{\mathcal{T}}$ satisfying the requirements (4.3.28, 4.3.29). The interest of this definition stems from the following well known lemma, the easy proof of which we omit.

Lemma 4.4.7 *Let $\mathcal{T}$ be a finite dimensional Hilbert space in $\mathcal{M}$ globally invariant under the action γ_H of the Hopf algebra H on $\mathcal{M}$. A basis of the above type gives rise to a unitary representation of H according to*

$$\gamma_a(\psi_i) = \sum_{i'=1}^{d} D_{i'i}(a)\, \psi_{i'}. \tag{4.4.30}$$

Our aim will now be to show that the extended algebras $\hat{\mathcal{F}}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ in fact contain such tensor multiplets for every irreducible representation of $D(G)$. In order to do this we make use of the representation theory of the double developed in [34]. ($D(G)$ being semisimple, every finite dimensional representation decomposes into a direct sum of irreducible ones.) The (equivalence classes of) irreducible representations are labeled by pairs (c, π), where $c \in C(G)$ is a conjugacy class and π is an irreducible representation of the normalizer group N_c. Here N_c is the abstract group corresponding to the mutually isomorphic normalizers N_g for $g \in c$, already encountered in Theorem 4.3.16. The representation $\hat{\pi}$ labeled by (c, π) is obtained by choosing an arbitrary $g_0 \in c$ and inducing up from the representation

$$\hat{\pi}(V_g\, U_h) = \delta_{g,g_0}\, \pi(h) \tag{4.4.31}$$

of the subalgebra $\mathcal{B}_{g_0}$ of $D(G)$ generated by $V(g)\,, g \in G$ and $U(h)\,, h \in N_{g_0}$. The representation space of $\hat{\pi}_{(c,\pi)}$ is thus $V_{(c,\pi)} = D(G) \otimes_{\mathcal{B}_{g_0}} V_\pi$. For a more complete discussion we refer to [34] remarking only that $\hat{\pi}_{(c,\pi)}(V_g\, U_h) = 0$ if $g \notin c$.

Definition 4.4.8 *The action γ of a group or Hopf algebra on a von Neumann algebra $\mathcal{M}$ is dominant iff the algebra of fixed points is properly infinite and the monoidal spectrum of γ is complete, i.e. for every finite dimensional unitary representation π of the group or Hopf algebra, respectively, there is a γ-invariant Hilbert space $\mathcal{T}$ in $\mathcal{M}$ such that $\gamma \restriction \mathcal{T}$ is equivalent to π.*

Proposition 4.4.9 *Let $\hat{\mathcal{M}}$ be a von Neumann algebra supporting an action of the quantum double $D(G)$. Assume further that there is a unitary representation of G in $\hat{\mathcal{M}}$ where $\bar{U}(g) \in \hat{\mathcal{M}}_g$ and $\alpha_h(\bar{U}(g)) = \bar{U}(hgh^{-1})$. Then the action of $D(G)$ on $\hat{\mathcal{M}}$ is dominant if and only if the action of G on $\mathcal{M} = \gamma_e(\hat{\mathcal{M}})$ is dominant.*

Proof. As a consequence of $\mathcal{M}^G = \hat{\mathcal{M}}^{D(G)}$ the conditions of proper infiniteness of the fixpoint algebras coincide. The 'only if' statement is easily seen by considering the representations of the double corresponding to the conjugacy class $c = \{e\}$. For these $N_c \cong G$ holds, implying that the representations of $D(G)$ with $c = \{e\}$ are in one-to-one correspondence to the representations of G. A multiplet in $\hat{\mathcal{M}}$ transforming according to $(\{e\}, \pi)$ is nothing but a π-multiplet in $\mathcal{M}$.

The 'if' statement requires more work. We have to show that for every pair (c, π), where π is an irreducible representation of the normalizer N_c, there exists a multiplet of isometries transforming according to $\hat{\pi}_{(c,\pi)}$. To begin with, choose $g \in c$ arbitrarily and find in $\mathcal{M}$ a multiplet of isometries $\psi_i,\ i=1,\ldots,d=\dim(\pi)$ transforming according to the representation π under the action of $N_g \subset G$. The existence of such a multiplet follows from the dominance of the group action on $\mathcal{M}$. Now, let $x_1,\ldots,x_n$ be representatives of the cosets G/N_g where $n=[G:N_g]=|c|$. Furthermore, the proper infiniteness of the fixpoint algebra allows us to choose a family $V_1,\ldots,V_n$ of isometries in $\mathcal{M}^G = \hat{\mathcal{M}}^{D(G)}$ satisfying $V_i^* V_j = \delta_{i,j},\ \sum_i V_i V_i^* = \mathbf{1}$. Defining

$$\Psi_{ij} = V_i\, \alpha_{x_i}(\bar{U}(g)\, \psi_j), \quad i=1,\ldots,n,\ j=1,\ldots,d \tag{4.4.32}$$

one verifies that the Ψ_{ij} constitute a complete family of mutually orthogonal isometries spanning a vectorspace of dimension $nd=\dim(\hat{\pi}_{(c,\pi)})$. That this space is mapped into itself by the action of the double follows from the fact that, for every $k\in G,\ k\,x_i$ can uniquely be written as $x_j\, h,\ h\in N_g$. Finally, the multiplet transforms according to the representation (c,π) of $D(G)$, which is evident from the definition of the latter in [34, (2.2.2)]. ■

Remark. Since in our field theoretic application the conditions of the proposition are satisfied thanks to Lemma 4.3.14 and the discussion in Subsection 4.4.2 we can conclude that $\hat{\mathcal{F}}(\mathcal{O}),\ \mathcal{O}\in\mathcal{K}$ has full $D(G)$-spectrum.

4.4.5 Commutation Relations and Statistics

Up to this point our investigations in this section have focused on the local inclusion $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ for any fixed region $\mathcal{O}$. Having clarified the relation between these algebras in terms of the action of the quantum double we can now complete our discussion of the latter. To this purpose we recall that the double construction has been introduced in [50] as a means of obtaining quasitriangular Hopf algebras (quantum groups) in the sense defined there, i.e. Hopf algebras possessing a 'universal R-matrix'. As it turns out the latter appears quite naturally in our approach when considering the spacelike commutation relations of irreducible $D(G)$-multiplets as defined above.

Proposition 4.4.10 *Assume the net $\mathcal{O}\mapsto\mathcal{F}(\mathcal{O})$ is bosonic, i.e. fulfills untwisted locality. Let $\mathcal{O}_2 < \mathcal{O}_1$ (i.e. $\mathcal{O}_2 \subset W_{LL}^{\mathcal{O}_1}$) and ψ^1,ψ^2 be $D(G)$-tensors in $\hat{\mathcal{F}}(\mathcal{O}_1)$, $\hat{\mathcal{F}}(\mathcal{O}_2)$, respectively. They then fulfill C-number commutation relations*

$$\psi_i^1\, \psi_j^2 = \sum_{i'j'} \psi_{j'}^2\, \psi_{i'}^1\ (D^1_{i'i} \otimes D^2_{j'j})(R), \tag{4.4.33}$$

where D^1, D^2 are the matrices of the respective representations and

$$R = \sum_{g\in G} V_g \otimes U_g \in D(G)\otimes D(G). \tag{4.4.34}$$

Proof. The equation $\sum_g V_g = \mathbf{1}$ in $D(G)$ implies $\sum_g \gamma_g = id$. We can thus compute

$$\begin{aligned}\psi_i^1\, \psi_j^2 &= \sum_{g\in G} \gamma_g(\psi_i^1)\, \psi_j^2 = \sum_{g\in G} \alpha_g(\psi_j^2)\, \gamma_g(\psi_i^1) \\ &= \sum_{g\in G}\sum_{i'j'} \psi_{j'}^2\, \psi_{i'}^1\ D^2_{j'j}(U_g)\, D^1_{i'i}(V_g),\end{aligned} \tag{4.4.35}$$

where the second identity follows from $\gamma_g(\psi_i^1) \in \mathcal{F}(\mathcal{O}_1)\, U_L^{\mathcal{O}_1}(g)$ and $Ad\, U_L^{\mathcal{O}_1}(g) \restriction \hat{\mathcal{F}}(\mathcal{O}_2) = \alpha_g$. The rest is clear. ■

Remarks. 1. Commutation relations of the above general type have apparently first been considered in [61]. For the special case of $Z(N)$ order disorder duality they date back at least to [107].

2. By this result the field extension of Definition 4.3.1 in conjunction with Theorem 4.4.4 may be considered a local version of the construction of the double. (If we had used the $U_R^{\mathcal{O}}(g)$ we would have ended up with R^{-1} which would do just as well.)

3. If the net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is fermionic an additional sign $\pm$ appears on the right hand side of (4.4.33) depending on the Bose/Fermi nature of the fields. Using the bosonization prescription of the next section this sign can be eliminated.

Let $\psi_i,\ i = 1 \dots d_r$ be a multiplet of isometries in $\hat{\mathcal{F}}(\mathcal{O})$ transforming according to the irreducible representation r of $D(G)$. Then the map

$$\rho(\cdot) = \sum_{i=1}^{d_r} \psi_i \cdot \psi_i^* \tag{4.4.36}$$

defines a unital *-endomorphism of $\hat{\mathcal{F}}$. The relative locality of $\mathcal{A}$ and $\hat{\mathcal{F}}$ implies the restriction of ρ to $\mathcal{A}$ to be localized in $\mathcal{O}$ in the sense that $\rho(A) = A\ \forall A \in \mathcal{A}(\mathcal{O}')$. Furthermore, ρ maps $\mathcal{A}(\mathcal{O}_1)$ into itself if $\mathcal{O}_1 \supset \mathcal{O}$ as follows from the $D(G)$-invariance of $\rho(x)$ for $x \in \mathcal{A}$. (The conventional argument using duality would allow us only to conclude $\rho(\mathcal{A}(\mathcal{O}_1)) \subset \hat{\mathcal{A}}(\mathcal{O}_1)$.)

Proposition 4.4.11 *In restriction to $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{O}_1 \supset \mathcal{O}$ the endomorphism ρ is irreducible.*

Proof. The proof is omitted as it is identical to the proof of [81, Prop. 6.9] where compact groups are considered. ■

Remarks. 1. In application to the net $\hat{\mathcal{A}}$ the endomorphisms ρ are localized only in wedge regions, i.e. they are of solitonic character.

2. Due to the spontaneous breakdown of the quantum symmetry the endomorphisms ρ which arise from non-group representations of $D(G)$ should not be considered as true superselection sectors of the net $\mathcal{A} \restriction \mathcal{H}_0$. This would be justified if the symmetry were unbroken. Nevertheless, one can analyze their statistics, as will be done in the rest of this section.

In order to study the statistics of endomorphisms one introduces [37, 56] the statistics operators

$$\varepsilon(\rho_1, \rho_2) = U_2^*\, \rho_1(U_2) \in (\rho_1\rho_2, \rho_2\rho_1), \tag{4.4.37}$$

where U_2 is a charge transporter intertwining ρ_2 and $\tilde{\rho}_2$, the latter being localized in the left spacelike complement of the localization region of ρ_1. With $U_2 = \sum_i \tilde{\psi}_i^{(2)}\, \psi_i^{(2)*}$ and using the spacelike commutation relations (4.4.33) we obtain

$$\varepsilon(\rho_1, \rho_2) = \sum_{ijkl} \psi_i^{(2)}\, \psi_l^{(1)}\, \psi_j^{(2)*}\, \psi_k^{(1)*}\, (D^1_{lk} \otimes D^2_{ij})(R). \tag{4.4.38}$$

Introducing the left inverse [35] of an endomorphism, which for morphisms implemented by a multiplet of field operators [45] is given by

$$\phi_\rho = \frac{1}{d_\rho} \sum_{i=1}^{d_\rho} \psi_i^* \cdot \psi_i, \tag{4.4.39}$$

one obtains for the statistics parameter [35]

$$\lambda_\rho = \phi_\rho(\varepsilon_{\rho,\rho}) = \frac{1}{d_\rho} \sum_{ijl} \psi_l \, \psi_j^* \ (D_{li} \otimes D_{ij})(R) = \frac{1}{d_\rho} \sum_{jl} \psi_l \, \psi_j^* \ M_{lj}, \tag{4.4.40}$$

where

$$M_{lj} = D_{lj}(m(R)) = D_{lj}(\sum_g V_g \, U_g)). \tag{4.4.41}$$

An easy calculation shows that $X = m(R) = \sum_g V_g \, U_g$ is a unitary element in the center of $D(G)$. This implies that it is represented by a phase ω_r in every irreducible representation r of $D(G)$, i.e. $M_{lj} = \delta_{lj} \, \omega_\rho$ which, applying the completeness relation (4.3.29), gives

$$\lambda_\rho = \frac{\omega_\rho}{d_\rho}. \tag{4.4.42}$$

Recalling Lemma 4.4.7 we see that in restriction to a field operator in a multiplet transforming according to the irreducible representation r the action of γ_X amounts to multiplication by ω_r. The unitary $X \in D(G)$ may thus be interpreted as the quantum double analogue of the group element k which distinguishes between bosons and fermions. This is reminiscent of the notion of ribbon elements in the framework of quantum groups, see Appendix C. In fact, the operator X defined above is just the inverse of Drinfel'd's $u = \sum_g V_g \, U_{g^{-1}}$ which itself is a ribbon element due to $S(u) = u$.

Appealing to the representation theory of $D(G)$ as expounded in [34] it is easy to compute the phase ω_r for the representation $r = (c, \pi)$. It is given by the scalar to which $g \in c$, obviously being contained in the center of the normalizer N_g, is mapped by the irreducible representation π of N_g. As an immediate consequence [33] ω_r is an n-th root of unity where n is the order of g.

Another consequence of (4.4.42) is that the statistical dimension of the sector ρ, defined as $d_\rho = |\lambda_\rho|^{-1}$ coincides with the dimension of the corresponding representation of the quantum double. This was to be expected and is in accord with the fact [81] that the action of finite dimensional Hopf algebras cannot give rise to noninteger dimensions.

We now turn to the calculation of the monodromy operator

$$\varepsilon_M(\rho_1, \rho_2) = \varepsilon(\rho_1, \rho_2) \, \varepsilon(\rho_2, \rho_1), \tag{4.4.43}$$

which measures the deviation from permutation group statistics. Inserting the statistics operators according to (4.4.38) and using twice the orthogonality relation we obtain

$$\varepsilon_M(\rho_1, \rho_2) = \sum_{\substack{ijkl \\ k'l'}} \ \psi_i^2 \, \psi_j^1 \, \psi_{k'}^{1*} \, \psi_{l'}^{2*} \ (D_{jl}^1 \otimes D_{ik}^2)(R) \, (D_{kl'}^2 \otimes D_{lk'}^1)(R). \tag{4.4.44}$$

The numerical factor to the right can be simplified to

$$\sum_{g,h\in G} D^1_{jk'}(V_g\,U_h)\,D^2_{il'}(U_g\,V_h) = (D^1_{jk'}\otimes D^2_{il'})(I), \tag{4.4.45}$$

where

$$I = R\,\sigma(R) \tag{4.4.46}$$

can be considered as the quantum group version of the monodromy operator. Finally we define the statistics characters [91] by

$$Y_{ij} = d_i d_j\,\phi_i(\varepsilon_M(\rho_i,\rho_j)^*). \tag{4.4.47}$$

Due to the fact that $\phi_i(\varepsilon_M(\rho_i,\rho_j)^*)$ is a selfintertwiner of ρ_j and the irreducibility of the latter, this gives a square matrix of c-numbers indexed by the superselection sectors, i.e. in our case the representations of the double. Inserting (4.4.44) and using $Y_{ij}\propto \mathbf{1}$ we obtain

$$Y_{ij} = (tr_i\otimes tr_j)\circ(D^i\otimes D^j)(I^*). \tag{4.4.48}$$

In [2] it was shown that Y is invertible, in fact $\frac{1}{|G|}Y$ is unitary. In conjunction with the known facts concerning the representation theory one concludes [2] that the quantum double $D(G)$ is a *modular Hopf algebra* in the sense of [94]. We are now in a position to complete our demonstration of the complete parallelism between quantum group theory and quantum field theory (which we claim only for the quantum double situation at hand!). What remains to be discussed is the Verlinde algebra structure [113] behind the fusion of representations of the double and the associated endomorphisms of $\hat{\mathcal{F}}$, respectively. The fusion rules are said to be diagonalized by a unitary matrix S if

$$N^k_{ij} = \sum_m \frac{S_{im}\,S_{jm}\,S^*_{km}}{S_{0m}}. \tag{4.4.49}$$

(For a comprehensive survey of fusion structures see [63].) One speaks of a Verlinde algebra if, in addition, S is symmetric, there is a diagonal matrix T of phases satisfying $TC = CT = T$ ($C_{ij}=\delta_{i\bar{j}}$ is the charge conjugation matrix) and S and T constitute a representation of $SL(2,\mathbb{Z})$ (in general not of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$), i.e.

$$S^2 = (ST)^3 = C. \tag{4.4.50}$$

On the one hand the representation categories of modular Hopf algebras are known [94] to be modular, i.e. to satisfy (4.4.49) and (4.4.50), where the phases in T are given by the values of the ribbon element X in the irreducible representations.

On the other hand this structure has been shown [91] to arise from the superselection structure of *every* rational quantum field theory in $1+1$ dimensions. In this framework the phases in T are given by the phases of the statistics parameters (4.4.42), whereas the matrix S arises from the statistics characters

$$T = \left(\frac{\sigma}{|\sigma|}\right)^{1/3}\,\mathrm{Diag}(\omega_i),\quad S = |\sigma|^{-1}\,Y. \tag{4.4.51}$$

For nondegenerate theories the number $\sigma = \sum_i \omega_i^{-1} d_i^2$ satisfies $|\sigma|^2 = \sum_i d_i^2$. Using the result [2] $\sigma = |G|$ this condition is seen to be fulfilled, for the semisimplicity of $D(G)$ gives $\sum_i d_i^2 = \dim(D(G)) = |G|^2$.

We thus observe, for the orbifold theories under study, a perfect parallelism between the general superselection theory [91] for quantum field theories in low dimensions and the representation theory of the quantum double [34]. This parallelism extends beyond the Verlinde structure. One observes, e.g., that the equations (2.4.2) of [34] and (2.30) in [91], both stating that the monodromy operator is diagonalized by certain intertwining operators, are identical although derived in apparently unrelated frameworks.

4.4.6 Complements on Fixpoint Nets and Solitons

In this subsection we apply the considerations of Section 3.7, where the extension of DHR representations of wedge-dual nets to the dual net were discussed in an abstract way, to the fixpoint net $\mathcal{A} = \mathcal{F}^G$. For simplicity we assume the net $\mathcal{F}$ to be bosonic, i.e. local. Considering $\mathfrak{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}) \restriction \mathcal{H}_0$ as the observables, we have seen that $\mathfrak{A}$ satisfies only wedge duality. Nevertheless, the result of [35] that the restrictions of $\mathcal{A}$ to the charged sectors, interpreted as representations of the abstract C^*-algebra $\mathcal{A}$ satisfy the DHR criterion and are connected to the vacuum by charged fields, remains true. Furthermore, we know that the dual net in the vacuum sector is given by

$$\mathfrak{A}^d(\mathcal{O}) = \hat{\mathcal{A}}_L(\mathcal{O}) \restriction \mathcal{H}_0 = \hat{\mathcal{A}}_R(\mathcal{O}) \restriction \mathcal{H}_0, \tag{4.4.52}$$

where

$$\hat{\mathcal{A}}_{L/R}(\mathcal{O}) = \hat{\mathcal{F}}_{L/R}(\mathcal{O})^G = \hat{\mathcal{F}}_{L/R}(\mathcal{O}) \cap U(G)', \tag{4.4.53}$$

the nonlocal nets $\hat{\mathcal{F}}_{L/R}(\mathcal{O})$ being obtained by adjoining to $\mathcal{F}(\mathcal{O})$ the disorder operators $U_L^{\mathcal{O}}(G)$ or $U_R^{\mathcal{O}}(G)$, respectively.

We will first discuss the case of abelian groups G. The disorder operators commuting with G, $\hat{\mathcal{A}}_{L/R}(\mathcal{O})$ is simply $\mathcal{A}(\mathcal{O}) \vee U_{L/R}^{\mathcal{O}}(G)''$. On the C^*-algebra $\hat{\mathcal{A}}_{L/R}$ there is an action of the dual group $\hat{G}$ which acts trivially on $\mathcal{A}$ and via

$$\hat{\alpha}_\chi(U_{L/R}^{\mathcal{O}}(g)) = \chi(g)\, U_{L/R}^{\mathcal{O}}(g) \quad \forall \mathcal{O} \in \mathcal{K} \tag{4.4.54}$$

on the disorder operators, cf. Section 4.4.1. Since this action commutes with the Poincaré group and since it is spontaneously broken ($\omega_0 \circ \hat{\alpha}_\chi \neq \omega_0 \ \forall \chi \neq e_{\hat{G}}$) it gives rise to inequivalent vacuum states on $\mathcal{A}$ via

$$\omega_\chi = \omega_0 \circ \hat{\alpha}_\chi. \tag{4.4.55}$$

Now, the sectors in $\mathcal{H}$ are labeled by the characters $\chi \in \hat{G}$ and the representation of $\mathcal{A}$ in $\mathcal{H}_\chi$ is of the form

$$\pi_\chi(A) = A \restriction \mathcal{H}_\chi \cong \pi_\chi^{\mathcal{O}}(A) = \psi\, A\, \psi^* \restriction \mathcal{H}_0, \tag{4.4.56}$$

where $\psi \in \mathcal{F}(\mathcal{O})$ and $\alpha_g(\psi) = \chi(g)\psi$. We can now consider the extensions $\hat{\pi}_{\chi,L}, \hat{\pi}_{\chi,R}$ of π_χ to the dual net $\mathcal{A}^d$. As is obvious from (4.4.56) and the commutation relation

(4.2.2) between fields and disorder operators, the extension $\hat{\pi}_{\chi,L}$ ($\hat{\pi}_{\chi,R}$) is nothing but a soliton sector interpolating between the vacua ω_0 and $\omega_{\chi^{-1}}$ (ω_χ and ω_0). The moral is that the net $\mathcal{A}^d$, while not having nontrivial localized representations by Theorem 3.4.1, admits soliton representations. Furthermore, with respect to $\mathcal{A}^d$, the charged fields ψ_χ are creation operators for solitons since they intertwine the representations of $\mathcal{A}^d$ on $\mathcal{H}_0$ and $\mathcal{H}_\chi$.

Due to $U_L^{\mathcal{O}}(g)\, U_R^{\mathcal{O}}(g) = U(g)$ and $U(g) \restriction \mathcal{H}_\chi = \chi(g)\mathbf{1}$ we have

$$U_L^{\mathcal{O}}(g) \restriction \mathcal{H}_\chi = \chi(g)\, U_R^{\mathcal{O}}(g^{-1}) \restriction \mathcal{H}_\chi, \tag{4.4.57}$$

so that the algebras $\hat{\mathcal{A}}_{L/R}(\mathcal{O}) \restriction \mathcal{H}_\chi$ are independent of whether we use the left or right localized disorder operators. In particular, in the vacuum sector $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g^{-1})$ coincide, but due to the different localization properties it is relevant whether $U_L^{\mathcal{O}}(g)$, considered as an element of $\mathcal{A}^d$, is represented on $\mathcal{H}_\chi$ by $U_L^{\mathcal{O}}(g)$ or by $\chi(g)\, U_R^{\mathcal{O}}(g^{-1})$. This reasoning shows that the two possibilities for extending a localized representation of a general non-dual net to a representation of the dual net correspond in the fixpoint situation at hand to the choice between the nets $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$ arising from the field extensions $\hat{\mathcal{F}}_L$ and $\hat{\mathcal{F}}_R$.

We now turn to non-abelian (finite) groups G where the outcome is less obvious a priori. Let $\hat{A} = \sum_{g\in G} F_g U_L^{\tilde{\mathcal{O}}}(g) \in \hat{\mathcal{A}}_L(\tilde{\mathcal{O}})$ (F_g must satisfy the conditions given in Theorem 4.3.16 and let $\psi_i \in \mathcal{F}(\mathcal{O})$, where $\mathcal{O} < \tilde{\mathcal{O}}$ be a multiplet of field operators transforming according to a finite dimensional representation of G. Then

$$\sum_i \psi_i\ (\sum_{g\in G} F_g U_L^{\tilde{\mathcal{O}}}(g))\ \psi_i^* = \sum_{g\in G}(\sum_i \psi_i \alpha_g(\psi_i^*))\, F_g U_L^{\tilde{\mathcal{O}}}(g). \tag{4.4.58}$$

In contrast to the abelian case where $\psi\alpha_g(\psi^*)$ is just a phase, $O_g \equiv \sum_i \psi_i \alpha_g(\psi_i^*)$ is a nontrivial unitary operator

$$O_g^{-1} = O_g^* = \sum_i \alpha_g(\psi_i)\psi_i^* \tag{4.4.59}$$

satisfying

$$\alpha_k(O_g) = O_{kgk^{-1}}. \tag{4.4.60}$$

In particular (4.4.58) is not contained in $\mathcal{A}^d(\tilde{\mathcal{O}})$ which implies that the map $\hat{A} \mapsto \sum \psi_i\, \hat{A}\, \psi_i^*$ does not reduce to a local symmetry on $\hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}})$. Rather, we obtain a monomorphism into $\hat{\mathcal{A}}_L(W_R^{\mathcal{O}})$. Defining $\hat{\mathcal{O}}$ and $\mathcal{O}_0$ as in Section 3.7 of Chapter 3 we clearly see that (4.4.58) is contained in $\mathcal{A}^d(\hat{\mathcal{O}})$. Furthermore, due to the relative locality of the net $\mathcal{A}$ with respect to $\mathcal{A}^d$ and $\mathcal{F}$, (4.4.58) commutes with $\mathcal{A}(\mathcal{O}_0)$. Thus we obtain precisely the localization properties which were predicted by our general analysis in Section 3.7.

4.5 Further Directions

4.5.1 Generalization to Continuous Groups (partial)

In this subsection we will generalize our considerations on quantum double actions to arbitrary locally compact groups (the quantum field theoretic framework gives rise only to compact groups.) In Section 4 we identified von Neumann algebras acted upon by the double $D(G)$ of a finite group with von Neumann algebras which are simultaneously graded by the group and automorphically acted upon by the latter, satisfying in addition the relation (4.4.8). The concept of group grading, however, loses its meaning for continuous groups. This problem is solved by appealing to the well known fact (see e.g. the introduction to [77]) that an algebra A (von Neumann or unital C^*) graded by a finite group G is the same as an algebra with a coaction of the group. A coaction is a homomorphism δ from A into $A \otimes \mathbb{C}G$ satisfying

$$(\delta \otimes id) \circ \delta = (id \otimes \delta_G) \circ \delta, \tag{4.5.1}$$

where $\delta_G : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$ is the coproduct given by $g \mapsto g \otimes g$. The correspondence between these notions is as follows. Given a G-graded algebra $A = \oplus_g A_g,\ A_g A_h \subset A_{gh}$ and defining $\delta(x) = x \otimes g$ for $x \in A_g$ one obtains a coaction. The converse is also true. The relation $\alpha_g(A_h) = A_{ghg^{-1}}$ between the group action and the grading obviously translates to

$$\delta \circ \alpha_g = (\alpha_g \otimes Ad\, g) \circ \delta. \tag{4.5.2}$$

The concept of coaction extends to continuous groups, where the group algebra $\mathbb{C}G$ is replaced by the von Neumann algebra $\mathcal{L}(G)$ (here we will treat only quantum double actions on von Neumann algebras) of the left regular representation which is generated by the operators $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$ on the Hilbert space $L^2(G)$.

In the next step we give a precise definition of the double of continuous group. To this purpose we have to put a topology on the crossed product of some algebra of functions on the group by the adjoint action of the latter. There are many ways of doing this, as is generally the case with infinite dimensional vector spaces. For compact *Lie* groups two different constructions, one of which appears to generalize to arbitrary compact groups, have been given in in [14]. The most important virtue of this work is that the topological Hopf algebras obtained there are reflexive as topological vector spaces, making the duality between $D(G)$ and $D(G)^*$ very explicit. From the technical point of view, however, the Fréchet topologies on which this approach relies are not very convenient. Yet another interesting approach can be found in [76] where also the representation theory of the quantum double in the (locally) compact case was studied. An application of the results expounded there in analogy to Section 4 should be possible but is deferred for reasons of space.

In the following we will define the quantum double in the framework of Kac algebras [52, 53]. The latter has been invented as a generalization of locally compact groups which is closed under duality. As the C^* and von Neumann versions of Kac algebras have been proved [53] equivalent (generalizing the equivalence between locally compact groups and measurable groups) it is just a matter of convenience which formulation we use. We therefore consider first the von Neumann version which is technically easier.

We start with the von Neumann algebra $M = L^\infty(G)$ of essentially bounded measurable functions acting on the Hilbert space $H = L^2(G)$ by pointwise multiplication. With the coproduct $\Gamma(f)(g, h) = f(gh)$ and the involution $\kappa(f)(g) = f(g^{-1})$ it is a coinvolutive Hopf von Neumann algebra. This means Γ is a coassociative isomorphism of M into $M \otimes M$, κ is an anti-automorphism (complex linear, antimultiplicative and $\kappa(x^*) = \kappa(x)^*$)

and $\Gamma \circ \kappa = \sigma \circ (\kappa \otimes \kappa) \circ \Gamma$ holds where σ is the flip. The weight φ, defined on M_+ by $\varphi(f) = \int_G dg\, f(g)$, is normal, faithful, semifinite (n.f.s.) and fulfills

1. For all $x \in M_+$ one has $(\imath \otimes \varphi)\Gamma(x) = \varphi(x)\mathbf{1}$.
2. For all $x, y \in \mathfrak{n}_\varphi$ one has $(\imath \otimes \varphi)((\mathbf{1} \otimes y^*)\Gamma(x)) = \kappa((\imath \otimes \varphi)(\Gamma(y^*)(\mathbf{1} \otimes x)))$.
3. $\kappa \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ \kappa\ \forall t \in \mathbb{R}$.

This makes $(M, \Gamma, \kappa, \varphi)$ a Kac algebra in the sense of [52], well known as $KA(G)$. The dual Kac algebra [52] of $KA(G)$ is $KS(G) = (\mathcal{L}(G), \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$, the von Neumann algebra of the left regular representation equipped with the coproduct $\hat{\Gamma}(\lambda(g)) = \lambda(g) \otimes \lambda(g)$, the coinvolution $\hat{\kappa}(\lambda(g)) = \lambda(g^{-1})$ and the weight $\hat{\varphi}$ which we do not bother to state (see e.g. [72]).

Defining now an action of G on M by the automorphisms $\alpha_g(f)(h) = f(g^{-1}hg)$ it is trivial to check weak continuity with respect to g. Furthermore, α_g is unitarily implemented by $u_g = \lambda(g)\rho(g)$, where $(\rho(g)\xi)(h) = \Delta(g)^{1/2}\xi(hg)$ is the right regular representation. We can thus consider the crossed product (in the usual von Neumann algebraic sense [111]) $\tilde{M} = M \rtimes_\alpha G$ on $H \otimes L^2(G)$ $(= L^2(G) \otimes L^2(G))$, generated by $\pi(M)$ and $\lambda_1(g) = \mathbf{1}_M \otimes \lambda(g),\ g \in G$.

Proposition 4.5.1 *There are mappings $\tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi}$ on $\tilde{M}$ such that the quadruple $(\tilde{M}, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$ is a Kac algebra, which we call the quantum double $\mathcal{D}(G)$. On the subalgebras $\pi(M)$ and $\lambda_1(G)'' = \mathbf{1}_M \otimes \mathcal{L}(G)$ the coproduct and the coinvolution act according to*

$$\tilde{\Gamma}(\pi(x)) = (\pi \otimes \pi)(\Gamma(x)), \quad \tilde{\kappa}(\pi(x)) = \pi(\kappa(x)), \quad x \in M, \tag{4.5.3}$$
$$\tilde{\Gamma}(\lambda_1(g)) = \lambda_1(g) \otimes \lambda_1(g), \quad \tilde{\kappa}(\lambda_1(g)) = \lambda_1(g^{-1}), \quad g \in G. \tag{4.5.4}$$

The Haar weight $\tilde{\varphi}$ is given by the dual weight [72]

$$\tilde{\varphi} = \varphi \circ \pi^{-1} \circ (\imath_{\tilde{M}} \otimes \hat{\varphi})(\tilde{\delta}(x)), \tag{4.5.5}$$

where $\tilde{\delta}$ is the dual coaction from $\tilde{M}$ to $\tilde{M} \otimes \mathcal{L}(G)$ which acts according to

$$\tilde{\delta}(\pi(x)) \;=\; \pi(x) \otimes \mathbf{1}_{\mathcal{L}(G)}, \quad x \in M \tag{4.5.6}$$
$$\tilde{\delta}(\lambda_1(g)) \;=\; \lambda_1(g) \otimes \lambda(g), \quad g \in G. \tag{4.5.7}$$

Proof. The automorphisms α_g of M are easily shown to satisfy $\Gamma \circ \alpha_g = (\alpha_g \otimes \alpha_g) \circ \Gamma$ and $\kappa \circ \alpha_g = \alpha_g \circ \kappa$. (The first identity is just $g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg)$, the second $(g^{-1}hg)^{-1} = g^{-1}h^{-1}g$.) Thus $\alpha : G \to Aut\, M$ constitutes an action of G on the Kac algebra $(M, \Gamma, \kappa, \varphi)$ in the sense of [31]. We can now apply [31, Théorème 1] to conclude that there exist a coproduct, a coinvolution and a Haar weight on $\tilde{M}$ such that the axioms of a Kac algebra are satisfied. The equations (4.5.3,4.5.4) are restatements of [31, Propositions 3.1, 3.3] whereas the Haar weight is as in [31, Définitions 1.9]. ■

Proposition 4.5.2 *The dual Kac algebra of the quantum double is*

$$\widehat{\mathcal{D}(G)} = (\mathcal{L}(G) \otimes L^\infty(G), \hat{\tilde{\Gamma}}, \hat{\tilde{\kappa}}, \hat{\varphi} \otimes \varphi). \tag{4.5.8}$$

The coproduct and the counit are

$$\begin{aligned}\hat{\hat{\Gamma}}(x) &= R\,(\mathbf{1}\otimes\sigma\otimes\mathbf{1})\,(\hat{\Gamma}\otimes\Gamma)(x)\,(\mathbf{1}\otimes\sigma\otimes\mathbf{1})\,R^*, && (4.5.9)\\ \hat{\hat{\kappa}}(x) &= V^*\,(\hat{\kappa}\otimes\kappa)(x)\,V, && (4.5.10)\end{aligned}$$

where R and V are given by

$$\begin{aligned}(R\xi)(g,h) &= (u_h\otimes\mathbf{1})\,\xi(g,h), && (4.5.11)\\ (V\xi)(g) &= u_g\,\xi(g). && (4.5.12)\end{aligned}$$

Proof. This is just the specialization of [31, Théorème 2] to the situation at hand. According to this theorem the von Neumann algebra underlying the dual of the crossed product Kac algebra $K \rtimes_\alpha G$ is $\hat{M}\otimes L^\infty(G)$ where $\hat{M}$ is the von Neumann algebra of $\hat{K}$. In our case $M = L^\infty(G)$ such that $\hat{M} = \mathcal{L}(G)$. The formulae for $\hat{\hat{\Gamma}}$ and $\hat{\hat{\kappa}}$ are stated in [31, Proposition 4.10]. ∎

Remark. If the group G is not finite the quantum double is neither compact nor discrete, for the weights $\tilde{\varphi}, \hat{\tilde{\varphi}} = \hat{\varphi}\otimes\varphi$ are both infinite.

We are now in a position to define a coaction of the dual double $\widehat{\mathcal{D}(G)}$ on an algebra A, provided A supports an action α and a coaction δ satisfying (4.5.2) (with g replaced by $\lambda(g)$). In order to remove the apparent asymmetry between $\alpha : A\times G \to A$ and $\delta : A \to A\otimes\mathcal{L}(G)$ we write the former as the homomorphism $\alpha : A \to A\otimes L^\infty(G)$ which maps $x\in A$ into $g\mapsto \alpha_g(x)\in L^\infty(G,A)$. We now show that the maps α and δ can be put together to yield a coaction.

Definition 4.5.3 *The map $\Delta : A \to A\otimes\mathcal{L}(G)\otimes L^\infty(G) = A\otimes\widehat{\mathcal{D}(G)}$ is defined by*

$$\Delta = (\imath_A\otimes\sigma)\circ(\alpha\otimes\imath_{\mathcal{L}(G)})\circ\delta, \tag{4.5.13}$$

where $\sigma : x\otimes y\mapsto y\otimes x$ is the flip map from $L^\infty(G)\otimes\mathcal{L}(G)$ to $\mathcal{L}(G)\otimes L^\infty(G)$.

Theorem 4.5.4 *The map Δ is a coaction of $\widehat{\mathcal{D}(G)}$ on A, i.e. it satisfies*

$$(\Delta\otimes\imath_{\hat{\mathcal{D}}})\circ\Delta = (\imath_A\otimes\hat{\hat{\Gamma}})\circ\Delta. \tag{4.5.14}$$

Proof. Appealing to the isomorphism $A\otimes L^\infty(G)\cong L^\infty(G,A)$ we identify $A\otimes\mathcal{L}(G)\otimes L^\infty(G)\otimes\mathcal{L}(G)\otimes L^\infty(G)$ with $L^\infty(G\times G, A\otimes\mathcal{L}(G)\otimes\mathcal{L}(G))$. We compute $(\Delta\otimes\imath)\circ\Delta(x)$ as follows (abbreviating $\imath_{\mathcal{L}(G)}$ by $\imath_{\mathcal{L}}$)

$$\begin{aligned}((\Delta\otimes\imath)\circ\Delta(x))(g,h) &= (\alpha_g\otimes\imath_{\mathcal{L}}\otimes\imath_{\mathcal{L}})\circ(\delta\otimes\imath_{\mathcal{L}})\circ(\alpha_h\otimes\imath_{\mathcal{L}})\circ\delta(x)\\ &= (\alpha_g\otimes\imath_{\mathcal{L}}\otimes\imath_{\mathcal{L}})\circ(\alpha_h\otimes Ad\,\lambda_h\otimes\imath_{\mathcal{L}})\circ(\delta\otimes\imath_{\mathcal{L}})\circ\delta(x) \qquad (4.5.15)\\ &= (\imath_A\otimes Ad\,\lambda_h\otimes\imath_{\mathcal{L}})\circ(\alpha_{gh}\otimes\hat{\Gamma})\circ\delta(x).\end{aligned}$$

The second equality follows from the connection (4.5.2) between the action α and the coaction δ whereas the third derives from the defining property (4.5.1) of the coaction $\hat{\Gamma}$. Now $(\alpha_{gh}\otimes\hat{\Gamma})\circ\delta(x)$ is seen to be nothing but $[(\mathbf{1}\otimes\sigma\otimes\mathbf{1})\,(\hat{\Gamma}\otimes\Gamma)(x)\,(\mathbf{1}\otimes\sigma\otimes\mathbf{1})](g,h)$, and the adjoint action of R in (4.5.9) is seen to have the same effect as $Ad\,(\imath_A\otimes Ad\,\lambda_h\otimes\imath_{\mathcal{L}})$ due to $\rho(g)\in\mathcal{L}(G)'$. ∎

Proposition 4.5.5 *The fixpoint algebra under the coaction Δ, defined as $A^\Delta = \{x \in A \mid \Delta(x) = x \otimes \mathbf{1}_{\hat{\mathcal{D}}}\}$, is given by*

$$A^\Delta = A^\alpha \cap A^\delta, \tag{4.5.16}$$

where A^α, A^δ are defined analogously.

Proof. Obvious consequence of Definition 4.5.3. ■

The coaction of the dual double $\widehat{\mathcal{D}(G)}$ on A constructed above is exactly the kind of output the theory of depth-2 inclusions [82, 54] would give when applied to the inclusion $A^{\mathcal{D}(G)} \subset A$, which in the quantum field theoretical application corresponds to $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$. Nevertheless it is perhaps not exactly what one might have desired from a generalization of the results of Section 4 to compact groups. At least to a physicist, some kind of bilinear map $\gamma : A \times D(G) \to A$, as it was defined above for finite G, would seem more intuitive. This map should be well defined on the whole algebra A. Such a map can be constructed, provided the von Neumann double $\mathcal{D}(G)$ is replaced by its C^*-variant, which is uniquely defined by the above mentioned results [7, 53]. The details will be given in a subsequent publication.

4.5.2 Bosonization

In this section we will show how the methods expounded in the preceding sections can be used to obtain an understanding of the Bose/Fermi correspondence in 1+1 dimensions in the framework of local quantum theory. This is so say, we will show how one can pass from a fermionic net of algebras with twisted duality to a bosonic net satisfying Haag duality *on the same Hilbert space*, and vice versa. Our method amounts to a continuum version of the Jordan-Wigner transformation and is reminiscent of Araki's approach to the XY-model [6].

Our starting point is as defined in Chapter 1, i.e. a net of field algebras with fermionic commutation relations (1.3.2) and twisted duality (1.3.6) augmented by the split property for wedge regions introduced in Section 2. As before there exists a selfadjoint unitary operator V distinguishing between even and odd operators. For the present investigations, however, the existence of further inner symmetries is ignored as they are irrelevant for the spacelike commutation relations. Therefore we now repeat the field extension of Section 3 replacing the group G by the subgroup $\mathbb{Z}_2 = \{e, k\}$. This amounts to simply extending the local algebras by the disorder operator associated with the only nontrivial group element k

$$\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee \{V^{\mathcal{O}}\}, \tag{4.5.17}$$

where $V^{\mathcal{O}} = U_L^{\mathcal{O}}(k)$. Again, the assignment $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ is isotonous, i.e. a net. This is of course the simplest instance of the situation discussed at the beginning of Section 4 where it was explained that there is an action of the dual group $\hat{G}$ on the extended net. We thus have an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the quasilocal algebra $\hat{\mathcal{F}}$ generated by $\alpha = Ad\, V$ and β

$$\begin{aligned} \alpha(F + GV^{\mathcal{O}}) &= F_+ - F_- + (G_+ - G_-)V^{\mathcal{O}}, & (4.5.18)\\ \beta(F + GV^{\mathcal{O}}) &= F - GV^{\mathcal{O}} & (4.5.19) \end{aligned}$$

where $F, G \in \mathcal{F}$. We now define $\tilde{\mathcal{F}}(\mathcal{O})$ as the fixpoint algebra under the *diagonal* action $\alpha \circ \beta = \beta \circ \alpha$:

$$\tilde{\mathcal{F}}(\mathcal{O}) = \{x \in \hat{\mathcal{F}}(\mathcal{O}) \mid x = \alpha \circ \beta(x)\}. \tag{4.5.20}$$

Obviously $\tilde{\mathcal{F}}(\mathcal{O})$ can be represented as the following sum:

$$\tilde{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_- V^{\mathcal{O}}. \tag{4.5.21}$$

It is instructive to compare $\tilde{\mathcal{F}}(\mathcal{O})$ with the twisted algebra

$$\mathcal{F}(\mathcal{O})^t = \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_- V, \tag{4.5.22}$$

the only difference being that in the former expression $V^{\mathcal{O}}$ appears instead of V. This reflects just the difference between Jordan-Wigner and Klein transformations. It is well known that the net $\mathcal{F}^t$ is local relative to $\mathcal{F}$. That the former cannot be local itself, however, follows clearly from the fact that it is unitarily equivalent to the latter by $\mathcal{F}(\mathcal{O})^t = Z\mathcal{F}(\mathcal{O})Z^*$.

Lemma 4.5.6 *Let W_L and W_R be left and right wedges, respectively. Then the wedge algebras of $\tilde{\mathcal{F}}$ are given by*

$$\tilde{\mathcal{F}}(W_L) = \mathcal{F}(W_L), \tag{4.5.23}$$
$$\tilde{\mathcal{F}}(W_R) = \mathcal{F}(W_R)^t. \tag{4.5.24}$$

Wedge duality holds for the net $\tilde{\mathcal{F}}$.

Proof. $V^{\mathcal{O}}$ is contained in $\mathcal{F}(W_L)_+$ for any $\mathcal{O} \subset W_L$. Thus, $\mathcal{F}(W_L)_- V^{\mathcal{O}} = \mathcal{F}(W_L)_-$, whence the first identity. Similarly we have $V_R^{\mathcal{O}} \in \mathcal{F}(W_R)_+$ for $\mathcal{O} \in W_R$, from which we obtain $\mathcal{F}(W_L)_- V^{\mathcal{O}} = \mathcal{F}(W_L)_- V$. Wedge duality for $\tilde{\mathcal{F}}$ now follows immediately from twisted duality for $\mathcal{F}$. ■

Proposition 4.5.7 *The net $\mathcal{O} \mapsto \tilde{\mathcal{F}}(\mathcal{O})$ is local.*

Proof. Let $\mathcal{O}_1, \mathcal{O}_2$ be mutually spacelike double cones. We may assume $\mathcal{O}_1 < \mathcal{O}_2$ such that $W_L^{\mathcal{O}_1}$ and $W_R^{\mathcal{O}_2}$ are mutually spacelike. The commutativity of $\tilde{\mathcal{F}}(\mathcal{O}_1)$ and $\tilde{\mathcal{F}}(\mathcal{O}_1)$ follows from the preceding lemma and twisted locality for $\mathcal{F}$ since $\mathcal{O}_1 \subset W_L^{\mathcal{O}_1}$ and $\mathcal{O}_2 \subset W_R^{\mathcal{O}_2}$. ■

Remark. A more intuitive proof goes as follows. Let $F_i \in \mathcal{F}(\mathcal{O}_i)_-, i = 1, 2$. Then commuting $F_1 V^{\mathcal{O}_1}$ through $F_2 V^{\mathcal{O}_2}$ gives exactly two factors of -1. The first arises from $F_1 F_2 = -F_2 F_1$ and the other from $V^{\mathcal{O}_2} F_1 = -F_1 V^{\mathcal{O}_2}$, whereas $V^{\mathcal{O}_1} F_2 = F_2 V^{\mathcal{O}_1}$.

Proposition 4.5.8 *The net $\tilde{\mathcal{F}}$ fulfills Haag duality for double cones.*

Proof. We have to prove $\tilde{\mathcal{F}}(\mathcal{O}) = \tilde{\mathcal{F}}(W_L^{\mathcal{O}}) \wedge \tilde{\mathcal{F}}(W_L^{\mathcal{O}})$. Using the lemma the right hand side is seen to equal $\mathcal{F}(W_L^{\mathcal{O}}) \wedge \mathcal{F}(W_R^{\mathcal{O}})^t$ which by (4.2.17) is unitarily equivalent to $\mathcal{F}(W_R^{\mathcal{O}})^t \otimes \mathcal{F}(W_L^{\mathcal{O}})$. On the other hand (4.2.19) leads to

$$\begin{aligned}\tilde{\mathcal{F}}(\mathcal{O}) &= \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_- V^{\mathcal{O}} \\ &\cong \mathcal{F}(W_R^{\mathcal{O}})_+ \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ + \mathcal{F}(W_R^{\mathcal{O}})_- V \otimes \mathcal{F}(W_L^{\mathcal{O}})_- \\ &\quad + [\mathcal{F}(W_R^{\mathcal{O}})_- \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ + \mathcal{F}(W_R^{\mathcal{O}})_+ V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-]\, V \otimes \mathbf{1} \\ &= \mathcal{F}(W_R^{\mathcal{O}})^t \otimes \mathcal{F}(W_L^{\mathcal{O}})\end{aligned} \tag{4.5.25}$$

which completes the proof. $\blacksquare$

It is obvious that the net $\tilde{\mathcal{F}}$ is Poincaré covariant with respect to the original representation of $\mathcal{P}$. Finally, the group G acts on $\tilde{\mathcal{F}}$ via the adjoint representation $g \mapsto Ad\, U(g)$. In particular $Ad\, U(k) = Ad\, V$ acts trivially on the first summand of the decomposition (4.5.21) and by multiplication with -1 on the second, i.e. the bosonized theory carries an action of $\mathbb{Z}_2$ in a natural way.

It should be clear that the same construction can be used to obtain a twisted dual fermionic net from a Haag dual bosonic net with a $\mathbb{Z}_2$ symmetry. It is not entirely trivial that these operations performed twice lead back to the net one started with, as the operators $V^{\mathcal{O}}$ constructed with the original and the bosonized net might differ. That this is not the case, however, can be derived from Lemma 4.5.6, the easy argument is left to the reader.

4.5.3 Chiral Theories on the Circle

For the foregoing analysis in this Chapter the split property for wedges was absolutely crucial. While this property has been proved only for free massive fields it is expected to be true for all reasonable theories with a mass gap. For conformally invariant theories in $1+1$ dimensions, however, it has no chance to hold. This is a consequence of the fact that two wedges $W_1 \subset W_2$ 'touch at infinity'. More precisely, there is an element of the conformal group transforming W_1, W_2 into double cones having a corner in common. For such regions there can be no interpolating type I factor, see e.g. [17]. On the other hand, for chiral theories on a circle, into which a 1+1 dimensional conformal theory should factorize, an appropriate kind of split property makes sense. For a general review of the framework, including a proof of the split property from the finiteness of the trace of $e^{-\tau L_0}$, we refer to [64]. We restrict ourselves to a concise statement of the axioms.

For every interval I on the circle such that $\overline{I} \neq S^1$, there is a von Neumann algebra $\mathfrak{A}(I)$ on the common Hilbert space $\mathcal{H}$. The assignment $I \mapsto \mathfrak{A}(I)$ fulfills isotony and locality:

$$I_1 \subset I_2 \quad \Rightarrow \quad \mathfrak{A}(I_1) \subset \mathfrak{A}(I_2), \tag{4.5.26}$$

$$I_1 \cap I_2 = \emptyset \quad \Rightarrow \quad \mathfrak{A}(I_1) \subset \mathfrak{A}(I_2)'. \tag{4.5.27}$$

Furthermore, there is a strongly continuous unitary representation of the Möbius group $SU(1,1)$ such that $\alpha_g(\mathfrak{A}(I)) = Ad\, U(g)(\mathfrak{A}(I)) = \mathfrak{A}(gI)$. Finally, the generator L_0 of the rotations is supposed to be positive and the existence of a unique invariant vector Ω is assumed.

Starting from these assumptions one can prove, among other important results, that the local algebras $\mathfrak{A}(I)$ are factors of type III_1 for which the vacuum is cyclic and separating. Furthermore, Haag duality [64] is fulfilled automatically:

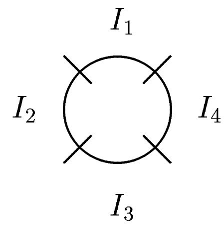
$$\mathfrak{A}(I)' = \mathfrak{A}(I'). \tag{4.5.28}$$

Given a chiral theory in its defining (vacuum) representation π_0 one may consider inequivalent representations. An important first result [22] states that all positive energy representations are locally equivalent to the vacuum representation, i.e. $\pi \restriction \mathfrak{A}(I) \cong \pi_0 \restriction \mathfrak{A}(I)\ \forall I$.

This implies that all superselection sectors are of the DHR type and can be analyzed accordingly [55, 57]. As a means of studying the superselection theory of a model it has been proposed [102] to examine the inclusion

$$\mathfrak{A}(I_1) \vee \mathfrak{A}(I_3) \subset (\mathfrak{A}(I_2) \vee \mathfrak{A}(I_4))' = \mathfrak{A}(I_{341}) \wedge \mathfrak{A}(I_{123}), \tag{4.5.29}$$

where $I_{1,\ldots,4}$ are quadrants of the circle and $I_{ijk} = \overline{I_i \cup I_j \cup I_k}$:



(4.5.30)

At least for strongly additive theories, where $\mathfrak{A}(I_1) \vee \mathfrak{A}(I_2) = \mathfrak{A}(I)$ if $\overline{I_1 \cup I_2} = I$, the inclusion (4.5.29) is easily seen to be irreducible. In the presence of nontrivial superselection sectors this inclusion is strict as the intertwiners between endomorphisms localized in I_1, I_3, respectively, are contained in the larger algebra of (4.5.29) by Haag duality but not in the smaller one. Furthermore, for rational theories the inclusion (4.5.29) is expected to have finite index.

While we have nothing to add in the way of model independent analysis the techniques developed in the preceding sections can be applied to a large class of interesting models. These are chiral nets obtained as fixpoints of a larger one under the action of a group. I. e. we start with a net $I \mapsto \mathcal{F}(I)$ on the Hilbert space $\mathcal{H}$ fulfilling isotony and locality, the latter possibly twisted. The Möbius group $SU(1,1)$ and the group G of inner symmetries are unitarily represented with common invariant vector Ω. Again, the net $\mathcal{F}$ is supposed to fulfill the split property (with the obvious modifications due to the different geometry). The net $I \mapsto \mathfrak{A}(I)$ is now defined by $\mathcal{A}(I) = \mathcal{F}(I) \wedge U(G)'$ and $\mathfrak{A}(I) = \mathcal{A}(I) \upharpoonright \mathcal{H}_0$ where $\mathcal{H}_0$ is the space of G-invariant vectors. The proof of Haag duality for chiral theories referred to above applies also to the net $\mathfrak{A}$, implying that there is no analogue of the violation of duality for the fixpoint net as occurs in 1+1 dimensions. This is easily understood as a consequence of the fact that the spacelike complement of an interval is again an interval, thus connected. However, our methods can be used to study the inclusion (4.5.29).

It is clear that due to the split property

$$\mathfrak{A}(I_1) \vee \mathfrak{A}(I_3) \cong \mathcal{F}(I_1) \otimes \mathcal{F}(I_3)^{G \times G} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0. \tag{4.5.31}$$

Our aim will now be to compute $(\mathfrak{A}(I_2) \vee \mathfrak{A}(I_4))'$. In analogy to the 1+1 dimensional case we use the split property to construct unitaries $Y_1, \ldots, Y_4 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing the following isomorphisms:

$$Y_i\, F_i F_{i+2}^t \, Y_i^* = F_i \otimes F_{i+2}^t \quad \forall F_i \in \mathcal{F}(I_i). \tag{4.5.32}$$

(One easily checks that $Y_{i+2} = T\, Y_i$ where $T\, x \otimes y = y \otimes x$.) These unitaries can in turn be used to define local implementers of the gauge transformations

$$U_i(g) = Y_i^* \,(U(g) \otimes \mathbf{1})\, Y_i \tag{4.5.33}$$

with the localization $U_i(g) \in \mathcal{F}(I_{i+2})'$. (The index arithmetic takes place modulo 4.) These operators satisfy

$$\begin{aligned} Ad\,U_i(g) \upharpoonright \mathcal{F}(I_i) &= \alpha_g, && (4.5.34)\\ [U_i(g), U_{i+2}(h)] &= 0, && (4.5.35)\\ U_i(g)\,U_{i+2}(g) &= U(g). && (4.5.36)\end{aligned}$$

By calculations similar to those in Section 4.3 one shows ($\mathcal{F}_i \equiv \mathcal{F}(I_i)$ etc.)

$$(\mathcal{A}_2 \vee \mathcal{A}_4)' = (\mathcal{F}_2 \vee \mathcal{F}_4)' \vee U_2(G)'' \vee U_4(G)''. \tag{4.5.37}$$

At this point we strengthen the property of Haag duality for the net $\mathcal{F}$ by requiring

$$(\mathcal{F}_1 \vee \mathcal{F}_3)' = (\mathcal{F}_2 \vee \mathcal{F}_4)^t, \tag{4.5.38}$$

which by the above considerations amounts to $\mathcal{F}$ having no nontrivial superselection sectors. This condition is fulfilled, e.g., by the CAR algebra on the circle which also possesses the split property. The chiral Ising model as discussed in [13] is covered by our general framework (with the group $\mathbb{Z}_2$).

While (4.5.38) is a strong restriction, it is the same as in [33] where the larger theory was supposed to be 'holomorphic', i.e. devoid of nontrivial representations. As we have seen, the assumptions (HD+SPW) made in the preceding sections of this chapter, where we studied massive theories, also imply the absence of charged sectors which, however, is much less obvious.

Making use of (4.5.38) we can now state quite explicitly how $(\mathfrak{A}_2 \vee \mathfrak{A}_4)'$ looks. In analogy to Theorem 4.3.10 we obtain

$$(\mathfrak{A}_2 \vee \mathfrak{A}_4)' = m\,(\mathcal{F}_1 \vee \mathcal{F}_3 \vee U_2(G)'') \upharpoonright \mathcal{H}_0. \tag{4.5.39}$$

Again, using (4.5.38) one can check that $\alpha_2(g) = Ad\,U_2(g)$ restrict to automorphisms of $\mathcal{F}_1 \vee \mathcal{F}_3$ rendering the algebra $\mathcal{F}_1 \vee \mathcal{F}_3 \vee U_2(G)''$ a crossed product. Recalling

$$\mathfrak{A}_1 \vee \mathfrak{A}_3 = m(\mathcal{F}_1) \vee m(\mathcal{F}_3) \upharpoonright \mathcal{H}_0 \tag{4.5.40}$$

we have the following natural sequence of inclusions:

$$\mathfrak{A}_1 \vee \mathfrak{A}_3 \subset m(\mathcal{F}_1 \vee \mathcal{F}_3) \upharpoonright \mathcal{H}_0 \subset (\mathfrak{A}_2 \vee \mathfrak{A}_4)' \tag{4.5.41}$$

both of which have index $|G|$. It is interesting to remark that the intermediate algebra $m(\mathcal{F}_1 \vee \mathcal{F}_3) \upharpoonright \mathcal{H}_0$ equals $(m(\mathcal{F}_2 \vee \mathcal{F}_4) \upharpoonright \mathcal{H}_0)'$. For general chiral theories the existence of such an intermediate subfactor between $\mathfrak{A}_1 \vee \mathfrak{A}_3$ and $(\mathfrak{A}_2 \vee \mathfrak{A}_4)'$ is not known. In the case of G being abelian where the $U_i(g)$ are invariant under global gauge transformations we obtain a square structure similar to the one encountered in Section 3.

$$\begin{array}{ccc} \mathfrak{A}_1 \vee \mathfrak{A}_3 \vee U_2(G)'' & \subset & (\mathfrak{A}_2 \vee \mathfrak{A}_4)' \\ \cup & & \cup \\ \mathfrak{A}_1 \vee \mathfrak{A}_3 & \subset & m(\mathcal{F}_1 \vee \mathcal{F}_3) \upharpoonright \mathcal{H}_0. \end{array} \tag{4.5.42}$$

It may be instructive to compare the above result with the situation prevailing in $2+1$ or more dimensions. There, as already mentioned in Chapter 1, the superselection theory for localized charges is isomorphic to the representation theory of a (unique) compact group. Furthermore, there is a net of field algebras acted upon by this group, such that the observables arise as the fixpoints. The analogue of the inclusion (4.5.29) then is

$$\mathfrak{A}(\mathcal{O}_1) \vee \mathfrak{A}(\mathcal{O}_2) \subset \mathfrak{A}(\mathcal{O}_1' \cap \mathcal{O}_2')', \tag{4.5.43}$$

where $\mathcal{O}_1$, $\mathcal{O}_2$ are spacelike separated double cones. Under natural assumptions it can be shown that the larger algebra equals $m(\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)) \upharpoonright \mathcal{H}_0$, implying that the inclusion (4.5.43) is of the type $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{G\times G} \subset (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mathrm{Diag}(G)}$ just as the first one in (4.5.41). That the index of the inclusion (4.5.29) is $|G|^2$ instead of $|G|$ as for (4.5.43) is a consequence of the low dimensional topology comparable to the phenomena occurring in $1+1$ dimensions.

Finally, we should mention that results similar to those of this subsection have been announced by Wassermann [114].

Chapter 5

Summary and Outlook

In this final chapter we summarize our results and indicate several directions for further investigations. We begin our discussion with Chapter 3 which was independent of aspects of group symmetry. There we have examined nets of observables satisfying the split property for wedges. Taking this property for granted for a moment, we have seen that in combination with Haag duality it has remarkable unifying power. Firstly, this combination of axioms implies several very desirable structural properties, to wit factoriality of the double cone algebras, n-regularity for all n, and the time-slice property. Even more remarkable are the implications for the superselection structure. As a consequence of the minimality of relative commutants of double cone algebras we obtain Haag duality in all irreducible, locally normal representations. The strongest result concerns the absence not only of locally generated superselection (DHR) sectors but also of charges localized in wedges. This in turn implies that soliton representations are characterized up to unitary equivalence by their asymptotic vacua.

The bulk of this dissertation was concerned with aspects of group symmetry in quantum field theories in $1+1$ dimensions. The starting point for the investigations of Chapter 4 was a field theory with an unbroken symmetry group, satisfying otherwise the same assumptions as the observables in Chapter 3 (Haag duality and the SPW). Under these conditions the fixpoint is easily proved not to satisfy Haag duality in the simple sectors. This argument does not apply to conformally covariant models since they cannot satisfy the SPW. In fact, in the conformal case duality does hold for fixpoint net [16]. Since in higher dimensions the fixpoint net always satisfies duality — irrespective of conformal covariance — one may wonder why things are different in two dimensions. The point is that in $\geq 2+1$ dimensions the spacelike complement of a double cone $\mathcal{O}$ is connected in Minkowski space as well as in the 'conformal spacetime' of [84], whereas in $1+1$ dimensions $\mathcal{O}'$ is connected only in the latter spacetime. Although these considerations already hint at qualitative differences between the superselection structures of massive and conformally covariant models in $1+1$ dimensions, it is only Theorem 3.4.1 which makes this precise. It shows that massive models can have non-trivial DHR sectors only if Haag duality is violated, whereas in the conformal case Haag duality is automatically satisfied in the vacuum sector and all positive energy representations are of the DHR type. (As always, massless models without conformal covariance, in particular gauge theories, are the most difficult, and we have nothing to say with respect to these.)

The basic tool for the investigations on massive fixpoint nets in Chapter 4 were disorder operators. They implement a global symmetry transformation on the fields localized in some wedge region and commute with the operators localized in the spacelike complement of a somewhat larger wedge. The canonical existence of such operators as well as their behavior under global gauge transformations follow from the SPW. Whereas disorder operators are only localized in wedge regions, they can in a natural way be associated to the bounded region where the interpolation between the global group action and the trivial action takes place. Extending the local algebras $\mathcal{F}(\mathcal{O})$ of the original theory by the disorder operators corresponding to the double cone $\mathcal{O}$ gives rise to a nonlocal net $\hat{\mathcal{F}}$. (This net is uniquely defined even if the SPW does not hold, provided disorder operators exist.) This nonlocal net is useful in two respects. Firstly, we have obtained an explicit characterization of the dual net $\mathcal{A}^d$, which is just the fixpoint net $\hat{\mathcal{A}} = \hat{\mathcal{F}}^G$ corresponding to the nonlocal extension $\hat{\mathcal{F}}$. Secondly, the net $\hat{\mathcal{F}}$ provides a very instructive example of a quantum field theory with quantum symmetry. It supports an action of the quantum double $D(G)$ which, however, is spontaneously broken in the sense that only the subalgebra $\mathbb{C}G$ is implemented by operators on the Hilbert space. Nevertheless, all other aspects of the quantum symmetry, like R-matrix commutation relations and the Verlinde algebra, show up and correspond nicely to the structures expected due to the general analysis [56, 91]. The spontaneous breakdown of the quantum symmetry is by no means a pathological phenomenon. On one hand, it allows for a deeper understanding of the violation of Haag duality for the fixpoint net $\mathcal{A}$ in analogy to Roberts' analysis in the group case. On the other hand it is in accord with the findings of [75] where it was argued (in the case of a cyclic group $Z(N)$) that the vacuum expectation values of order and disorder variables can vanish jointly, as they must in the case of unbroken quantum symmetry, only if there is no mass gap. This, however, is excluded by the SPW.

In view of the fact that in the situation studied in this paper 'one half' of the quantum double symmetry is spontaneously broken, it is natural to study also the dual situation, where one starts with a Haag dual net $\mathcal{B}(\mathcal{O})$ of C^*-algebras with abelian (for the moment) symmetry group K and a vacuum state ω_0 which is *not* gauge invariant. Let $\mathcal{H}_0$ be the associated GNS-Hilbert space and let us assume that the symmetry is completely broken, i.e. $\omega_k = \omega_0 \circ \alpha_k \neq \omega_0 \ \forall k \neq e$. (Examples for this situation are provided by the dual nets associated with the fixpoint nets under abelian groups G, cf. Section 4.4.1. I.e. $\mathcal{B}(\mathcal{O}) := \hat{\mathcal{A}}(\mathcal{O}) \upharpoonright \mathcal{H}_0$ and $K := \hat{G}$.) Assuming again the SPW, it is known [101] that there are soliton states interpolating any two of these inequivalent vacua. One may then consider the reducible representation $\pi = \oplus_{k\in K}\pi_k$ of $\mathcal{B}$ on the Hilbert space $\mathcal{H} = \oplus_{k\in K}\mathcal{H}_k$ where π_k is the GNS-representation corresponding to a soliton state which connects the vacua ω_0 and ω_k. (Of these representations only π_e is a vacuum representation.) Furthermore, in strong analogy to [36] one can define 'soliton creation operators' $\Psi_k,\ k \in K$ which intertwine π_e and π_k and which generate an irreducible 'field algebra' $\mathcal{D}$ on $\mathcal{H}$. Finally, there is an action β_l of $\hat{K}$ on $\mathcal{D}$ via

$$\beta_l(\Psi_k) = \langle l, k\rangle \, \Psi_k, \ \ k \in K, l \in \hat{K} \tag{5.1.1}$$

which commutes with the action of K. The operators Ψ_k are true soliton operators since their vacuum expectation values vanish – in contrast to the 'fields' ψ_l in $\mathcal{B}$ which transform nontrivially under K via

$$\alpha_k(\psi_l) = \langle l, k\rangle \, \psi_l, \;\; k \in K, l \in \hat{K}. \tag{5.1.2}$$

Finally, the operators $\psi_l, \Psi_k, \;\; k \in K, l \in \hat{K}$ obey order-disorder commutation relations. Thus, the resulting situation is in a sense the mirror image – or 'dual' – of that in Section 4.4.1, K and $\hat{K}$ playing the roles of $\hat{G}$ and G, respectively. In fact, we claim that in this situation the following is true. Forgetting about the net $\mathcal{F}$ on the large Hilbert space $\mathcal{H}$ and retaining only $\mathcal{B}(\mathcal{O}) := \hat{\mathcal{A}}(\mathcal{O}) \restriction \mathcal{H}_0$ and the action of $K := \hat{G}$, the construction sketched above just recovers the original net $\mathcal{F}$ and the action of G. Namely, $\hat{K} = \hat{\hat{G}} = G$ and $\mathcal{F}(\mathcal{O}) = \mathcal{D}(\mathcal{O})^K$. Using the methods of [35] the proof should be straightforward, but is deferred to a future publication. Instead, we refer to Section 4.4.6 where it was shown that the extension to the dual net $\mathcal{A}^d$ of the DHR representation $\pi_\chi, \; \chi \in \hat{G}$ of the fixpoint net $\mathcal{A}$ leads just to a soliton sector which interpolates ω_0 and $\omega_0 \circ \hat{\alpha}_\chi$. In this context we should mention that we did not yet study the residual symmetry of $\mathcal{A}^d = \hat{\mathcal{A}}$ in the case where G is non-abelian.

The duality between the scenario sketched above and the one in Section 4.4 can easily be understood in physical terms. It is well known that quantum field theories in $1+1$ dimensions, like the $\mathcal{P}(\phi)_2$ models, typically possess a unique symmetric vacuum for some range of the parameters whereas spontaneous symmetry breakdown and vacuum degeneracy occur for other choices. In both cases either the order or the disorder symmetry is spontaneously broken in accordance with the result of [75]. Yet, the construction sketched above shows that the algebraic structure of order-disorder duality is the same in both massive regimes.

From a mathematical point of view the construction of disorder fields from the order variables (Section 4.4) or conversely (above) may be considered as a local version of the construction of the quantum double. We recall that the quantum double was invented by Drinfel'd as a means to obtain quasitriangular Hopf algebras, and in [93] it was shown to be 'factorizable', see Appendix C. Factorizable Hopf algebras are self-dual structures [85] which generalize the obviously self-dual algebra $C(G \times \hat{G})$ where G is an abelian group. An unbroken implementation of the quantum double symmetry can only occur in massless representations. For the $\mathcal{P}(\phi)_2$ theory with interaction $\lambda\phi^4 - \delta\phi^2$ it is furthermore known that there is a critical point at the interface between the symmetric and broken phases. Unfortunately, little is rigorously known about the possible conformal invariance of the theory at this point.

Before we turn to the discussion of Chapter 2 an amusing remark seems appropriate. The investigations in Chapter 4 were to a good deal motivated by the works [33, 34] on conformal quantum field theories. In [33] conformal group fixpoint theories ('orbifold theories') were considered in the special case where the original theory is 'holomorphic', i.e. has no representations besides the vacuum. For technical reasons (existence and uniqueness up to local fields of disorder operators) we were led to require Haag duality and the split property for wedges, the latter being incompatible with conformal covariance. Yet, it turned out (Theorem 3.4.1) that these requirements are essentially equivalent to the holomorphicity assumption!

In view of the results of the Chapters 3 and 4, it seems quite urgent to verify the SPW for at least one nontrivial (i.e. not free) massive field theory lest these parts of present work be just a theory about free fields. For the free massive scalar field the modified nuclearity condition of Appendix B can be verified explicitly as will be shown in a forthcoming publication.

We now turn to Chapter 2, were we began with the proof of two intuitively reasonable properties of the Doplicher-Roberts construction in $\geq 2+1$ dimensions. The (essentially unique) complete field net which describes all DHR sectors has itself no localized sectors, and it can also be obtained from an intermediate, thus incomplete, field net by an application of the DR construction. Clearly, the complete (w.r.t. the DHR sectors) field net may still have nontrivial representations with the weaker Buchholz-Fredenhagen localization property.

Turning to $1+1$ dimensions, and *not* assuming the SPW such that there may be DHR sectors, the situation is in a sense quite similar. The degenerate sectors may be considered 'better localized' than generic DHR sectors insofar as they arise from *local* fields, in contrast to what is to be expected in the general case. Non-local charged fields played a role, e.g., in [87] where, however, the underlying quantum symmetry was spontaneously broken. As will be shown elsewhere, the symmetry breakdown encountered there is generic in massive models. As was mentioned above, the peculiar nature of the superselection structure of massive models manifests itself also in an analysis which starts from the observables [88]. For this reason, the considerations in Sections 3 and 4 were aimed primarily at conformally covariant theories in $1+1$ dimensions. This leads us to the following concluding remark.

It is well known [84] that conformal theories live on a suitably compactified Minkowski space. This compactification renders the spacetime non-simply connected, which in turn implies the existence of a center in the algebra of observables [57]. Triviality of the center was however an essential requirement for the Doplicher-Roberts analysis, in particular [42, 45]. In our first approach we circumvented this problem by working with the restriction of the net to Minkowski space. Since this 'removal of a point at infinity' may destroy Haag duality, an analysis on the compactified spacetime seems desirable. It should be obvious that in this case the DR construction may produce fields which live only on a covering space.

What remains to be done is, of course, to provide a solution of the 'quantum symmetry problem' which is as thorough as the DR reconstruction theory. By the above result one needs to consider only the non-degenerate case. Since every finite dimensional factorizable Hopf algebra can be obtained as a quotient of a quantum double by a two-sided ideal one may expect that quantum doubles will play an important role in an extension of the constructions in [47] to low dimensional theories. In the literature there exist several Tannaka-Krein-type reconstruction theorems for braided monoidal categories. As in the symmetric case they are, however, not sufficient for the problem at hand since in the quantum field theoretic application one does not have a representation functor, i.e. the finite dimensional representation spaces are not explicitly given. Even though relaxing this assumption does not lead to a more general structure than compact groups in the symmetric case, it is by no means clear that compact quantum groups, say, are already the end of the story.

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Appendix A

Local Symmetries and the Translation Group

In this appendix we expose some consequences of the fact that in $1+1$ dimensions the Poincaré group has finite dimensional unitary representations contrary to the situation in higher dimensions. In [41, Theorem 10.4] it has been shown that the group

$$G_{\max} = \{U \in \mathcal{U}(\mathcal{H}) \mid U\Omega = \Omega,\, U\mathcal{F}(\mathcal{O})U^* = \mathcal{F}(\mathcal{O})\ \forall \mathcal{O}\} \tag{A.1.1}$$

of all local symmetries of a quantum field theory with the distal split property commutes elementwise with the Poincaré transformations $U(\tilde{\mathcal{P}})$, as a consequence of which [16, Theorem 3.1] there can be only one representation of the Poincaré group compatible with the local structure. These results hinge upon the fact that there are no finite dimensional unitary representations of $\tilde{\mathcal{P}}$ which, however, is not true in $1+1$ dimensions. Instead we have

Lemma A.1.9 *Every finite dimensional unitary representation of the Poincaré group $\mathcal{P}$ in 1+1 dimensions factors through the homomorphism $\mathcal{P} \to \mathcal{L}:\ (a,\Lambda) \mapsto \Lambda$, i.e. is trivial on the normal subgroup of translations. It is thus unitarily equivalent to a direct sum of one dimensional representations $(a,\Lambda) \mapsto e^{is\Lambda}$ which are parameterized by $s \in \mathbb{R}$.*

Proof. Let P^0, P^1, K be the hermitian generators of the translations and the boosts. As P^0 and P^1 commute they can be jointly diagonalized by a unitary transformation. The defining relations $[P^0, K] = P^1,\ [P^1, K] = P^0$ now take the form $P_i^0\, K_{ij} - K_{ij}\, P_j^0 = \delta_{ij}\, P_i^1$ which for $i = j$ reduces to $P_i^1 = 0$ and similarly for P^0. The proof is finished by observing that the Lorentz group in $1+1$ dimensions is isomorphic to $\mathbb{R}$. ■

We can now state the 1+1 dimensional version of [41, Theorem 10.4].

Proposition A.1.10 *Let a quantum field theory in 1+1 dimensions be given where the split property holds for a pair of regions $\mathcal{O}_1 \subset \mathcal{O}_2$. Then the gauge group G_{max} commutes elementwise with the translations $U(\mathbb{R}^2)$.*

Proof. The proof is identical with that of [41, Theorem 10.4] with the exception of the last step. Instead of finite dimensional representations of $\mathcal{P}_+^\uparrow$ being absent as in higher dimensions all such representations are trivial on the translations according to the lemma. ■

In [39] it was shown that the representation of the Poincaré group is unique in charged DHR sectors with finite statistics. In $1+1$ dimensions we have only

Lemma A.1.11 *Let ρ be an irreducible sector and let $U_1^\rho(\Lambda,a), U_2^\rho(\Lambda,a)$ be ρ-covariant representations of the Poincaré group, i.e. $Ad\,U_i^\rho(\Lambda,a)\circ\rho = \rho\circ\alpha_{\Lambda,a}$. Then there exists $s\in\mathbb{R}$ such that*

$$U_1^\rho(\Lambda,a) = U_2^\rho(\Lambda,a)\, e^{is\Lambda}. \tag{A.1.2}$$

Proof. We have $X(\Lambda,a) = U_1^\rho(\Lambda,a)^* U_2^\rho(\Lambda,a) \in \rho(\mathcal{A})' = \mathbb{C}\mathbf{1}$. Thus, $X(\Lambda,a)$ is a one-dimensional representation of the Poincaré group and, by Lemma A.1.9, is of the form $e^{is\Lambda}$. ■

Appendix B

The Split Property for Wedges and Nuclearity

The nuclearity criterion [12] formalizes the idea that in order to be physically realistic a quantum field theory should satisfy some restrictions on the number of its local degrees of freedom in the sense that there should be only a finite number of quantum states occupying a finite volume of phase space. As phase space is an elusive concept in quantum field theory the formalization of this condition is not trivial. The formulation we give here is the one which has proved most suitable [25]. For any bounded region $\mathcal{O}$ define the linear map $\Theta_{\mathcal{O},\beta} : \mathcal{R}(\mathcal{O}) \to \mathcal{H}$ as

$$\Theta_{\mathcal{O},\beta}(A) = e^{-\beta H} A\Omega,$$

where Ω is a vacuum vector and H the Hamiltonian (assumed to be positive). This map is required to be compact and of arbitrarily small order if $\beta > 0$ is sufficiently large. The order q of a continuous linear map Θ of a Banach space $\mathcal{E}$ into another Banach space $\mathcal{F}$ is the non-negative number (if it exists)

$$q = \limsup_{\varepsilon \searrow 0} \frac{\ln \ln N(\varepsilon)}{\ln 1/\varepsilon},$$

where $N(\varepsilon)$, the ε-content of Θ, is the maximum number of elements E_i in the unit ball of $\mathcal{E}$ such that $\|\Theta(E_i - E_k)\| > \varepsilon$ if $i \neq k$. Criteria of this type have been proved [19, 25, 26] to imply the split property whereas it is not known to which extent the converse holds (at least one has: split $\Rightarrow$ compactness of the map Θ.)

As introduced in Chapter 1, the usual split property states that the von Neumann algebras $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2')''$ and $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2')''$ are isomorphic whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$. In this formulation, one of the regions ($\mathcal{O}_2'$) is infinite. In this appendix we are concerned with the question whether both regions may be infinite, viz. wedges. From the physical point of view one would expect two subsystems separated from each other by a finite spacelike distance to decouple in the above-mentioned sense that their states can be prepared independently. In this respect it should not matter* whether the regions are of finite or infinite extension. On the other hand, as witnessed by the Chapters 3-4 and the recent construction [101] of soliton states, the SPW is extremely useful for the model

*This heuristic idea has to be taken with caution, however, see below.

independent analysis of phenomena, well-known to exist in models, such that it should be understood from a more fundamental point of view.

B.1 Nuclearity Criterion for Lorentz Boosts

Before trying to prove the split property for wedge regions we should reflect for a moment whether it has a chance to hold at all. There is an easy argument by Araki (see [17]) to the effect that there can be no product state for the spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2$ if there exists a translation a such that $\mathcal{O}_1 + a \subseteq \mathcal{O}_1$ and $\mathcal{O}_2 + a \subseteq \mathcal{O}_2$. This in fact rules out the split property for wedges in more than 2 dimensions as any translation parallel to the edges of the wedges does the job. In 1+1 dimensions, however, this argument does not apply. As to the proof of the split property, one easily convinces oneself that the known proofs for finite regions would carry over immediately to the situation at hand should the nuclearity criterion stated above apply also to wedge shaped regions. This can however not be the case as will be shown elsewhere [90].

Motivated by the nuclearity criterion in terms of modular operators [24, 25] and by the geometric action of the modular operators associated with wedges proved by Borchers [15] we state our first result.

Proposition B.1.1 *Let Ω be the vacuum vector and K the generator (typically neither positive nor negative definite) of the group of Lorentz boosts. Suppose for a wedge $W \subset W_R$ that there exists a $\lambda > 0$ such that the vectors $A\Omega$ are in the domain of definition of the unbounded operator $e^{\lambda K}$ for every $A \in \mathcal{R}(W)$ and that the map $\Theta_{W,\lambda} : \mathcal{R}(W) \to \mathcal{H}$ defined by*

$$\Theta_{W,\lambda}(A) = e^{\lambda K} A\Omega \tag{B.1.1}$$

is of order $q_\lambda < 1/3$. Then the inclusion $\mathcal{R}(W) \subset \mathcal{R}(W_R)$ is split.

Proof. To simplify notation we denote $\mathcal{R}(W)$ by $\mathcal{A}$ and $\mathcal{R}(W_R)$ by $\mathcal{B}$. For fixed $A \in \mathcal{A}$ and $B' \in \mathcal{B}'$ we consider the function

$$F(z) = \langle \Omega, B' e^{-izK} A\Omega \rangle \tag{B.1.2}$$

which as a consequence of the assumptions is bounded and analytic in the interior of the strip $0 \leq \mathrm{Im} z \leq \lambda$ and continuous at the boundary. We observe that $\alpha_t(A) = e^{-itK} A e^{itK}$ will leave $\mathcal{A}$ for sufficiently large $|t|$ but stay in $\mathcal{B}$ implying $[e^{-itK} A e^{itK}, B'] = 0 \ \ \forall t \in \mathbb{R}$. Making use of the invariance of the vacuum we thereby have

$$F(z) = \langle e^{izK} A^* \Omega, B'^* \Omega \rangle \tag{B.1.3}$$

for real z. In this representation it is obvious that $F(z)$ continues analytically to the strip $-\lambda \leq \mathrm{Im} z \leq 0$, too, the domain of analyticity containing the real line by standard arguments. Applying the three line theorem we obtain the estimate

$$|F(0)| \leq |F(i\lambda)|^{1/2} \cdot |F(-i\lambda)|^{1/2} \tag{B.1.4}$$

and, hence,

$$|\langle \Omega, AB'\Omega \rangle| \leq \|B'\| \cdot \|e^{\lambda K} A\Omega\|^{1/2} \cdot \|e^{\lambda K} A^*\Omega\|^{1/2}. \tag{B.1.5}$$

With the definition of the map $\Xi_*(\cdot) : \mathcal{A} \to \mathcal{B}'_*$

$$\Xi_*(A)(\cdot) = \langle \Omega, A \cdot \Omega \rangle \tag{B.1.6}$$

we have as in [26]

$$\|\Xi_*(A \pm A^*)\| \leq \|e^{\lambda K}(A \pm A^*)\Omega\|, \tag{B.1.7}$$

from which it follows that the order q_* of the map $\Xi_*(\cdot) : \mathcal{A} \to \mathcal{B}'_*$ is at most equal to the order of the map $A \mapsto e^{\lambda K} A\Omega$ from $\mathcal{A}$ to $\mathcal{H}$. Applying the arguments in the proof of Proposition 4.1 in [25] the proof is complete. We remark that the argument is simplified considerably by the fact [48] that the local algebra of any wedge automatically is a factor which allows to use directly the results of [24] without having to deal with the possible occurrence of a center. In the case of bounded regions there is no general argument to this effect. ■

Remark. The expert reader will recognize the similarity of the above result to Proposition 2.2. of [26], the main difference being that in the case of energy nuclearity the time translations lead out of the larger region for large $|t|$ resulting in the appearance of two cuts along the real line.

B.2 Connections with Modular Nuclearity

The conventional nuclearity criteria as well as the one put forward in the preceding section depend crucially on the existence of a representation of the Poincaré group which is a global object. At least for some applications, for example quantum field theory on curved spacetimes, one would prefer a condition applying to some kind of local dynamics. A natural candidate in this respect are the automorphism groups associated (see e.g. [25]) by the Tomita-Takesaki theory to the local algebras and any vector being cyclic and separating for these. Having thereby for any wedge W (and the vacuum vector) a positive selfadjoint operator $\Delta_{W,\Omega}$ we define for any pair of wedges $W \subset \hat{W}$ and any $0 < \lambda < 1/2$ the map $\Xi_\lambda : \ \mathcal{R}(W) \to \mathcal{H}$ by

$$\Xi_\lambda(A) = \Delta^\lambda_{\hat{W},\Omega} A\Omega. \tag{B.2.1}$$

In terms of these maps we have at once the following nuclearity criterion.

Proposition B.2.1 *Suppose the order q_λ of the map Ξ_λ is smaller than $1/3$. Then the inclusion $\mathcal{R}(W) \subset \mathcal{R}(\hat{W})$ is split.*

Proof. Follows immediately from Lemma 3.1 in [25], remark 1 in [24] and the factoriality of the wedge algebras. ■

By the relation

$$\frac{q_{1/4}}{2} \leq q_* \leq q_\lambda \leq \max\left(\frac{1}{2\lambda}, \frac{1}{1-2\lambda}\right) \cdot \frac{q_{1/4}}{2}, \quad 0 < \lambda < 1/2 \tag{B.2.2}$$

between the orders of the maps (B.2.1) and (B.1.6) given in [25, Lemma 3.1] it is clear that the above local (modular) nuclearity is implied by some sufficiently strong (global) nuclearity for boosts.

For the direction from modular to Lorentz nuclearity one needs an additional piece of information on the action of the modular groups associated to wedge regions. To this purpose the authors of [25] had to assume a property [12] proved only for nets of local observables derived from a Wightman field theory to hold in general. The situation is more favorable in $1+1$ dimensions thanks to the important results of Borchers [15] of which we summarize those relevant to our investigation. Let $V(\tau) = e^{i\tau K}$ be the unitary representation of the Lorentz boosts. Then the unitary one parameter groups

$$R(\tau) := V(\tau) \cdot \Delta_{W_R}^{i\tau/2\pi}, \tag{B.2.3}$$

$$L(\tau) := V(\tau) \cdot \Delta_{W_L}^{-i\tau/2\pi} \tag{B.2.4}$$

implement automorphisms of all wedge algebras open to the right and left sides, respectively. In other words, the modular groups $\Delta_{W_R}^{i\tau/2\pi}$ and $\Delta_{W_L}^{-i\tau/2\pi}$ act geometrically on the right and left wedges, respectively, differing from the Lorentz boosts only by an internal symmetry transformation. In theories fulfilling wedge duality (i.e. $\mathcal{R}(W_L) = \mathcal{R}(W_R)'$) these groups coincide: $R(\tau) = L(\tau)$. Obviously the global and the modular nuclearity criteria coincide should the groups $R(\tau), L(\tau)$ happen to be trivial as we then have

$$e^{\tau K} = \Delta_{W_R}^{\tau/2\pi} = \Delta_{W_L}^{-\tau/2\pi}. \tag{B.2.5}$$

In general, however, this will not be the case. In order to conclude from modular nuclearity for the right wedge, say, that nuclearity for boosts holds, we need some knowledge about the spectral properties of U, the generator of the group $R(t) = e^{itU}$. Still quite easy is the case where U is bounded which is equivalent to the map $t \mapsto R(t)$ being normcontinuous. In this case $R(t)$ continues to an entire function taking values in $\mathcal{B}(\mathcal{H})$. As composition with bounded maps preserves nuclearity properties we see that also in this case modular nuclearity implies nuclearity for boosts. For unbounded U no such conclusion can be drawn, however.

Appendix C

Quantum Groups and Quantum Doubles

A Hopf algebra is an algebra H which at the same time is a coalgebra, i.e. there are homomorphisms $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{C}$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{C.1.1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}, \tag{C.1.2}$$

with the usual identification $H \otimes \mathbb{C} = \mathbb{C} \otimes H = H$. Furthermore, there is an antipode, i.e. an antihomomorphism $S : H \to H$ for which

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon(\cdot)\mathbf{1}, \tag{C.1.3}$$

where $m : H \otimes H \to H$ is the multiplication map of the algebra.
Remark. By (C.1.2) the counit, which is simply a one dimensional representation, is the 'neutral element' with respect to the comultiplication.

For the quantum double $D(G)$ defined in Definition 4.4.1 these maps are given by

$$\begin{aligned}
\varepsilon(V(g)U(h)) &= \delta_{g,e}, && \text{(C.1.4)}\\
\Delta(V(g)U(h)) &= \sum_k V(hk)U(h) \otimes V(k^{-1})U(h), && \text{(C.1.5)}\\
S(V(g)U(h)) &= V(h^{-1}g^{-1}h)U(h^{-1}) && \text{(C.1.6)}
\end{aligned}$$

on the basis $\{V(g)U(h)\, g, h \in G\}$ and extended to $D(G)$ by linearity.

A Hopf algebra H is *quasitriangular*, or simply a quantum group, if there is an element $R \in H \otimes H$ satisfying

$$\Delta'(\cdot) = R\,\Delta(\cdot)\,R^{-1}, \tag{C.1.7}$$

where $\Delta' = \sigma \circ \Delta$ with $\sigma(a \otimes b) = b \otimes a$ and

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(R) &= R_{13}\,R_{23}, && \text{(C.1.8)}\\
(\mathrm{id} \otimes \Delta)(R) &= R_{13}\,R_{12}. && \text{(C.1.9)}
\end{aligned}$$

Here $R_{12} = R \otimes \mathbf{1}, R_{23} = \mathbf{1} \otimes R$ and $R_{13} = (id \otimes \sigma)(R \otimes \mathbf{1})$. As a consequence, R satisfies the Yang-Baxter equation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}. \tag{C.1.10}$$

It is easy to verify that the R-matrix (4.4.34) satisfies these requirements.
Remark. As shown by Drinfel'd, for quantum groups the square of the antipode is inner, i.e. $S^2(a) = uau^{-1}$ where u is given by $u = m \circ (S \otimes \mathrm{id}) \circ \sigma(R)$. The operator u satisfies $\varepsilon(u) = 1,\ \Delta(u) = (\sigma(R)R)^{-1}\,(u \otimes u) = (u \otimes u)(\sigma(R)R)^{-1}$. For quantum doubles of finite groups the antipode is even involutive ($S^2 = \mathrm{id}$, equivalently u is central). This holds for all finite dimensional Hopf-*-algebras, whether quantum groups or not.

A quantum group is called *factorizable* [93] if the map $H^* \to H$ given by $H^* \ni x \mapsto \langle x \otimes id, I\rangle$ is nondegenerate, where I is as in (4.4.46). Quantum doubles are automatically factorizable.

A quasitriangular Hopf algebra possessing a (non unique) central element v satisfying the conditions

$$v^2 = u\, S(u),\ \varepsilon(v) = 1,\ S(v) = v, \tag{C.1.11}$$

$$\Delta(v) = (\sigma(R)R)^{-1}\,(v \otimes v), \tag{C.1.12}$$

where u is the operator defined in the above remark, is called a *ribbon Hopf algebra* [94].

Finally, *modular Hopf algebras* [94] are defined by some restrictions on their representation structure, the most important of which is the nondegeneracy of the matrix Y defined in (4.4.48). Obviously, the conditions of factorizability and modularity are strongly related.

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