On monotone complete C^*-algebras and a characterization of von Neumann algebras

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Abstract

1 Introduction

1.1 DEFINITION *A C^*-algebra A is a W^*-algebra if there is a Banach space V such that $V^* \cong A$ as Banach spaces.*

By [2, Theorem 4.2.9], every von Neumann algebra is a W^*-algebra. As I mentioned, the converse is also true:

1.2 THEOREM *A C^*-algebra is isomorphic to a von Neumann algebra if and only if it is a W^*-algebra.*

See [4, Theorem III.3.5] for a proof. (Takesaki defines W^*-algebras as C^*-algebras A having a faithful representation (H, π) such that $\pi(A) \subseteq B(H)$ is a von Neumann algebra. Thus he must state the theorem slightly differently. Of course both definitions are equivalent as consequence of the theorem.)

This is nice since it gives a representation-free characterization of von Neumann algebras. But in practice another characterization of von Neumann algebras often is more useful, and we will sketch it in the next section.

Every commutative von Neumann algebra A is a commutative unital C^*-algebra, thus there is a compact Hausdorff space X such that $A \cong C(X, \mathbb{C})$ as C^*-algebras. It is natural to ask what can be said about X as consequence of A being von Neumann. (Unfortunately, Murphy does not go into this at all.) In fact, there is a very satisfactory characterization of the spaces X for which $C(X)$ is a commutative W^*-algebra. We will discuss this in Section 3.

2 Monotone complete C^*-algebras

Recall that we have a partial ordering on the set A_{sa} of self-adjoint elements of a C^*-algebra A, given by $a \leq b \Leftrightarrow 0 \leq b - a$. I assume known the notion of a least upper bound in a partially ordered set.

2.1 DEFINITION *A C^*-algebra A is called monotone complete if every bounded increasing net $\{a_\iota\} \subseteq A_{sa}$ has a least upper bound.*

2.2 LEMMA *Every von Neumann algebra is monotone complete.*

Proof. Let $A \subseteq B(H)$ be a von Neumann and $\{a_\iota\}_{\iota \in I}$ an increasing net of positive elements. (The general case of a net in A_{sa} is easily reduced to this, as Murphy does.) By Vigier's theorem, a_ι converges strongly to an $a \in B(H)_{sa}$, which is in A by strong closedness. It is easy to check that a is a least upper bound for $\{a_\iota \mid \iota \in I\}$. (After all, a was constructed using the fact that every increasing net $\{x_\iota\}$ in $\mathbb{R}$ that is bounded above converges to its supremum.) ■

It is easy to see that not every C^*-algebra is monotone complete. In addition, not every monotone complete C^*-algebra is isomorphic to a von Neumann algebra.

2.3 Definition *Let A be a monotone complete C^*-algebra. A positive functional $\varphi \in A^*$ is called normal if $\lim_\iota \varphi(a_\iota) = \varphi(a)$ holds for every bounded increasing net $\{a_\iota\} \subseteq A_+$ and its least upper bound a.*

We say that A has enough normal states if for every $0 \neq a \in A_+$ there is a normal state such that $\varphi(a) > 0$.

2.4 Lemma *Every von Neumann algebra has enough normal states.*

Proof. Let A be a von Neumann algebra and $0 < a \in A$. In view of $A \cong (A_*)^*$ there is a $\varphi \in A_*$ (thus an ultraweakly continuous $\varphi \in A^*$) such that $\varphi(a) \neq 0$. Since φ is a linear combination of finitely many positive normal functionals, there must be a positive ultraweakly continuous functional φ with $\varphi(a) > 0$. Now φ is also normal, cf. [2, Exercise IV.4]. ■

2.5 Proposition *Let A be a monotone complete C^*-algebra and $\varphi \in A^*$ normal. Let (H_φ, π_φ) be the corresponding GNS representation. Then π_φ is normal (defined as for functionals) and $\pi_\varphi(A) \subseteq B(H_\varphi)$ is a von Neumann algebra.*

Proof. Not very difficult. See [4, Proposition 3.15], which is a slight generalization of [2, Theorem 4.3.4]. ■

2.6 Theorem *A monotone complete C^*-algebra is isomorphic to a von Neumann algebra (thus is a W^*-algebra) if and only if it has enough normal functionals.*

Proof. If A is isomorphic to a von Neumann algebra then it has enough normal functionals by Lemma 2.4.

Now assume that A has enough normal functionals. Consider the universal representation $(H, \pi) = \bigoplus_\varphi (H_\varphi, \pi_\varphi)$, where φ runs through the set of normal states of A. By Proposition 2.5, each π_φ is strongly continuous, and the same holds for $\pi = \bigoplus \pi$. Thus $\pi(A)$ is a von Neumann algebra. If $0 \neq a \in A$ then $a^*a > 0$, thus by assumption there is a normal φ with $\varphi(a^*a) > 0$. Now it is straightforward to see that $\pi_\varphi(a) \neq 0$. This proves that π is faithful. ■

3 The abelian case

3.1 Definition *A topological space X is called totally disconnected if its connected components all are singletons. (Equivalently, X has no connected subspace Y with more than one element.) A Stone space is a totally disconnected compact Hausdorff space.*

3.2 Definition *A topological space X is called extremely disconnected (some authors write 'extremally disconnected') if the closure of every open $U \subseteq X$ is open (and closed, thus 'clopen'). A stonean space is an extremely disconnected compact Hausdorff space.*

3.3 Lemma *Every extremely disconnected space X is totally disconnected. In particular, every stonean space is a Stone space.*

Proof. Let $x, y \in X$, $x \neq y$. By the Hausdorff property there are open U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Now $C = \overline{U}$ satisfies $C \cap V = \emptyset$ (since $U \subseteq X \backslash V$ and $X \backslash V$ is closed) and is clopen by the extreme disconnectedness of X. Now we have $x \in C, y \in X \backslash C$. But this implies that no set $Y \subset X$ containing $\{x, y\}$ can be connected. Thus X is totally disconnected. ■

Not every Stone space is stonean.

3.4 Lemma *If X is a topological space, let* $\mathrm{clop}(X) = \{Y \subseteq X \mid Y \text{ is clopen}\}$. *Then* $(\mathrm{clop}(X), \cup, \cap, \neg, 0, 1)$*, where $\neg Y = X \backslash Y$, is a Boolean algebra.*

Proof. Immediate. ■

Every Boolean algebra has a partial order $\leq$ defined by $Y \leq Z \Leftrightarrow Y \vee Z = Z$. For a Boolean algebra of subsets, $\leq$ just is inclusion $\subseteq$.

3.5 Definition *A Boolean algebra B is complete if any directed subset of B has a least upper bound.*

3.6 Theorem *For a compact Hausdorff space X the following are equivalent:*

(i) X is extremely disconnected (thus stonean).

(ii) X is a Stone space and the Boolean algebra $\mathrm{clop}(X)$ *is complete.*

(iii) The commutative C^*-algebra $C(X, \mathbb{R})$ is monotone complete (thus also $C(X, \mathbb{C})$).

Note that we cannot omit 'Stone' in (ii)! If X is connected then $\mathrm{clop}(X) = \{\emptyset, X\}$, which is complete, but X certainly is not extremely disconnected.

Proof. Not terribly difficult, but since this is 'just' point set topology, we omit it. The equivalence (i)$\Leftrightarrow$(ii) can be found in [1, Proposition 11.1.30]. I will add the remaining implication once I have the time. For that, see [4, Proposition III.1.7] or [3, Theorem 2.3.7]. ■

As we have seen in the preceding section, a monotone complete C^*-algebra is isomorphic to a von Neumann algebra if and only if it has enough normal states. In the abelian case, this becomes an additional requirement on the stonean space X:

3.7 Definition *A positive Borel measure μ on X is called normal if $\lim_\iota \int f_\iota \, d\mu = \int \sup\{f_\iota\} d\mu$ for every increasing bounded net of positive continuous functions on X.*

A stonean space X is called hyperstonean if it admits sufficiently many normal positive Borel measures, thus for every non-zero positive $f \in C(X)$ there is a normal positive Borel measure μ such that $\int f \, d\mu > 0$.

Now one has:

3.8 Theorem *Let X be a compact Hausdorff space. Then $C(X, \mathbb{C})$ is isomorphic to a von Neumann algebra (i.e. is a W^*-algebra) if and only if X is hyperstonean.*

Proof. We know from Theorem 3.6 that $C(X)$ is monotone complete if and only if X is stonean. Thus by Theorem 2.6, $C(X)$ is isomorphic to a von Neumann algebra if and only if X is stonean and $C(X)$ admits sufficiently many normal positive functionals. Since positive functionals on $C(X)$ come from positive Borel measures by Riesz' representation theorem, there are enough positive normal functionals on $C(X)$ if and only if there are enough positive normal measures on X. See also [4, Theorem III.1.18]. ■

References

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[4] M. Takesaki: *Theory of operator algebras I*. Springer, 1979.