A remark on the invariance of dimension

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Abstract

Combining Kulpa's proof of the cubical Sperner lemma and a dimension theoretic idea of van Mill we give a very short proof of the invariance of dimension, i.e. the statement that cubes $[0,1]^n$ and $[0,1]^m$ are homeomorphic if and only if $n = m$.

1 Introduction

In introductory courses to (general) topology, as also this author has given them many times, it is standard to show that the connected subsets of $\mathbb{R}$ are precisely the convex ones ('intervals') and to exhibit the intermediate value theorem from calculus as an easy consequence. Brouwer's fixed-point theorem for $I = [0,1]$ follows easily. Connectedness arguments also readily prove that $\mathbb{R} \ncong \mathbb{R}^n, I \ncong I^n$ if $n \geq 2$, where $\cong$ means homeomorphism, i.e. isomorphism in the category of topological spaces. If the course covers the fundamental group, the non-vanishing of the latter for the circle S^1 is used to prove the fixed-point theorem for the 2-disk and to distinguish $\mathbb{R}^2$ from $\mathbb{R}^n$ for $n \geq 3$. But the generalization of these results to higher dimensions is usually omitted, referring to courses in algebraic topology. Typical representatives of this approach are [14, 15].

There are certainly proofs of the fixed-point theorem that do not (explicitly) invoke algebraic topology. On the one hand, there is a long tradition of proofs that use a combination of calculus and linear algebra, the nicest perhaps being the one of Lax [11]. However, these proofs seem vaguely inappropriate considering that basic topology should be a more elementary subject than calculus. (At this stage, the student would not be helped by the information that behind such proofs there is de Rham cohomology.) On the other hand, there is the combinatorial approach via Sperner's lemma [16, 7]. (Cf. [3, pp. 411-417] for a nice presentation.) But also the combinatorial approach is not entirely satisfactory since it requires introducing simplicial language and is complicated due to its use of barycentric subdivision. Neither of these complaints applies to the beautiful and elementary proof of the fixed-point theorem published by Kulpa in 1997, cf. [9]. But one still would like to have an accessible proof of the invariance of dimension.

It is well known that the invariance of dimension can be deduced from Brouwer's fixed-point theorem in at least two different ways. On the one hand, one can use Borsuk's theory of maps into spheres (cohomotopy) to prove the 'invariance of domain', another result of Brouwer, from which invariance of dimension readily follows. This approach is followed, e.g., in Eilenberg's and Steenrod's *Foundations of algebraic topology*, in the topology books of Kuratowski and of Dugundji, and in Engelking's and Siekluchi's *Topology. A geometric approach*. A drawback of this approach is that it invariably requires some use of simplicial techniques beyond those typically employed in the proof of the fixed-point theorem. In [10], Kulpa simplified these methods somewhat and obtained a proof of the invariance of dimension in about seven pages. However, this is still more involved than one might wish.

The second way to deduce the invariance of dimension from the fixed-point theorem relies on dimension theory. Dimension theory (cf. [4] for the most up-to-date account) associates to a space X an element $\dim(X) \in \{-1, 0, 1, \ldots, \infty\}$ in a homeomorphism-invariant manner. (There are actually several competing definitions of dimension.) Invariance of the dimension results as soon as one proves $\dim(I^n) = n$. The proof of $\dim(I^n) \leq n$ is easy, but the opposite inequality requires a certain non-separation result, cf. Corollary 4.2 below. The latter result can be deduced from Brouwer's fixed-point theorem, and this is done in virtually all expositions of dimension theory. However, setting up enough of dimension theory (for any one of the existing definitions) to prove $\dim(I^n) = n$ again requires several pages. In [12], van Mill exhibits a very elegant and efficient alternative approach. He uses yet another notion of dimension, for which proving the upper bound is very easy, and which has the virtue that Corollary 4.2 immediately gives the lower bound. We call this notion the 'separation-dimension', cf. Definition 4.4. (For a brief history of the latter, cf. Remark 4.8.)

The purpose and only original contribution of this otherwise expository note is the observation that what Kulpa 'really' proves in [9] is Theorem 3.1 below, from which Corollary 4.2 follows as immediately as does the Poincaré-Miranda theorem. This allows to cut out any reference to the latter (and to Brouwer's theorem) and to give a proof of dimension invariance that easily fits into four pages and can be explained in one lecture of 90 minutes. This makes it even shorter than Brouwer's first proof [1], which was neither self-contained nor easy to read. It also bears emphasizing that deducing both the fixed-point theorem and the invariance of dimension from the higher-dimensional connectedness of the cube asserted by Theorem 3.1 makes these deductions entirely analogous to the elementary ones in dimension one. The price to pay for this efficient approach is that it does not provide a proof of the invariance of domain.

The heart of this note is Section 4, but for the reader's convenience, we include an appendix with Kulpa's deduction of the theorems of Poincaré-Miranda and Brouwer and some corollaries.

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2 The cubical Sperner lemma

In the entire section $n \in \mathbb{N}$ is fixed and $I = [0, 1]$. The faces of the n-cube I^n are given by

$$I_i^- = \{x \in I^n \mid x_i = 0\}, \qquad I_i^+ = \{x \in I^n \mid x_i = 1\}.$$

We need some more notations:

- For $k \in \mathbb{N}$, we put $\mathbb{Z}_{/k} = k^{-1}\mathbb{Z} = \{n/k \mid n \in \mathbb{Z}\}$. Clearly $\mathbb{Z}^n_{/k} \subset \mathbb{R}^n$.
- $e_i \in \mathbb{Z}^n_{/k}$ is the vector whose coordinates are all zero, except the i-th, which is $1/k$.
- $C(k) = I^n \cap \mathbb{Z}^n_{/k} = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}^n$. (The combinatorial n-cube.)
- $C_i^\pm(k) = I_i^\pm \cap \mathbb{Z}^n_{/k}$. (The faces of the combinatorial n-cube $C(k)$.)
- $\partial C(k) = \bigcup_i (C_i^+(k) \cup C_i^-(k))$. (The boundary of the combinatorial n-cube $C(k)$.)
- A subcube of $C(k)$ is a set $C = \{z_0 + \sum_{i=1}^n a_i e_i \mid a \in \{0, 1\}^n\} \subset C(k)$, where $z_0 \in C(k)$.

Sperner's lemma in its original form [16, 7] concerns simplices. The cubical version stated below was first proven in [8] and differently in [18, Lemma 1]. The following proof is the one given in [9].

Proposition 2.1 *Let $\varphi : C(k) \to \{0, \ldots, n\}$ be a map such that (i) $x \in C_i^-(k) \Rightarrow \varphi(x) < i$ and (ii) $x \in C_i^+(k) \Rightarrow \varphi(x) \neq i-1$. Then there is a subcube $C \subset C(k)$ such that $\varphi(C) = \{0, \ldots, n\}$.*

Definition 2.2 *An* <u>*n-simplex*</u> *in* $\mathbb{Z}^n_{/k}$ *is an* $(n+1)$*-tuple* $S = (z_0, \ldots, z_n) \subset \mathbb{Z}^n_{/k}$ *such that*

$$z_1 = z_0 + e_{\alpha(1)}, \quad z_2 = z_1 + e_{\alpha(2)}, \quad \ldots, \quad z_n = z_{n-1} + e_{\alpha(n)},$$

where α *is a permutation of* $\{1, \ldots, n\}$. *The subset* $F_i(S) = (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \subset S$, *where* $i \in \{0, \ldots, n\}$, *is called the i-th* <u>*face of the n-simplex S*</u>.

A finite ordered set $F \subset \mathbb{Z}^n_{/k}$ *is called a* <u>*face*</u> *if it is a face of some simplex.*

Note that the faces $F_i(S)$ are $(n-1)$-simplices in the above sense only if $i = 0$ or $i = n$.

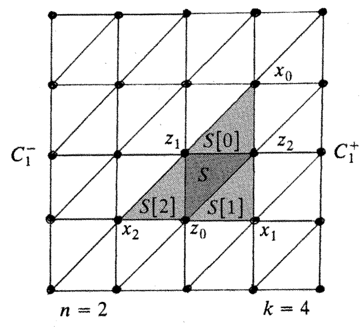

Figure 1: The neighbors of a simplex in $\mathbb{Z}^2_{/4}$. From [9] in Amer. Math. Monthly.

Lemma 2.3 *(i) Let* $S = (z_0, \ldots, z_n) \subset \mathbb{Z}^n_{/k}$ *be an n-simplex. Then for every* $i \in \{0, \ldots, n\}$ *there is a unique n-simplex* $S[i]$, *the* <u>*i-th neighbor*</u> *of S, such that* $S \cap S[i] = F_i(S)$.

(ii) If $S \subset C(k)$ *and* $i \in \{0, \ldots, n\}$ *then* $S[i] \subset C(k)$ *holds if and only if* $F_i(S) \not\subset \partial C(k)$.

Proof. (i) Existence: We define the i-th neighbor $S[i]$ as follows:

(a) $S[0] = (z_1, \ldots, z_n, x_0)$, where $x_0 = z_n + (z_1 - z_0)$.

(b) $0 < i < n$: Take $S[i] = (z_0, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n)$, where $x_i = z_{i-1} + (z_{i+1} - z_i)$.

(c) $S[n] = (x_n, z_0, \ldots, z_{n-1})$, where $x_n = z_0 - (z_n - z_{n-1})$.

It is obvious that $\#(S \cap S[i]) = n$ in all three cases. In the three cases, the distances between consecutive points of $S[i]$ are given by (a) $e_{\alpha(2)}, \ldots, e_{\alpha(n)}, e_{\alpha(1)}$, (c) $e_{\alpha(n)}, e_{\alpha(1)}, \ldots, e_{\alpha(n-1)}$, and (b) $e_{\alpha(1)}, \ldots, e_{\alpha(i-1)}, e_{\alpha(i+1)}, e_{\alpha(i)}, e_{\alpha(i+2)}, \ldots, e_{\alpha(n)}$. (Figure 1 should make this quite clear.) Thus $S[i]$ is a legal n-simplex for each $i \in \{0, \ldots, n\}$. Uniqueness: It remains to show that these are the only ways of defining $S[i]$ consistently with $S \cap S[i] = \{z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}$. In the cases $i = 0$ or $i = n$, the latter condition implies that $S[i]$ has a string of n consecutive z_i's in common with S, and therefore also their differences given by $n-1$ mutually different vectors e_j. This means that only one such vector is left, and the only way to use it so that $S[i]$ is an n-simplex different from S is to use it at the other end of the string of z's. This shows the uniqueness of the above definitions in cases (a) and (c). In the case $0 < i < n$, S and $S[i]$ have two corresponding substrings of z's. A little thought shows that the order of these two substrings must be the same in $S[i]$ as in S, so that all we can do is exchange two adjacent difference vectors $e_{\alpha(i)}, e_{\alpha(i+1)}$, as done in the definition of $S[i]$ in case (b).

(ii) We must check whether $S[i] \subset C(k)$, which amounts to checking whether the new point x_i is in I^n. In case (a), we have $S = (z_1 - e_{\alpha(1)}, z_1, z_2, \ldots, z_n)$ and $S(0) = (z_1, \ldots, z_n, z_n + e_{\alpha(1)})$. If $F_0(s) = (z_1, \ldots, z_n) \subset C_j^\varepsilon(k)$ then $z_1, \ldots, z_n$ all have the same j-coordinate c, thus we must have $\alpha(1) = j$ and $c = 1$ (since $S \subset I^n$). But then $S[0] \not\subset I^n$. Conversely, if both S and $S[0]$ are in I^n, then z_1 must have $\alpha(1)$-th coordinate > 0 and z_n must have $\alpha(1)$-th coordinate < 1. All other coordinates of $z_1, \ldots, z_n$ are non-constant since the vectors $e_{\alpha(2)}, \ldots, e_{\alpha(n)}$ appear as differences. Thus $F_0(S)$ is not contained in any face I_i^ε. The cases (b) and (c) are checked similarly. ■

Proof of Proposition 2.1. For later use, we note the following fact (*): If $S \subset I^n$ satisfies $\varphi(S \cap I_i^\varepsilon) = \{0, \ldots, n-1\}$ then $i = n$ and $\varepsilon = -$. [The statement $\varphi(S \cap I_i^\varepsilon) = \{0, \ldots, n-1\}$ is contradicted by assumption (ii) if $\varepsilon = +$ and by (i) if $\varepsilon = -$ and $i < n$.]

We call a subset $S \subset C(k)$ with $l+1$ elements <u>full</u> if $\varphi(S) = \{0, \ldots, l\}$. By (vi), a full n-simplex S meets all $H_i^\pm$. We will prove that the number N_k of full n-simplices in $C(k)$ is odd, thus non-zero, for all k. The proof of $N_k \equiv 1 \pmod 2$ proceeds by induction over the dimension n of $C(k)$ (for fixed k). For $n = 0$ we have $C(k) = \{0\}$, and there is exactly one full n-simplex, namely $S = (z_0 = 0)$. Thus $N_0 = 1$.

For an n-simplex $S \subset C(k)$, let $N(S)$ denote the number of full $(n-1)$-faces of S. If S is full then $N(S) = 1$. [Since $\varphi(S) = \{0, \ldots, n\}$ and the only full $(n-1)$-face is obtained by omitting the unique z_i for which $\varphi(z_i) = n$.] If S is not full then $N(S) = 0$ in the case $\{0, \ldots, n-1\} \not\subset \varphi(S)$ [since omitting a z_i cannot give a full (n-1)-face] or $N(S) = 2$ in the case $\varphi(S) = \{0, \ldots, n-1\}$ [since there are $i \neq i'$ such that $z_i = z_{i'}$, so that S becomes full upon omission of either z_i or $z_{i'}$]. Thus

$$N_k \equiv \sum_S N(S) \pmod 2, \tag{2.1}$$

where the summation extends over all n-simplices in $C(k)$.

Now by the Lemma, an $(n-1)$-face $F \subset C(k)$ belongs to one or two n-simplices in $C(k)$, depending on whether $F \subset \partial C(k)$ or not. Thus only the full faces $F \subset \partial C(k)$ contribute to (2.1):

$$N_k \equiv \#\{F \subset \partial C(k) \text{ full } (n-1)\text{−face}\} \pmod 2.$$

If $F \subset \partial C(k)$ is a full $(n-1)$-face then (*) implies $F \subset C_n^-(k)$. We can identify $C_n^-(k) = C(k) \cap I_n^-$ with $C_{n-1}(k)$, and under this identification F is a full $(n-1)$-simplex in $\mathbb{Z}_{/k}^{n-1}$. Thus $N_k \equiv N_{k-1} \pmod 2$. By the induction hypothesis, N_{k-1} is odd, thus N_k is odd.

Thus there is a full n-simplex $S = (z_0, \ldots, z_n)$, and if $C = \{z_0 + \sum_{i=1}^n a_i e_i \mid a \in \{0,1\}^n\}$ we have $S \subset C$ so that $\varphi(C) = \{0, \ldots, n\}$. ■

Remark 2.4 Combinatorial proofs of Sperner's lemma, whether simplicial or cubical, may appear mysterious. Homological proofs tend to be more transparent, cf. e.g. [6] in the simplicial case, but here the point of course is to avoid the heavy machinery of homology. □

3 Higher connectedness of the cube

Now we are in a position to prove, still following [9], this beautiful theorem, which for $n = 1$ is just the connectedness of $[0, 1]$:

Theorem 3.1 *For $i = 1, \ldots, n$, let $H_i^+, H_i^- \subset I^n$ be closed sets such that for all i one has $I_i^\pm \subset H_i^\pm$ and $H_i^- \cup H_i^+ = I^n$. Then $\bigcap_i (H_i^- \cap H_i^+) \neq \emptyset$.*

Proof. We define $F_0 = I^n$ and $F_i = H_i^+ \backslash I_i^-$ for all $i \in \{1, \ldots, n\}$. Now define a map $\varphi : I^n \to \{0, \ldots, n\}$ by

$$\varphi(x) = \max\left\{ j : \ x \in \bigcap_{k=0}^{j} F_k \right\}.$$

Since $I_i^- \cap F_i = \emptyset$, we have $x \in I_i^- \ \Rightarrow \ \varphi(x) < i$. On the other hand, if $x \in I_i^+$ then $\varphi(x) \neq i-1$. Namely, $\varphi(x) = i-1$ would mean that $x \in \bigcap_{k=0}^{i-1} F_k$ and $x \notin F_i$. But this is impossible since $x \in I_i^+ \subset H_i^+$ and $I_i^+ \cap I_i^- = \emptyset$, thus $x \in F_i$.

By the above, the restriction of φ to $C(k) \subset I^n$ satisfies the assumptions of Proposition 2.1. Thus for every $k \in \mathbb{N}$ there is a subcube $C_k \subset C(k)$ such that $\varphi(C_k) = \{0, \ldots, n\}$. Now, if $\varphi(y) = i \in \{1, \ldots, n\}$ then $y \in F_i = H_i^+ \backslash I_i^- \subset H_i^+$. On the other hand, if $\varphi(x) = i - 1 \in \{0, \ldots, n-1\}$ then $x \notin F_i = H_i^+ \backslash I_i^-$, which is equivalent to $x \notin H_i^+ \vee x \in I_i^-$. The first alternative implies $x \in H_i^-$ (since $H_i^+ \cup H_i^- = I^n$), as does the second (since $I_i^- \subset H_i^-$). In either case, $x \in H_i^-$. Combining these facts, we find that $\varphi(C_k) = \{0, \ldots, n\}$ implies $C_k \cap H_i^+ \neq \emptyset \neq C_k \cap H_i^-$ for all i, thus the subcube C_k meets all the $H_i^\pm$. We clearly have $\mathrm{diam}(C_k) = \sqrt{n}/k$, and since k was arbitrary, applying the following lemma to $X = I^n$, $\{K_1, \ldots, K_{2n}\} = \{H_i^\pm\}$ and $S_k = C_k$ gives $\bigcap_i (H_i^- \cap H_i^+) \neq \emptyset$. ∎

Lemma 3.2 *Let (X, d) be a metric space and $K_1, \ldots, K_m$ compact subsets. Let $\{S_k \subset X\}_{k \in \mathbb{N}}$ satisfy* $\mathrm{diam}(S_k) \stackrel{k \to \infty}{\longrightarrow} 0$ *and $S_k \cap K_i \neq \emptyset$ for all $k \in \mathbb{N}$, $i = 1, \ldots, m$. Then $\bigcap_i K_i \neq \emptyset$.*

Proof. Consider $K = \prod_i K_i$ equipped with the metric $d_K(x, y) = \sum_i d(x_i, y_i)$. For every $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, choose an $x_{k,i} \in S_k \cap K_i$ and define $x_k = (x_{k,1}, \ldots, x_{k,m}) \in K$. By compactness of K there exists a point $z = (z_1, \ldots, z_m) \in K$ every neighborhood of which contains x_k for infinitely many k. Now, $d(z_i, z_j) \leq d(z_i, x_{k,i}) + d(x_{k,i}, x_{k,j}) + d(x_{k,j}, z_j) \leq 2d_K(z, x_k) + \mathrm{diam}(S_k)$. Since by construction every neighborhood of z contains points x_k with arbitrarily large k, we can make both terms on the r.h.s. arbitrarily small and conclude that $z = (x, \ldots, x)$ for some $x \in X$. Since $z_i \in K_i$ for all i, we have $x \in \bigcap_i K_i$, and are done. ∎

4 The dimension of I^n

Definition 4.1 *If $A, B, C \subset X$ are closed sets such that $X \backslash C = U \cup V$, where U, V are disjoint open sets such that $A \subset U$ and $B \subset V$, we say that C separates A and B.*

The following result plays an essential rôle in virtually all accounts of dimension theory. While it is usually derived from Brouwer's fixed-point theorem, we obtain it more directly as an obvious corollary of Theorem 3.1.

Corollary 4.2 *Whenever $C_1, \ldots, C_n \subset I^n$ are closed sets such that C_i separates I_i^- and I_i^+ for each i, then $\bigcap_i C_i \neq \emptyset$.*

Proof. In view of Definition 4.1, we have open sets $U_i^\pm$ such that $I_i^\pm \subset U_i^\pm$, $U_i^+ \cap U_i^- = \emptyset$ and $U_i^+ \cup U_i^- = X \backslash C_i$ for all i. Define $H_i^\pm = U_i^\pm \cup C_i$. Then $X \backslash H_i^\pm = U_i^\mp$, thus $H_i^\pm$ is closed. By construction, $I_i^\pm \subset H_i^\pm$ and $H_i^+ \cup H_i^- = I^n$, $H_i^+ \cap H_i^- = C_i$, for all i. Now Theorem 3.1 gives $\bigcap_i C_i = \bigcap_i (H_i^- \cap H_i^+) \neq \emptyset$. ∎

The preceding result will provide a lower bound on the dimension of I^n. The next result, taken from [12], will provide the upper bound:

Proposition 4.3 *Let $A_1, B_1, \ldots, A_{n+1}, B_{n+1} \subset I^n$ be closed sets such that $A_i \cap B_i = \emptyset$ for all i. Then for all i there exist closed sets C_i separating A_i and B_i and satisfying $\bigcap_i C_i = \emptyset$.*

Proof. Pick real numbers $r_1, r_2, \ldots$ such that $r_i - r_j \notin \mathbb{Q}$ for $i \neq j$. (It suffices to take $r_k = k\sqrt{2}$.) Then the sets $E_i = r_i + \mathbb{Q}$ are mutually disjoint dense subsets of $\mathbb{R}$.

Let $A, B \subset I^n$ be disjoint closed sets and $E \subset \mathbb{R}$ dense. Then for every $x \in A$ we can find an open neighborhood $U_x = I^n \cap \prod_{i=1}^n (a_i, b_i)$ with $a_i, b_i \in E$ such that $\overline{U_x}$ is disjoint from B. Since $A \subset I^n$ is closed, thus compact, there are $x_1, \ldots, x_k \in A$ such that $U = U_{x_1} \cup \cdots \cup U_{x_k} \supset A$. Now $C = \partial U \subset I^n$ is closed and $X \backslash C = U \cup V$, where $V = I^n \backslash \overline{U}$. Now U, V are open and disjoint such that $A \subset U, B \subset V$, thus C separates A and B. If $x \in \partial(I^n \cap \prod_i (a_i, b_i))$ then at least one of the coordinates x_i of x equals a_i or b_i, and thus is in E. Now, $C = \partial U \subset \partial U_{x_1} \cup \cdots \cup \partial U_{x_k} \subset \{x \in I^n \mid \exists j \in \{1, \ldots, n\} : x_j \in E\}$.

We can thus find, for each pair (A_i, B_i) a closed set $C_i \subset \{x \in I^n \mid \exists j : x_j \in E_i\}$ that separates A_i and B_i. Let now $x \in \bigcap_i C_i$. Then for every $i \in \{1, \dots, n+1\}$ there is a $j_i \in \{1, \dots, n\}$ such that $x_{j_i} \in E_i$. By the 'pigeonhole principle' (a map $A \to B$ with $|A| > |B|$ cannot be injective), there are $i, i' \in \{1, \dots, n+1\}$ such that $i \neq i'$ and $j_i = j_{i'} = j$. But this means that $x_j \in E_i \cap E_{i'} \in \emptyset$, which is absurd. Thus $\bigcap_i C_i = \emptyset$. ■

Proposition 4.3 should be compared with Corollary 4.2. In order to do this systematically, the following is convenient:

Definition 4.4 *Let X be a topological space. We define the* <u>*separation-dimension*</u> $s\text{-}\dim(X) \in \{-1, 0, 1, \dots, \infty\}$ *as follows:*

- *We put $s\text{-}\dim(X) = -1$ if and only if $X = \emptyset$.*
- *If $X \neq \emptyset$ and $n \in \mathbb{N}_0$, we say that $s\text{-}\dim(X) \leq n$ if, given closed sets $A_1, B_1, \dots, A_{n+1}, B_{n+1}$ such that $A_i \cap B_i = \emptyset$ for all i, there exist closed C_i separating A_i and B_i and satisfying $\bigcap_i C_i = \emptyset$. (This is consistent: If $s\text{-}\dim(X) \leq n$ and $n < m$ then $s\text{-}\dim(X) \leq m$.)*
- *If $s\text{-}\dim(X) \leq n$ holds, but $s\text{-}\dim(X) \leq n-1$ does not, we say $s\text{-}\dim(X) = n$.*
- *If there is no $n \in \mathbb{N}$ such that $s\text{-}\dim(X) \leq n$ then $s\text{-}\dim(X) = \infty$.*

Remark 4.5 It is obvious that a homeomorphism $X \cong Y$ implies $s\text{-}\dim(X) = s\text{-}\dim(Y)$. □

Theorem 4.6 *We have $s\text{-}\dim(I^n) = n$.*

Proof. Proposition 4.3 implies $s\text{-}\dim(I^n) \leq n$. On the other hand, it is clear that $s\text{-}\dim(I^n) \geq n$ holds if and only if there are closed sets $A_1, B_1, \dots, A_n, B_n \subset X$ satisfying $A_i \cap B_i = \emptyset$ for all i such that any closed sets C_i separating A_i and B_i satisfy $\bigcap_i C_i \neq \emptyset$. This is exactly what is asserted by Corollary 4.2. ■

Corollary 4.7 *We have $I^n \cong I^m$ if and only if $n = m$.*

Remark 4.8 1. A family $\{(A_1, B_1), \dots, (A_{n+1}, B_{n+1})\}$ as above is called 'essential'. Therefore one could also speak of the 'essential family dimension', but this does not seem to be in widespread use.

2. In 1938, Eilenberg and Otto [2] proved that the separation dimension coincides with the covering dimension in the case of separable metrizable spaces. This was generalized to normal spaces by Hemmingsen (1946). For a modern proof and more history see [4], in particular Theorem 3.2.6.

3. It is immediate from the definition that $s\text{-}\dim(Y) \leq s\text{-}\dim(X)$ for closed $Y \subset X$, implying $s\text{-}\dim(\mathbb{R}^n) \geq n$ and $s\text{-}\dim(S^n) \geq n$. In order to prove the converse inequalities, and thereby the invariance of dimension for spheres and Euclidean spaces, one needs a 'sum theorem' for the separation dimension. This is the statement that if $X = \bigcup_{i \in \mathbb{N}} Y_i$ with $Y_i \subset X$ closed and $s\text{-}\dim(Y_i) \leq d$ then $s\text{-}\dim(X) \leq d$. Such a result follows from the combination of the first half of this remark and the known sum theorem for the covering dimension, cf. [4, Theorem 3.1.8]. (This is "just" point set topology, with no simplicial or combinatorial methods involved.) However, from an aesthetic perspective it would be desirable to give a direct proof of the sum theorem for s-dim. □

A The theorems of Poincaré-Miranda and Brouwer

In order to further illustrate the power of Theorem 3.1, and for the benefit of the reader, we include deductions of many important classical results about continuous functions on I^n, following [9]. But we emphasize that none of this is needed for the proof of the invariance of dimension given in Section 4.

Corollary A.1 (Poincaré-Miranda theorem) *Let* $f = (f_1, \ldots, f_n) \in C(I^n, \mathbb{R}^n)$*. If* $f_i(I_i^-) \subset (-\infty, 0]$*,* $f_i(I_i^+) \subset [0, \infty)$ *for all* i*, then there is* $x \in I^n$ *such that* $f(x) = 0$*.*

Proof. Put $H_i^- = f_i^{-1}((-\infty, 0]), H_i^+ = f_i^{-1}([0, \infty))$. Then clearly $I_i^\pm \subset H_i^\pm$ and $H_i^- \cup H_i^+ = I^n$, for all i. By Theorem 3.1, there exists $x \in \bigcap_i (H_i^- \cap H_i^+)$, and it is clear that $f(x) = 0$. ■

Corollary A.2 (Brouwer's fixed-point theorem) *Let* $g \in C(I^n, I^n)$*. Then there exists* $x \in I^n$ *such that* $g(x) = x$*. (I.e.,* I^n *has the fixed-point property.)*

Proof. Put $f(x) = x - g(x)$. Then the assumptions of Corollary A.1 are satisfied, so that there is $x \in I^n$ for which $f(x) = 0$. Thus $g(x) = x$. ■

Corollary A.3 *Let* $g \in C(I^n, I^n)$ *satisfy* $g(I_i^\pm) \subset I_i^\pm$ *for all* i*. (E.g.* $g \restriction \partial I^n = \mathrm{id}$*.) Then* $g(I^n) = I^n$*.*

Proof. Let $p \in I^n$, and put $f(x) = g(x) - p$. Then f satisfies the assumptions of Corollary A.1, thus there is $x \in I^n$ with $f(x) = 0$. This means $g(x) = p$, so that g is surjective. ■

Remark A.4 1. The history of the above results is quite convoluted and interesting. See the introduction of [9] for a glimpse.

2. Since compact convex subsets of $\mathbb{R}^n$ are homeomorphic to I^m for some $m \leq n$, they also have the fixpoint property. The convexity assumption cannot be omitted as is shown by a non-trivial rotation of $S^1 \subset \mathbb{R}^2$. On the other hand, Corollary A.3 and the resulting non-existence of retractions to the boundary extend to arbitrary compact subsets of $\mathbb{R}^n$, cf. [9].

3. The Poincaré-Miranda theorem seems to be much more popular with analysts than with topologists. One may indeed argue that Brouwer's theorem is more fundamental, asserting the fixed-point property of any n-cell irrespective of its shape (e.g. disk, cube or simplex). But apart from being a particularly natural higher dimensional generalization of the intermediate value theorem, the Poincaré-Miranda theorem often is the more convenient point of departure for other proofs. (The Poincaré-Miranda theorem can be deduced from Brouwer's theorem, cf. [13], but the argument is more involved. Cf. also [5, p. 118].)

4. It is also true that the Poincaré-Miranda theorem implies Theorem 3.1: Given $H_i^\pm$ as in the latter, the functions $f_i(x) = \mathrm{dist}(x, H_i^-) - \mathrm{dist}(x, H_i^+)$ are continuous and satisfy $f_i(I_i^-) \subset [-1, 0]$, $f_i(I_i^+) \subset [0, 1]$. Now the Poincaré-Miranda theorem gives an $x \in I^n$ such that $f(x) = 0$. The assumption $H_i^- \cup H_i^+ = I^n$ implies that $\mathrm{dist}(x, H_i^-), \mathrm{dist}(x, H_i^+)$ cannot both be non-zero. Thus $f_i(x) = 0$ is equivalent to $x \in H_i^- \cap H_i^+$. Thus $x \in \bigcap_i (H_i^- \cap H_i^+)$.

5. Combining the above facts and some other well-known implications, we see that the following statements are 'equivalent' (in the sense of being easily deducible from each other):

(i) the non-existence of a retraction $r : D^n \to \partial D^n$,

(ii) the non-contractibility of $\partial D^n = S^{n-1}$,

(iii) $\pi_{n-1}(S^{n-1}) \neq 0$,

(iv) the fixed point property of D^n,

(v) the statement $f : D^n \to D^n,\ f \restriction \partial D^n = \mathrm{id} \ \Rightarrow\ f(D^n) = D^n$,

(vi) the Poincaré-Miranda theorem,

(vii) Theorem 3.1.

(A similar statement appears in [17, Theorem 6.6.1], where however Corollary 4.2 is listed instead of the more convenient Theorem 3.1 and correspondingly, the Poincaré-Miranda type theorem given there makes the stronger assumptions $f_i(I_i^-) \subset (-\infty, 0)$, $f_i(I_i^+) \subset (0, \infty)\ \forall i$.)

We observe that (i)-(vi) all involve continuous maps, whereas (vii) only involves the faces and the topology of the cube I^n. The latter therefore seems closest in spirit to point set topology, as is also supported by its interpretation as higher-dimensional connectedness and its rôle in the above proof of dimension invariance. □

References

[1] L. E. J. Brouwer: Beweis der Invarianz der Dimensionenzahl. Math. Ann. **70**, 161-165 (1911).

[2] S. Eilenberg, E. Otto: Quelques propriétés caractéristiques de la dimension. Fund. Math. **31**, 149-153 (1938).

[3] R. Engelking: *General topology.* Heldermann Verlag, 1989.

[4] R. Engelking: *Theory of dimensions: Finite and infinite.* Heldermann Verlag, 1995.

[5] V. I. Istrătescu: *Fixed point theory.* D. Reidel Publishing Company, 1981.

[6] N. V. Ivanov: Sperner's lemma, the Brouwer fixed-point theorem, and cohomology. arXiv:0906.5193.

[7] B. Knaster, C. Kuratowski, S. Mazurkiewicz: Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. Fund. Math. **14**, 132-137 (1929).

[8] H. W. Kuhn: Some combinatorial lemmas in topology. IBM Journ. Res. Devel. **4**, 518-524 (1960).

[9] W. Kulpa: The Poincaré-Miranda theorem. Amer. Math. Monthly **104**, 545-550 (1997).

[10] W. Kulpa: Poincaré and domain invariance theorem. Acta Univ. Carol. Math. et Phys. **39**, 127-136 (1998).

[11] P. D. Lax: Change of variables in multiple integrals. Amer. Math. Monthly **106**, 497-501 (1999).

[12] J. van Mill: *Infinite-dimensional topology.* North-Holland, 1989.

[13] C. Miranda: Un'osservazione su un teorema di Brouwer. Boll. Un. Mat. Ital. **3**, 5-7 (1940).

[14] J. R. Munkres: *Topology.* Second edition. Prentice-Hall, 2000.

[15] V. Runde: *A taste of topology.* Springer Universitext, 2005.

[16] E. Sperner: Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. Abh. Math. Sem. Univ. Hamb. **6**, 265-272 (1928).

[17] T. Tom Dieck: *Algebraic topology.* European Mathematical Society, 2008.

[18] L. A. Wolsey: Cubical Sperner lemmas as applications of generalized complementary pivoting. J. Combin. Theor. (A) **23**, 78-87 (1977).