and conclude that the matrix for T_f, cf. E 3.2.16, is constant on diagonals. Insert $f = e_k$, $k \in \mathbb{Z}$, and check that the corresponding Toeplitz operators are kth powers of the unilateral shift (3.2.16) or its adjoint.

E 3.3.15. Take f in $L^\infty(\mathbb{T})$, and let T_f be the corresponding Toeplitz operator; cf. E 3.3.14. Show that T_f is a compact operator on H^2 only if $f = 0$.

Hint: If $T_f \in \mathbf{B}_0(H^2)$, then, since $e_n \to 0$ weakly (cf. E 3.1.10). $\|T_f e_n\| \to 0$. In particular, $(T_f e_n | e_{n+k}) \to 0$ for each k in $\mathbb{Z}$. Now apply E 3.3.14 to show that $\hat{f} = 0$, whence $f = 0$.

E 3.3.16. Take f and g in $L^\infty(\mathbb{T})$, and let T_f and T_g denote the corresponding Toeplitz operators; cf. E 3.3.14. Show that $T_f T_g - T_g T_f$ and $T_f T_g - T_{fg}$ are both compact operators on H^2 if either f or g is continuous.

Hint: Show that the two operators have finite rank if $f = e_k$ for some k in $\mathbb{Z}$. If $f \in C(\mathbb{T})$, use the fact that f can be uniformly approximated by trigonometric polynomials, and apply 3.3.3. (P.S. Don't miss E 4.3.11 later on.)

E 3.3.17. If $f \in C(\mathbb{T})$, show that the Toeplitz operator T_f, cf. E 3.3.14, is a Fredholm operator if f is invertible in $C(\mathbb{T})$. Show in this case that index T_f = index T_u, where $u = f|f|^{-1}$.

Hint: Use first E 3.3.16. Then use that a self-adjoint Fredholm operator has zero index, so that 3.3.19 applies.

E 3.3.18. Show that functions f and g in $C(\mathbb{T}, \mathbb{T})$ that are homotopic (E 1.4.19) inside $C(\mathbb{T}, \mathbb{T})$ give Toeplitz operators of Fredholm type with index T_f = index T_g.

Hint: If $f_t \colon \mathbb{T} \to \mathbb{T}$ is a continuous path in $C(\mathbb{T}, \mathbb{T})$ with $f_0 = f$ and $f_1 = g$, then index T_{f_s} = index T_{f_t} when $|s - t|$ is small enough by 3.3.18.

E 3.3.19. Take f invertible in $C(\mathbb{T})$ and consider the Toeplitz operator T_f; cf. E 3.3.14. Show that the winding number of f around 0 equals $-$index T_f. Compare with E 4.1.19.

Hint: Use E 3.3.17 and E 3.3.18 plus the fact (to be proved or taken at face value) that the homotopy classes in $C(\mathbb{T}, \mathbb{T})$ are labeled by the winding number. Check the formula with $f = e_k$, where $k \in \mathbb{Z}$; cf. E 3.3.14.

3.4. The Trace

Synopsis. Definition and invariance properties of the trace. The trace class operators and the Hilbert–Schmidt operators. The dualities among $\mathbf{B}_0(\mathfrak{H})$, $\mathbf{B}^1(\mathfrak{H})$, and $\mathbf{B}(\mathfrak{H})$. Hilbert–Schmidt operators as integral operators. The Fredholm equation. The Sturm–Liouville problem. Exercises.

3.4.1. In search for analogies between the theory of functions and the theory of operators on a complex(!) Hilbert space $\mathfrak{H}$, we have already (in 3.3.1 and 3.3.4) mentioned that $\mathbf{B}_f(\mathfrak{H})$ corresponds to the continuous functions with compact supports and $\mathbf{B}_0(\mathfrak{H})$ corresponds to the continuous functions vanishing at infinity. The class $\mathbf{B}(\mathfrak{H})$ plays a double role: sometimes it mimics the set of all bounded continuous functions and sometimes it behaves like an L^∞-space. The latter behavior assumes the existence of an analogue on $\mathfrak{H}$ to Lebesgue measure, an analogue we will now exhibit.

3.4.2. Choose an orthonormal basis $\{e_j | j \in J\}$ for the Hilbert space $\mathfrak{H}$ (cf. 3.1.12), and for every positive operator T in $\mathbf{B}(\mathfrak{H})$ define the *trace* of T by

$$\operatorname{tr}(T) = \sum (Te_j | e_j),$$

with values in $[0, \infty]$.

3.4.3. Proposition. *For every T in $\mathbf{B}(\mathfrak{H})$ we have*

$$\operatorname{tr}(T^*T) = \operatorname{tr}(TT^*).$$

PROOF. For each i and j we have

$$(Te_i | e_j)(e_j | Te_i) = (T^*e_j | e_i)(e_i | T^*e_j) \geq 0.$$

Summing the first expression over j we get

$$\sum_j ((Te_i | e_j)e_j | Te_i) = (Te_i | Te_i) = (T^*Te_i | e_i).$$

Summing the second expression over i we similarly have

$$\sum_i ((T^*e_j | e_i)e_i | T^*e_j) = (T^*e_j | T^*e_j) = (TT^*e_j | e_j).$$

Since the elements in the series are positive, the sum over both i and j does not depend on the order of the summation, whence

$$\operatorname{tr}(T^*T) = \sum_i (T^*Te_i | e_i) = \sum_j (TT^*e_j | e_j) = \operatorname{tr}(TT^*).$$ □

3.4.4. Corollary. *If U is unitary and $T \geq 0$, then*

$$\operatorname{tr}(UTU^*) = \operatorname{tr}(T).$$

In particular, the definition of tr *is independent of the choice of basis, and, therefore,* $\|T\| \leq \operatorname{tr} T$.

PROOF. Since $T = (T^{1/2})^2$ by 3.2.11, we may replace T by $UT^{1/2}$ in 3.4.3. The last assertions follow from 3.1.14 and 3.2.25 (or E 3.2.1). □

3.4.5. Lemma. *If $T \in \mathbf{B}(\mathfrak{H})$ such that* $\operatorname{tr}(|T|^p) < \infty$ *for some* $p > 0$, *then T is compact.*

PROOF. Given an orthonormal basis $\{e_j | j \in J\}$ and $\varepsilon > 0$ there is a finite subset λ of J such that $\sum_{j \notin \lambda} (|T|^p e_j | e_j) < \varepsilon$. If P_λ denotes the projection of $\mathfrak{H}$ onto the span of $\{e_j | j \in \lambda\}$, then by (**) in 3.2.3 and 3.4.4

$$\| |T|^{p/2}(I - P_\lambda) \|^2 = \|(I - P_\lambda)|T|^p(I - P_\lambda)\|$$
$$\leq \operatorname{tr}((I - P_\lambda)|T|^p(I - P_\lambda)) < \varepsilon.$$

Since ε is arbitrary, we conclude from 3.3.3(i) that $|T|^{p/2} \in \mathbf{B}_0(\mathfrak{H})$. Thus, for a suitable orthonormal basis (which we still denote by $\{e_j | j \in J\}$) we have

$$|T|^{p/2} = \sum \lambda_j e_j \odot e_j$$

by 3.3.8 (cf. 3.3.9) and the λ_j's vanish at infinity. For integer values of p it is clear that

$$|T| = \sum \lambda_j^{2/p} e_j \odot e_j. \tag{$*$}$$

To establish the validity of the formula (*) in general one will have to define the symbol $|T|^p$ for all real $p > 0$; and we must postpone this task until we have the spectral theorem (4.4.1) at hand. Assuming (*) it is clear that $|T| \in \mathbf{B}_0(\mathfrak{H})$, and from the polar decomposition $T = U|T|$, cf. 3.2.17, it follows that T belongs to the ideal $\mathbf{B}_0(\mathfrak{H})$. □

3.4.6. We define the sets of *trace class* operators and *Hilbert–Schmidt* operators as

$$\mathbf{B}^1(\mathfrak{H}) = \operatorname{span}\{T \in \mathbf{B}_0(\mathfrak{H}) | T \geq 0, \operatorname{tr}(T) < \infty\},$$
$$\mathbf{B}^2(\mathfrak{H}) = \{T \in \mathbf{B}_0(\mathfrak{H}) | \operatorname{tr}(T^*T) < \infty\}.$$

Since, evidently, $\operatorname{tr}(T_1 + T_2) = \operatorname{tr}(T_1) + \operatorname{tr}(T_2)$ and $\operatorname{tr}(\alpha T_1) = \alpha \operatorname{Tr}(T_1)$ for all positive operators T_1 and T_2 and each $\alpha \geq 0$; and since $T = \sum_{k=0}^{3} \mathrm{i}^k T_k$, with $T_k \geq 0$, for every T in $\mathbf{B}^1(\mathfrak{H})$, it follows that the definition $\operatorname{tr}(T) = \sum \mathrm{i}^k \operatorname{tr}(T_k)$ extends tr to a linear functional on $\mathbf{B}^1(\mathfrak{H})$. From now on we may therefore apply the function tr to any operator in the set $\mathbf{B}(\mathfrak{H})_+ + \mathbf{B}^1(\mathfrak{H})$ (with the convention that $\alpha + \infty = \infty$ for every α in $\mathbb{C}$).

3.4.7. Just as for vectors in $\mathfrak{H}$, there is a *parallellogram law* for operators in $\mathbf{B}(\mathfrak{H})$, viz.,

$$(S + T)^*(S + T) + (S - T)^*(S - T) = 2(S^*S + T^*T), \tag{$*$}$$

easily verified by computation. From this one derives the useful estimate

$$(S + T)^*(S + T) \leq 2(S^*S + T^*T). \tag{$**$}$$

By direct computation we also verify the following *polarization identity* for operators on a complex Hilbert space:

$$4T^*S = \sum_{k=0}^{3} \mathrm{i}^k (S + \mathrm{i}^k T)^*(S + \mathrm{i}^k T). \tag{$***$}$$

3.4.8. Proposition. *The classes* $\mathbf{B}^1(\mathfrak{H})$ *and* $\mathbf{B}^2(\mathfrak{H})$ *are self-adjoint ideals in* $\mathbf{B}(\mathfrak{H})$ *and*

$$\mathbf{B}_f(\mathfrak{H}) \subset \mathbf{B}^1(\mathfrak{H}) \subset \mathbf{B}^2(\mathfrak{H}) \subset \mathbf{B}_0(\mathfrak{H}).$$

PROOF. If $T \geq 0$ with $\operatorname{tr}(T) < \infty$, and $S \in \mathbf{B}(\mathfrak{H})$, then by (***) in 3.4.7

$$4TS = 4T^{1/2}T^{1/2}S = \sum \mathrm{i}^k(S + \mathrm{i}^k I)^* T(S + \mathrm{i}^k I).$$

By 3.4.3 and 3.2.11 we further have

$$\begin{aligned}\operatorname{tr}(V^*TV) &= \operatorname{tr}(V^*T^{1/2}T^{1/2}V)\\ &= \operatorname{tr}(T^{1/2}VV^*T^{1/2}) \leq \|VV^*\| \operatorname{tr}(T);\end{aligned}$$

and applied with $V = S + \mathrm{i}^k I$ it shows that $TS \in \mathbf{B}^1(\mathfrak{H})$. Thus, $\mathbf{B}^1(\mathfrak{H})$ is a self-adjoint right ideal and therefore a twosided ideal (4.1.2).

We claim that

$$\mathbf{B}^1(\mathfrak{H}) = \{T \in \mathbf{B}(\mathfrak{H}) \,|\, \operatorname{tr}(|T|) < \infty\}. \qquad (*)$$

If $|T| \in \mathbf{B}^1(\mathfrak{H})$, then from the polar decomposition $T = U|T|$ (3.2.17) we see from the first part of the proof that $T \in \mathbf{B}^1(\mathfrak{H})$. Conversely, $|T| = U^*T$, so if $T \in \mathbf{B}^1(\mathfrak{H})$, then $|T| \in \mathbf{B}^1(\mathfrak{H})$.

It follows from (**) in 3.4.7 that $\mathbf{B}^2(\mathfrak{H})$ is a linear subspace of $\mathbf{B}_0(\mathfrak{H})$, and 3.4.3 shows that this subspace is self-adjoint. Since $\mathbf{B}^1(\mathfrak{H})$ is an ideal in $\mathbf{B}(\mathfrak{H})$, it follows from the definition of $\mathbf{B}^2(\mathfrak{H})$ that this set is also an ideal.

If $T \in \mathbf{B}_f(\mathfrak{H})$, then $|T|$ is a diagonalizable operator of finite rank, whence $|T|$ (and T) belongs to $\mathbf{B}^1(\mathfrak{H})$. If $T \in \mathbf{B}^1(\mathfrak{H})$, then by 3.2.11

$$T^*T = |T|^2 = |T|^{1/2}|T|\,|T|^{1/2} \leq \|T\|\,|T|,$$

which shows that $\operatorname{tr}(T^*T) < \infty$, i.e. $T \in \mathbf{B}^2(\mathfrak{H})$. The last assertion (used freely throughout the proof) is contained in 3.4.5. □

3.4.9. Theorem. *The ideal* $\mathbf{B}^2(\mathfrak{H})$ *of Hilbert–Schmidt operators form a Hilbert space under the inner product*

$$(S|T)_{\mathrm{tr}} = \operatorname{tr}(T^*S), \quad S, T \in \mathbf{B}^2(\mathfrak{H}).$$

PROOF. That $T^*S \in \mathbf{B}^1(\mathfrak{H})$ follows from (***) in 3.4.7. Thus the sesquilinear form $(\cdot|\cdot)_{\mathrm{tr}}$ is well-defined, self-adjoint, and positive. Moreover, it gives an inner product on $\mathbf{B}^2(\mathfrak{H})$ because the associated 2-norm satisfies

$$\|T\|_2^2 = \operatorname{tr}(T^*T) \geq \|T^*T\| = \|T\|^2,$$

by 3.4.4. This inequality also implies that every Cauchy sequence (T_n) in $\mathbf{B}^2(\mathfrak{H})$ for the 2-norm will converge in norm to an element T in $\mathbf{B}_0(\mathfrak{H})$. For every projection P on a finite-dimensional subspace of $\mathfrak{H}$ we estimate

$$\begin{aligned}\|P(T - T_n)\|_2^2 &= \operatorname{tr}((T - T_n)^* P(T - T_n)) = \operatorname{tr}(P(T - T_n)(T - T_n)^* P)\\ &= \lim_m \operatorname{tr}(P(T_m - T_n)(T_m - T_n)^* P)\end{aligned}$$

$$= \lim_m \operatorname{tr}((T_m - T_n)^* P(T_m - T_n))$$

$$\leq \limsup_m \operatorname{tr}((T_m - T_n)^*(T_m - T_n)) = \limsup_m \|T_m - T_n\|_2^2,$$

and, since P is arbitrary, we conclude that

$$\|T - T_n\|_2 \leq \limsup_m \|T_m - T_n\|_2;$$

which implies that $T \in \mathbf{B}^2(\mathfrak{H})$ and that $T_n \to T$ in 2-norm. □

3.4.10. Lemma. *If* $T \in \mathbf{B}^1(\mathfrak{H})$ *and* $S \in \mathbf{B}(\mathfrak{H})$, *then*

$$|\operatorname{tr}(ST)| \leq \|S\| \operatorname{tr}(|T|).$$

Proof. Let $T = U|T|$ be the polar decomposition of T (3.2.17). Then $(SU|T|^{1/2})^* \in \mathbf{B}^2(\mathfrak{H})$ [because $|T|^{1/2} \in \mathbf{B}^2(\mathfrak{H})$], so by the Cauchy–Schwarz inequality (for the trace) we have

$$\begin{aligned}|\operatorname{tr}(ST)|^2 &= |\operatorname{tr}(SU|T|^{1/2}|T|^{1/2})|^2 = |(|T|^{1/2}|(SU|T|^{1/2})^*)_{\operatorname{tr}}|^2 \\ &\leq \||T|^{1/2}\|_2^2 \|(SU|T|^{1/2})^*\|_2^2 = \operatorname{tr}(|T|)\operatorname{tr}(|T|^{1/2}U^*S^*SU|T|^{1/2}) \\ &\leq \operatorname{tr}(|T|)\operatorname{tr}(\|U^*S^*SU\||T|) \leq \|S\|^2(\operatorname{tr}(|T|))^2;\end{aligned}$$

using 3.4.3 and 3.2.9 on the way. □

3.4.11. Lemma. *If* S *and* T *belong to* $\mathbf{B}^2(\mathfrak{H})$, *then*

$$\operatorname{tr}(ST) = \operatorname{tr}(TS).$$

The same formula holds when $S \in \mathbf{B}(\mathfrak{H})$ *and* $T \in \mathbf{B}^1(\mathfrak{H})$.

Proof. The polarization identity [(***) in 3.4.7] in conjunction with 3.4.3 gives

$$\begin{aligned}4\operatorname{tr}(T^*S) &= \sum \mathrm{i}^k \operatorname{tr}((S + \mathrm{i}^k T)^*(S + \mathrm{i}^k T)) \\ &= \sum \mathrm{i}^k \operatorname{tr}((S^* + \mathrm{i}^{-k} T^*)^*(S^* + \mathrm{i}^{-k} T^*)) \\ &= \sum \mathrm{i}^k \operatorname{tr}((T^* + \mathrm{i}^k S^*)^*(T^* + \mathrm{i}^k S^*)) = 4\operatorname{tr}(ST^*),\end{aligned}$$

which proves the first assertion. For the second, we may assume that $T \geq 0$ (the equation is linear in T), and then from the first result we have

$$\begin{aligned}\operatorname{tr}(ST) &= \operatorname{tr}((ST^{1/2})T^{1/2}) = \operatorname{tr}(T^{1/2}(ST^{1/2})) \\ &= \operatorname{tr}((T^{1/2}S)T^{1/2}) = \operatorname{tr}(T^{1/2}(T^{1/2}S)) = \operatorname{tr}(TS).\end{aligned}$$ □

3.4.12. Theorem. *The ideal* $\mathbf{B}^1(\mathfrak{H})$ *of trace class operators form a Banach algebra under the norm*

$$\|T\|_1 = \operatorname{tr}(|T|), \quad T \in \mathbf{B}^1(\mathfrak{H}).$$

PROOF. Clearly $\|\cdot\|_1$ is a homogeneous function on $\mathbf{B}^1(\mathfrak{H})$, which is faithful because $\|\cdot\|_1 \geq \|\cdot\|$; cf. 3.4.4. To prove subadditivity take S and T in $\mathbf{B}^1(\mathfrak{H})$ with polar decomposition $S + T = W|S + T|$. Then by 3.4.10

$$\begin{aligned}\|S + T\|_1 &= \operatorname{tr}(W^*(S + T)) \leq |\operatorname{tr}(W^*S)| + |\operatorname{tr}(W^*T)| \\ &\leq \|W^*\|(\operatorname{tr}(|S|) + \operatorname{tr}(|T|)) \leq \|S\|_1 + \|T\|_1.\end{aligned}$$

The corresponding inequality for the product is obtained from the polar decomposition $ST = V|ST|$, which gives

$$\begin{aligned}\|ST\|_1 &= \operatorname{tr}(V^*ST) \leq \|V^*S\| \operatorname{tr}(|T|) \\ &\leq \||S|\| \operatorname{tr}(|T|) \leq \operatorname{tr}(|S|)\operatorname{tr}(|T|) = \|S\|_1 \|T\|_1.\end{aligned}$$

If (T_n) is a Cauchy sequence in $\mathbf{B}^1(\mathfrak{H})$ for the 1-norm it must converge in norm to an element T in $\mathbf{B}_0(\mathfrak{H})$. With polar decomposition $T - T_n = U|T - T_n|$ we have, for each finite-dimensional projection P on $\mathfrak{H}$, that

$$\begin{aligned}\operatorname{tr}(P|T - T_n|) &= \operatorname{tr}(PU^*(T - T_n)) \\ &= \lim_m \operatorname{tr}(PU^*(T_m - T_n)) \leq \limsup_m \|T_m - T_n\|_1,\end{aligned}$$

by 3.4.10, since $\|PU^*\| \leq 1$. Since P is arbitrary, we conclude that

$$\|T - T_n\|_1 \leq \limsup_m \|T_m - T_n\|_1,$$

which shows that $T \in \mathbf{B}^1(\mathfrak{H})$ and that $T_n \to T$ in 1-norm. □

3.4.13. Theorem. *The bilinear form*

$$\langle S, T\rangle = \operatorname{tr}(ST)$$

implements the dualities between the pair of Banach spaces $\mathbf{B}_0(\mathfrak{H})$ *and* $\mathbf{B}^1(\mathfrak{H})$ *and the pair* $\mathbf{B}^1(\mathfrak{H})$ *and* $\mathbf{B}(\mathfrak{H})$. *Thus, (with* $*$ *as in* 2.3.1)

$$(\mathbf{B}_0(\mathfrak{H}))^* = \mathbf{B}^1(\mathfrak{H}) \quad \textit{and} \quad (\mathbf{B}^1(\mathfrak{H}))^* = \mathbf{B}(\mathfrak{H}).$$

PROOF. Clearly every T in $\mathbf{B}^1(\mathfrak{H})$ gives rise to a bounded functional $\varphi_T = \langle\cdot, T\rangle$ on $\mathbf{B}_0(\mathfrak{H})$, and $\|\varphi_T\| \leq \|T\|_1$ by 3.4.10. Conversely, if $\varphi \in (\mathbf{B}_0(\mathfrak{H}))^*$, we take S in $\mathbf{B}^2(\mathfrak{H})$ and estimate

$$|\varphi(S)| \leq \|\varphi\| \|S\| \leq \|\varphi\| \|S\|_2.$$

Since $\mathbf{B}^2(\mathfrak{H})$ is a Hilbert space (3.4.9), there is by 3.1.9 a unique element T^* in $\mathbf{B}^2(\mathfrak{H})$ such that $\varphi(S) = \operatorname{tr}(TS) = \operatorname{tr}(ST)$ for all S in $\mathbf{B}^2(\mathfrak{H})$. However, for each projection P on $\mathfrak{H}$ of finite rank we have (with $T = U|T|$) that

$$|\operatorname{tr}(P|T|)| = |\operatorname{tr}(PU^*T)| = |\varphi(PU^*)| \leq \|\varphi\|.$$

Since P is arbitrary, this implies that $T \in \mathbf{B}^1(\mathfrak{H})$ with $\|T\|_1 \leq \|\varphi\|$. Evidently, the correspondence $\varphi \leftrightarrow T$ is a bijective isometry, whence $(\mathbf{B}_0(\mathfrak{H}))^* = \mathbf{B}^1(\mathfrak{H})$.

Clearly every S in $\mathbf{B}(\mathfrak{H})$ determines a bounded functional $\psi_S = \langle S, \cdot\rangle$ on

$\mathbf{B}^1(\mathfrak{H})$, and $\|\psi_S\| \le \|S\|$ by 3.4.10. Conversely, if $\psi \in (\mathbf{B}^1(\mathfrak{H}))^*$, we define a sesquilinear form B on $\mathfrak{H}$ by

$$B(x, y) = \psi(x \odot y), \quad x, y \in \mathfrak{H},$$

with $x \odot y$ as the rank one operator defined in 3.3.9. Straightforward computations show that

$$\begin{aligned}|x \odot y| &= ((x \odot y)^*(x \odot y))^{1/2} = ((y \odot x)(x \odot y))^{1/2} \\ &= (\|x\|^2 y \odot y)^{1/2} = \|x\|\,\|y\|(\|y\|^{-1} y \odot \|y\|^{-1} y);\end{aligned}$$

and, therefore, the form B is bounded, as

$$|B(x, y)| \le \|\psi\|\,\|x \odot y\|_1 = \|\psi\| \operatorname{tr}(|x \odot y|) = \|\psi\|\,\|x\|\,\|y\|.$$

By 3.2.2 there is then a unique operator S in $\mathbf{B}(\mathfrak{H})$ such that $\|S\| \le \|\psi\|$ and

$$\psi(x \odot y) = B(x, y) = (Sx|y).$$

Every self-adjoint T in $\mathbf{B}^1(\mathfrak{H})$ has a diagonal form $T = \sum \lambda_j e_j \odot e_j$ for some orthonormal basis $\{e_j | j \in J\}$ and real eigenvalues λ_j with $\sum |\lambda_j| = \|T\|_1$. Thus,

$$\begin{aligned}\psi(T) &= \sum \lambda_j \psi(e_j \odot e_j) = \sum \lambda_j (Se_j | e_j) \\ &= \sum (STe_j | e_j) = \operatorname{tr}(ST).\end{aligned}$$

Since $\mathbf{B}^1(\mathfrak{H})$ is self-adjoint, the formula $\psi(T) = \operatorname{tr}(ST)$ holds for all T; and again we have constructed a bijective isometry $\psi \leftrightarrow S$, so that $(\mathbf{B}^1(\mathfrak{H}))^* = \mathbf{B}(\mathfrak{H})$. □

3.4.14. Proposition. *For every orthonormal basis* $\{e_j | j \in J\}$ *in* $\mathfrak{H}$, *the set*

$$\{e_i \odot e_j | (i, j) \in J^2\}$$

of rank one operators form an orthonormal basis for $\mathbf{B}^2(\mathfrak{H})$.

PROOF. Since $(e_i \odot e_j)^* = e_j \odot e_i$ and $(e_i \odot e_j)(e_k \odot e_l) = \delta_{jk} e_i \odot e_l$, it is clear that the operators $e_i \odot e_j$ form an orthonormal set in $\mathbf{B}^2(\mathfrak{H})$. However, if $T \in \mathbf{B}^2(\mathfrak{H})$, then

$$\begin{aligned}(T | e_i \odot e_j)_{\mathrm{tr}} &= \operatorname{tr}((e_j \odot e_i)T) = \operatorname{tr}(e_j \odot T^* e_i) \\ &= \sum_l (e_l | T^* e_i)(e_j | e_l) = (Te_j | e_i).\end{aligned}$$

This shows that the orthogonal complement to the span of the $e_i \odot e_j$'s is $\{0\}$ which means that they form a basis. □

3.4.15. The result above gives a particularly concrete realization of the Hilbert–Schmidt operators in the case where the underlying Hilbert space has the form $L^2(X)$ with respect to some Radon integral $\int$ on a locally compact Hausdorff space X; see 6.1. If namely $\int \otimes \int$ denotes the product integral on X^2 (6.6.3), we consider the Hilbert space $L^2(X^2)$. If $\{e_j | j \in J\}$ is an orthonormal