Thus

$$u(x,t)=\sum_{m=1}^{\infty} A_m \cos mt \sin mx, \tag{8}$$

and note that this series converges absolutely. The solution can also be expressed in terms of traveling waves. In fact

$$u(x,t)=\frac{f(x+t)+f(x-t)}{2}. \tag{9}$$

Here $f(x)$ is defined for all x as follows: first, f is extended to $[-\pi,\pi]$ by making it odd, and then f is extended to the whole real line by making it periodic of period 2π, that is, $f(x+2\pi k)=f(x)$ for all integers k.

Observe that (8) implies (9) in view of the trigonometric identity

$$\cos v \sin u = \frac{1}{2}[\sin(u+v)+\sin(u-v)].$$

As a final remark, we should note an unsatisfactory aspect of the solution to this problem, which however is in the nature of things. Since the initial data $f(x)$ for the plucked string is not twice continuously differentiable, neither is the function u (given by (9)). Hence u is not truly a solution of the wave equation: while $u(x,t)$ does represent the position of the plucked string, it does not satisfy the partial differential equation we set out to solve! This state of affairs may be understood properly only if we realize that u does solve the equation, but in an appropriate generalized sense. A better understanding of this phenomenon requires ideas relevant to the study of "weak solutions" and the theory of "distributions." These topics we consider only later, in Books III and IV.

2 The heat equation

We now discuss the problem of heat diffusion by following the same framework as for the wave equation. First, we derive the time-dependent heat equation, and then study the steady-state heat equation in the disc, which leads us back to the basic question (7).

2.1 Derivation of the heat equation

Consider an infinite metal plate which we model as the plane $\mathbb{R}^2$, and suppose we are given an initial heat distribution at time $t=0$. Let the temperature at the point (x,y) at time t be denoted by $u(x,y,t)$.

Consider a small square centered at (x_0,y_0) with sides parallel to the axis and of side length h, as shown in Figure 9. The amount of heat energy in S at time t is given by

$$H(t)=\sigma \iint_S u(x,y,t)\,dx\,dy\,,$$

where $\sigma>0$ is a constant called the specific heat of the material. Therefore, the heat flow into S is

$$\frac{\partial H}{\partial t}=\sigma\iint_S \frac{\partial u}{\partial t}\,dx\,dy\,,$$

which is approximately equal to

$$\sigma h^2 \frac{\partial u}{\partial t}(x_0,y_0,t),$$

since the area of S is h^2. Now we apply Newton's law of cooling, which states that heat flows from the higher to lower temperature at a rate proportional to the difference, that is, the gradient.

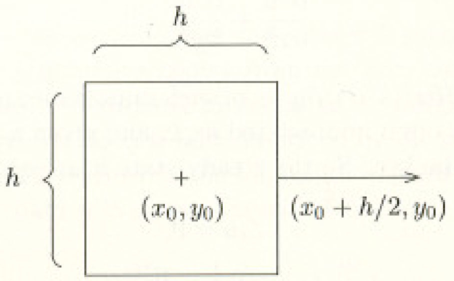

Figure 9. Heat flow through a small square

The heat flow through the vertical side on the right is therefore

$$-\kappa h \frac{\partial u}{\partial x}(x_0+h/2,y_0,t)\,,$$

where $\kappa>0$ is the conductivity of the material. A similar argument for the other sides shows that the total heat flow through the square S is given by

$$\kappa h\left[\frac{\partial u}{\partial x}(x_0+h/2,y_0,t)-\frac{\partial u}{\partial x}(x_0-h/2,y_0,t)\right.$$
$$\left.+\frac{\partial u}{\partial y}(x_0,y_0+h/2,t)-\frac{\partial u}{\partial y}(x_0,y_0-h/2,t)\right].$$

Applying the mean value theorem and letting h tend to zero, we find that

$$\frac{\sigma}{\kappa}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2};$$

this is called the **time-dependent heat equation**, often abbreviated to the heat equation.

2.2 Steady-state heat equation in the disc

After a long period of time, there is no more heat exchange, so that the system reaches thermal equilibrium and $\partial u/\partial t = 0$. In this case, the time-dependent heat equation reduces to the **steady-state heat equation**

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{10}$$

The operator $\partial^2/\partial x^2+\partial^2/\partial y^2$ is of such importance in mathematics and physics that it is often abbreviated as $\triangle$ and given a name: the Laplace operator or **Laplacian**. So the steady-state heat equation is written as

$$\triangle u = 0,$$

and solutions to this equation are called **harmonic functions**.

Consider the unit disc in the plane

$$D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\},$$

whose boundary is the unit circle C. In polar coordinates (r,θ), with $0\le r$ and $0\le\theta<2\pi$, we have

$$D=\{(r,\theta): 0\le r<1\} \quad\text{and}\quad C=\{(r,\theta): r=1\}.$$

The problem, often called the **Dirichlet problem** (for the Laplacian on the unit disc), is to solve the steady-state heat equation in the unit disc subject to the boundary condition $u=f$ on C. This corresponds to fixing a predetermined temperature distribution on the circle, waiting a long time, and then looking at the temperature distribution inside the disc.

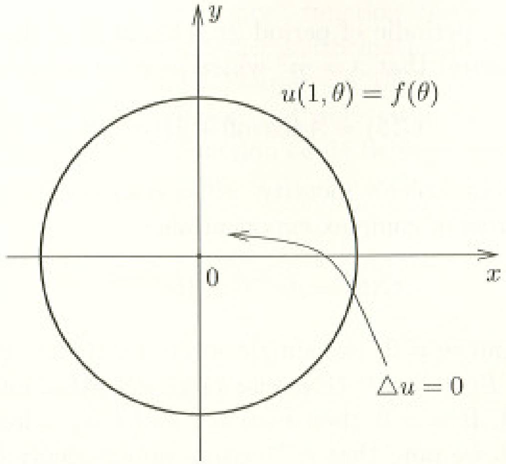

Figure 10. The Dirichlet problem for the disc

While the method of separation of variables will turn out to be useful for equation (10), a difficulty comes from the fact that the boundary condition is not easily expressed in terms of rectangular coordinates. Since this boundary condition is best described by the coordinates (r,θ), namely $u(1,\theta)=f(\theta)$, we rewrite the Laplacian in polar coordinates. An application of the chain rule gives (Exercise 10):

$$\triangle u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

We now multiply both sides by r^2, and since $\triangle u = 0$, we get

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}=-\frac{\partial^2 u}{\partial\theta^2}.$$

Separating these variables, and looking for a solution of the form $u(r,\theta)=F(r)G(\theta)$, we find

$$\frac{r^2F''(r)+rF'(r)}{F(r)}=-\frac{G''(\theta)}{G(\theta)}.$$

Since the two sides depend on different variables, they must both be constant, say equal to λ. We therefore get the following equations:

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0\,, \\ r^2 F''(r) + rF'(r) - \lambda F(r) = 0. \end{cases}$$

Since G must be periodic of period 2π, this implies that $\lambda \geq 0$ and (as we have seen before) that $\lambda = m^2$ where m is an integer; hence

$$G(\theta) = \tilde{A} \cos m\theta + \tilde{B} \sin m\theta.$$

An application of Euler's identity, $e^{ix} = \cos x + i \sin x$, allows one to rewrite G in terms of complex exponentials,

$$G(\theta) = Ae^{im\theta} + Be^{-im\theta}.$$

With $\lambda = m^2$ and $m \neq 0$, two simple solutions of the equation in F are $F(r) = r^m$ and $F(r) = r^{-m}$ (Exercise 11 gives further information about these solutions). If $m = 0$, then $F(r) = 1$ and $F(r) = \log r$ are two solutions. If $m > 0$, we note that r^{-m} grows unboundedly large as r tends to zero, so $F(r)G(\theta)$ is unbounded at the origin; the same occurs when $m = 0$ and $F(r) = \log r$. We reject these solutions as contrary to our intuition. Therefore, we are left with the following special functions:

$$u_m(r, \theta) = r^{|m|} e^{im\theta}, \qquad m \in \mathbb{Z}.$$

We now make the important observation that (10) is *linear*, and so as in the case of the vibrating string, we may superpose the above special solutions to obtain the presumed general solution:

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}.$$

If this expression gave all the solutions to the steady-state heat equation, then for a reasonable f we should have

$$u(1, \theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta).$$

We therefore ask again in this context: given any reasonable function f on $[0, 2\pi]$ with $f(0) = f(2\pi)$, can we find coefficients a_m so that

$$f(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \ ?$$

Historical Note: D'Alembert (in 1747) first solved the equation of the vibrating string using the method of traveling waves. This solution was elaborated by Euler a year later. In 1753, D. Bernoulli proposed the solution which for all intents and purposes is the Fourier series given by (4), but Euler was not entirely convinced of its full generality, since this could hold only if an "arbitrary" function could be expanded in Fourier series. D'Alembert and other mathematicians also had doubts. This viewpoint was changed by Fourier (in 1807) in his study of the heat equation, where his conviction and work eventually led others to a complete proof that a general function could be represented as a Fourier series.

3 Exercises

1. If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define

$$|z| = (x^2 + y^2)^{1/2}$$

and call this quantity the **modulus** or **absolute value** of z.

(a) What is the geometric interpretation of $|z|$?

(b) Show that if $|z| = 0$, then $z = 0$.

(c) Show that if $\lambda \in \mathbb{R}$, then $|\lambda z| = |\lambda||z|$, where $|\lambda|$ denotes the standard absolute value of a real number.

(d) If z_1 and z_2 are two complex numbers, prove that

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

(e) Show that if $z \neq 0$, then $|1/z| = 1/|z|$.

2. If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define the **complex conjugate** of z by

$$\overline{z} = x - iy.$$

(a) What is the geometric interpretation of $\overline{z}$?

(b) Show that $|z|^2 = z\overline{z}$.

(c) Prove that if z belongs to the unit circle, then $1/z = \overline{z}$.

Corollary 5.3 *If f is integrable on the circle and $\hat{f}(n) = 0$ for all n, then $f = 0$ at all points of continuity of f.*

The proof is immediate since all the partial sums are 0, hence all the Cesàro means are 0.

Corollary 5.4 *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

This means that if f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $\epsilon > 0$, then there exists a trigonometric polynomial P such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } -\pi \leq x \leq \pi.$$

This follows immediately from the theorem since the partial sums, hence the Cesàro means, are trigonometric polynomials. Corollary 5.4 is the periodic analogue of the Weierstrass approximation theorem for polynomials which can be found in Exercise 16.

5.3 Abel means and summation

Another method of summation was first considered by Abel and actually predates the Cesàro method.

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantities $A(r)$ are called the **Abel means** of the series. One can prove that if the series converges to s, then it is Abel summable to s. Moreover, the method of Abel summability is even more powerful than the Cesàro method: when the series is Cesàro summable, it is always Abel summable to the same sum. However, if we consider the series

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1),$$

then one can show that it is Abel summable to $1/4$ since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2},$$

but this series is not Cesàro summable; see Exercise 13.

5.4 The Poisson kernel and Dirichlet's problem in the unit disc

To adapt Abel summability to the context of Fourier series, we define the Abel means of the function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

Since the index n takes positive and negative values, it is natural to write $c_0 = a_0$, and $c_n = a_n e^{in\theta} + a_{-n} e^{-in\theta}$ for $n > 0$, so that the Abel means of the Fourier series correspond to the definition given in the previous section for numerical series.

We note that since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \leq r < 1$. Just as in the case of Cesàro means, the key fact is that these Abel means can be written as convolutions

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ is the **Poisson kernel** given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{4}$$

In fact,

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \, d\varphi \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi - \theta)} \right) d\varphi, \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series.

Lemma 5.5 *If $0 \leq r < 1$, then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

The Poisson kernel is a good kernel,[8] *as r tends to 1 from below.*

Proof. The identity $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$ has already been derived in Section 1.1. Note that

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta).$$

Hence if $1/2 \leq r \leq 1$ and $\delta \leq |\theta| \leq \pi$, then

$$1 - 2r\cos\theta + r^2 \geq c_\delta > 0.$$

Thus $P_r(\theta) \leq (1-r^2)/c_\delta$ when $\delta \leq |\theta| \leq \pi$, and the third property of good kernels is verified. Clearly $P_r(\theta) \geq 0$, and integrating the expression (4) term by term (which is justified by the absolute convergence of the series) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1,$$

thereby concluding the proof that P_r is a good kernel.

Combining this lemma with Theorem 4.1, we obtain our next result.

Theorem 5.6 *The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f.*

We now return to a problem discussed in Chapter 1, where we sketched the solution of the steady-state heat equation $\Delta u = 0$ in the unit disc with boundary condition $u = f$ on the circle. We expressed the Laplacian in terms of polar coordinates, separated variables, and expected that a solution was given by

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}, \tag{5}$$

where a_m was the m^{th} Fourier coefficient of f. In other words, we were led to take

$$u(r,\theta) = A_r(f)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)P_r(\theta - \varphi)\, d\varphi.$$

We are now in a position to show that this is indeed the case.

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[8] In this case, the family of kernels is indexed by a continuous parameter $0 \leq r < 1$, rather than the discrete n considered previously. In the definition of good kernels, we simply replace n by r and take the limit in property (c) appropriately, for example $r \to 1$ in this case.

Theorem 5.7 *Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by the Poisson integral*

$$u(r,\theta) = (f * P_r)(\theta) \tag{6}$$

has the following properties:

(i) *u has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.*

(ii) *If θ is any point of continuity of f, then*

$$\lim_{r\to 1} u(r,\theta) = f(\theta).$$

If f is continuous everywhere, then this limit is uniform.

(iii) *If f is continuous, then $u(r,\theta)$ is the unique solution to the steady-state heat equation in the disc which satisfies conditions* (i) *and* (ii).

Proof. For (i), we recall that the function u is given by the series (5). Fix $\rho < 1$; inside each disc of radius $r < \rho < 1$ centered at the origin, the series for u can be differentiated term by term, and the differentiated series is uniformly and absolutely convergent. Thus u can be differentiated twice (in fact infinitely many times), and since this holds for all $\rho < 1$, we conclude that u is twice differentiable inside the unit disc. Moreover, in polar coordinates,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

so term by term differentiation shows that $\Delta u = 0$.

The proof of (ii) is a simple application of the previous theorem. To prove (iii) we argue as follows. Suppose v solves the steady-state heat equation in the disc and converges to f uniformly as r tends to 1 from below. For each fixed r with $0 < r < 1$, the function $v(r,\theta)$ has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \quad \text{where} \quad a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta)e^{-in\theta}\, d\theta.$$

Taking into account that $v(r,\theta)$ solves the equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \tag{7}$$

we find that

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0. \tag{8}$$

Indeed, we may first multiply (7) by $e^{-in\theta}$ and integrate in θ. Then, since v is periodic, two integrations by parts give

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial\theta^2}(r,\theta)e^{-in\theta}\,d\theta = -n^2 a_n(r).$$

Finally, we may interchange the order of differentiation and integration, which is permissible since v has two continuous derivatives; this yields (8).

Therefore, we must have $a_n(r) = A_n r^n + B_n r^{-n}$ for some constants A_n and B_n, when $n \neq 0$ (see Exercise 11 in Chapter 1). To evaluate the constants, we first observe that each term $a_n(r)$ is bounded because v is bounded, therefore $B_n = 0$. To find A_n we let $r \to 1$. Since v converges uniformly to f as $r \to 1$ we find that

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta.$$

By a similar argument, this formula also holds when $n = 0$. Our conclusion is that for each $0 < r < 1$, the Fourier series of v is given by the series of $u(r,\theta)$, so by the uniqueness of Fourier series for continuous functions, we must have $u = v$.

Remark. By part (iii) of the theorem, we may conclude that if u solves $\triangle u = 0$ in the disc, and converges to 0 uniformly as $r \to 1$, then u must be identically 0. However, if uniform convergence is replaced by pointwise convergence, this conclusion may fail; see Exercise 18.

6 Exercises

1. Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1}[\hat{f}(n) + \hat{f}(-n)]\cos n\theta + i[\hat{f}(n) - \hat{f}(-n)]\sin n\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

3. We return to the problem of the plucked string discussed in Chapter 1. Show that the initial condition f is *equal* to its Fourier sine series

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{with} \quad A_m = \frac{2h}{m^2}\frac{\sin mp}{p(\pi - p)}.$$

[Hint: Note that $|A_m| \leq C/m^2$.]

4. Consider the 2π-periodic odd function defined on $[0,\pi]$ by $f(\theta) = \theta(\pi - \theta)$.

(a) Draw the graph of f.

(b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi}\sum_{k \text{ odd } \geq 1}\frac{\sin k\theta}{k^3}.$$

5. On the interval $[-\pi,\pi]$ consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta & \text{if } |\theta| \leq \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty}\frac{1 - \cos n\delta}{n^2\pi\delta}\cos n\theta.$$

6. Let f be the function defined on $[-\pi,\pi]$ by $f(\theta) = |\theta|$.

This function is complex-valued as opposed to the examples R and W above, and so the nowhere differentiability of f_α does not imply the same property for its real and imaginary parts. However, a small modification of our proof shows that, in fact, the real part of f_α,

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x,$$

as well as its imaginary part, are both nowhere differentiable. To see this, observe first that by the same proof, Lemma 3.2 has the following generalization: if g is a continuous function which is differentiable at x_0, then

$$\triangle_N(g)'(x_0 + h) = O(\log N) \quad \text{whenever } |h| \leq c/N.$$

We then proceed with $F(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n x$, noting as above that $\triangle_{2N}(F) - \triangle_N(F) = 2^{-n\alpha} \cos 2^n x$; as a result, assuming that F is differentiable at x_0, we get that

$$|2^{n(1-\alpha)} \sin(2^n(x_0 + h))| = O(\log N)$$

when $2N = 2^n$, and $|h| \leq c/N$. To get a contradiction, we need only choose h so that $|\sin(2^n(x_0 + h))| = 1$; this is accomplished by setting δ equal to the distance from $2^n x_0$ to the nearest number of the form $(k + 1/2)\pi$, $k \in \mathbb{Z}$ (so $\delta \leq \pi/2$), and taking $h = \pm\delta/2^n$.

Clearly, when $\alpha > 1$ the function f_α is continuously differentiable since the series can be differentiated term by term. Finally, the nowhere differentiability we have proved for $\alpha < 1$ actually extends to $\alpha = 1$ by a suitable refinement of the argument (see Problem 8 in Chapter 5). In fact, using these more elaborate methods one can also show that the Weierstrass function W is nowhere differentiable if $ab \geq 1$.

4 The heat equation on the circle

As a final illustration, we return to the original problem of heat diffusion considered by Fourier.

Suppose we are given an initial temperature distribution at $t = 0$ on a ring and that we are asked to describe the temperature at points on the ring at times $t > 0$.

The ring is modeled by the unit circle. A point on this circle is described by its angle $\theta = 2\pi x$, where the variable x lies between 0 and 1. If $u(x, t)$ denotes the temperature at time t of a point described by the angle θ, then considerations similar to the ones given in Chapter 1 show that u satisfies the differential equation

$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}. \tag{7}$$

The constant c is a positive physical constant which depends on the material of which the ring is made (see Section 2.1 in Chapter 1). After rescaling the time variable, we may assume that $c = 1$. If f is our initial data, we impose the condition

$$u(x, 0) = f(x).$$

To solve the problem, we separate variables and look for special solutions of the form

$$u(x, t) = A(x)B(t).$$

Then inserting this expression for u into the heat equation we get

$$\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)}.$$

Both sides are therefore constant, say equal to λ. Since A must be periodic of period 1, we see that the only possibility is $\lambda = -4\pi^2 n^2$, where $n \in \mathbb{Z}$. Then A is a linear combination of the exponentials $e^{2\pi i n x}$ and $e^{-2\pi i n x}$, and $B(t)$ is a multiple of $e^{-4\pi^2 n^2 t}$. By superposing these solutions, we are led to

$$u(x, t) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}, \tag{8}$$

where, setting $t = 0$, we see that $\{a_n\}$ are the Fourier coefficients of f.

Note that when f is Riemann integrable, the coefficients a_n are bounded, and since the factor $e^{-4\pi^2 n^2 t}$ tends to zero extremely fast, the series defining u converges. In fact, in this case, u is twice differentiable and solves equation (7).

The natural question with regard to the boundary condition is the following: do we have $u(x, t) \to f(x)$ as t tends to 0, and in what sense? A simple application of the Parseval identity shows that this limit holds in the mean square sense (Exercise 11). For a better understanding of the properties of our solution (8), we write it as

$$u(x, t) = (f * H_t)(x),$$

where H_t is the **heat kernel for the circle**, given by

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}, \tag{9}$$

and where the convolution for functions with period 1 is defined by

$$(f * g)(x) = \int_0^1 f(x-y)g(y)\,dy.$$

An analogy between the heat kernel and the Poisson kernel (of Chapter 2) is given in Exercise 12. However, unlike in the case of the Poisson kernel, there is no elementary formula for the heat kernel. Nevertheless, it turns out that it is a good kernel (in the sense of Chapter 2). The proof is not obvious and requires the use of the celebrated Poisson summation formula, which will be taken up in Chapter 5. As a corollary, we will also find that H_t is everywhere positive, a fact that is also not obvious from its defining expression (9). We can, however, give the following heuristic argument for the positivity of H_t. Suppose that we begin with an initial temperature distribution f which is everywhere ≤ 0. Then it is physically reasonable to expect $u(x,t) \leq 0$ for all t since heat travels from hot to cold. Now

$$u(x,t) = \int_0^1 f(x-y)H_t(y)\,dy.$$

If H_t is negative for some x_0, then we may choose $f \leq 0$ supported near x_0, and this would imply $u(x_0,t) > 0$, which is a contradiction.

5 Exercises

1. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a parametrization for the closed curve Γ.

(a) Prove that γ is a parametrization by arc-length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ is precisely $s-a$, that is,

$$\int_a^s |\gamma'(t)|\,dt = s - a.$$

(b) Prove that any curve Γ admits a parametrization by arc-length. [Hint: If η is any parametrization, let $h(s) = \int_a^s |\eta'(t)|\,dt$ and consider $\gamma = \eta \circ h^{-1}$.]

2. Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a parametrization for a closed curve Γ, with $\gamma(t) = (x(t), y(t))$.

(a) Show that

$$\frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,ds = \int_a^b x(s)y'(s)\,ds = -\int_a^b y(s)x'(s)\,ds.$$

(b) Define the **reverse parametrization** of γ by $\gamma^- : [a,b] \to \mathbb{R}^2$ with $\gamma^-(t) = \gamma(b+a-t)$. The image of γ^- is precisely Γ, except that the points $\gamma^-(t)$ and $\gamma(t)$ travel in opposite directions. Thus γ^- "reverses" the orientation of the curve. Prove that

$$\int_\gamma (x\,dy - y\,dx) = -\int_{\gamma^-} (x\,dy - y\,dx).$$

In particular, we may assume (after a possible change in orientation) that

$$\mathcal{A} = \frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,ds = \int_a^b x(s)y'(s)\,ds.$$

3. Suppose Γ is a curve in the plane, and that there exists a set of coordinates x and y so that the x-axis divides the curve into the union of the graph of two continuous functions $y = f(x)$ and $y = g(x)$ for $0 \leq x \leq 1$, and with $f(x) \geq g(x)$ (see Figure 6). Let Ω denote the region between the graphs of these two functions:

$$\Omega = \{(x,y) :\ 0 \leq x \leq 1 \text{ and } g(x) \leq y \leq f(x)\}.$$

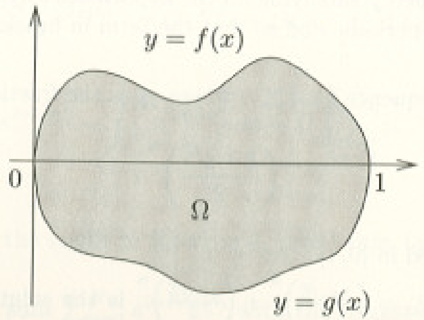

Figure 6. Simple version of the area formula

With the familiar interpretation that the integral $\int h(x)\,dx$ gives the area under the graph of the function h, we see that the area of Ω is $\int_0^1 f(x)\,dx -$