1 The gamma function

For $s > 0$, the **gamma function** is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt. \tag{1}$$

The integral converges for each positive s because near $t = 0$ the function t^{s-1} is integrable, and for t large the convergence is guaranteed by the exponential decay of the integrand. These observations allow us to extend the domain of definition of Γ as follows.

Proposition 1.1 *The gamma function extends to an analytic function in the half-plane* $\mathrm{Re}(s) > 0$, *and is still given there by the integral formula* (1).

Proof. It suffices to show that the integral defines a holomorphic function in every strip

$$S_{\delta,M} = \{\delta < \mathrm{Re}(s) < M\},$$

where $0 < \delta < M < \infty$. Note that if σ denotes the real part of s, then $|e^{-t}t^{s-1}| = e^{-t}t^{\sigma-1}$, so that the integral

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\, dt,$$

which is defined by the limit $\lim_{\epsilon\to 0}\int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}\, dt$, converges for each $s \in S_{\delta,M}$. For $\epsilon > 0$, let

$$F_\epsilon(s) = \int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}\, dt.$$

By Theorem 5.4 in Chapter 2, the function F_ϵ is holomorphic in the strip $S_{\delta,M}$. By Theorem 5.2, also of Chapter 2, it suffices to show that F_ϵ converges uniformly to Γ on the strip $S_{\delta,M}$. To see this, we first observe that

$$|\Gamma(s) - F_\epsilon(s)| \le \int_0^\epsilon e^{-t}t^{\sigma-1}\, dt + \int_{1/\epsilon}^\infty e^{-t}t^{\sigma-1}\, dt.$$

The first integral converges uniformly to 0, as ϵ tends to 0 since it can be easily estimated by ϵ^δ/δ whenever $0 < \epsilon < 1$. The second integral converges uniformly to 0 as well, since

$$\left|\int_{1/\epsilon}^\infty e^{-t}t^{\sigma-1}\, dt\right| \le \int_{1/\epsilon}^\infty e^{-t}t^{M-1} dt \le C\int_{1/\epsilon}^\infty e^{-t/2}\, dt \to 0,$$

and the proof is complete.

1.1 Analytic continuation

Despite the fact that the integral defining Γ is not absolutely convergent for other values of s, we can go further and prove that there exists a meromorphic function defined on all of $\mathbb{C}$ that equals Γ in the half-plane $\mathrm{Re}(s) > 0$. In the same sense as in Chapter 2, we say that this function is the analytic continuation[2] of Γ, and we therefore continue to denote it by Γ.

To prove the asserted analytic extension to a meromorphic function, we need a lemma, which incidentally exhibits an important property of Γ.

Lemma 1.2 *If* $\mathrm{Re}(s) > 0$, *then*

$$\Gamma(s+1) = s\Gamma(s). \tag{2}$$

As a consequence $\Gamma(n+1) = n!$ *for* $n = 0, 1, 2, \ldots$.

Proof. Integrating by parts in the finite integrals gives

$$\int_\epsilon^{1/\epsilon} \frac{d}{dt}(e^{-t}t^s)\, dt = -\int_\epsilon^{1/\epsilon} e^{-t}t^s\, dt + s\int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}\, dt,$$

and the desired formula (2) follows by letting ϵ tend to 0, and noting that the left-hand side vanishes because $e^{-t}t^s \to 0$ as t tends to 0 or ∞. Now it suffices to check that

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1,$$

and to apply (2) successively to find that $\Gamma(n+1) = n!$.

Formula (2) in the lemma is all we need to give a proof of the following theorem.

Theorem 1.3 *The function* $\Gamma(s)$ *initially defined for* $\mathrm{Re}(s) > 0$ *has an analytic continuation to a meromorphic function on* $\mathbb{C}$ *whose only singularities are simple poles at the negative integers* $s = 0, -1, \ldots$. *The residue of* Γ *at* $s = -n$ *is* $(-1)^n/n!$.

[2]Uniqueness of the analytic continuation is guaranteed since the complement of the poles of a meromorphic function forms a connected set.

Proof. It suffices to extend Γ to each half-plane $\text{Re}(s) > -m$, where $m \geq 1$ is an integer. For $\text{Re}(s) > -1$, we define

$$F_1(s) = \frac{\Gamma(s+1)}{s}.$$

Since $\Gamma(s+1)$ is holomorphic in $\text{Re}(s) > -1$, we see that F_1 is meromorphic in that half-plane, with the only possible singularity a simple pole at $s = 0$. The fact that $\Gamma(1) = 1$ shows that F_1 does in fact have a simple pole at $s = 0$ with residue 1. Moreover, if $\text{Re}(s) > 0$, then

$$F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s)$$

by the previous lemma. So F_1 extends Γ to a meromorphic function on the half-plane $\text{Re}(s) > -1$. We can now continue in this fashion by defining a meromorphic F_m for $\text{Re}(s) > -m$ that agrees with Γ on $\text{Re}(s) > 0$. For $\text{Re}(s) > -m$, where m is an integer ≥ 1, define

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}.$$

The function F_m is meromorphic in $\text{Re}(s) > -m$ and has simple poles at $s = 0, -1, -2, \ldots, -m+1$ with residues

$$\begin{aligned} \text{res}_{s=-n} F_m(s) &= \frac{\Gamma(-n+m)}{(m-1-n)!(-1)(-2)\cdots(-n)} \\ &= \frac{(m-n-1)!}{(m-1-n)!(-1)(-2)\cdots(-n)} \\ &= \frac{(-1)^n}{n!}. \end{aligned}$$

Successive applications of the lemma show that $F_m(s) = \Gamma(s)$ for $\text{Re}(s) > 0$. By uniqueness, this also means that $F_m = F_k$ for $1 \leq k \leq m$ on the domain of definition of F_k. Therefore, we have obtained the desired continuation of Γ.

Remark. We have already proved that $\Gamma(s+1) = s\Gamma(s)$ whenever $\text{Re}(s) > 0$. In fact, by analytic continuation, this formula remains true whenever $s \neq 0, -1, -2, \ldots$, that is, whenever s is not a pole of Γ. This is because both sides of the formula are holomorphic in the complement of the poles of Γ and are equal when $\text{Re}(s) > 0$. Actually, one can go further, and note that if s is a negative integer $s = -n$ with $n \geq 1$, then both sides of the formula are infinite and moreover

$$\text{res}_{s=-n}\Gamma(s+1) = -n \ \text{res}_{s=-n}\Gamma(s).$$

Finally, note that when $s = 0$ we have $\Gamma(1) = \lim_{s\to 0} s\Gamma(s)$.

An alternate proof of Theorem 1.3, which is interesting in its own right and whose ideas recur later, is obtained by splitting the integral for $\Gamma(s)$ defined on $\text{Re}(s) > 0$ as follows:

$$\Gamma(s) = \int_0^1 e^{-t}t^{s-1}\,dt + \int_1^\infty e^{-t}t^{s-1}\,dt.$$

The integral on the far right defines an entire function; also expanding e^{-t} in a power series and integrating term by term gives

$$\int_0^1 e^{-t}t^{s-1}\,dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}.$$

Therefore

$$(3) \qquad \Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t}t^{s-1}\,dt \quad \text{for } \text{Re}(s) > 0.$$

Finally, the series defines a meromorphic function on $\mathbb{C}$ with poles at the negative integers and residue $(-1)^n/n!$ at $s = -n$. To prove this, we argue as follows. For a fixed $R > 0$ we may split the sum in two parts

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} = \sum_{n=0}^N \frac{(-1)^n}{n!(n+s)} + \sum_{n=N+1}^\infty \frac{(-1)^n}{n!(n+s)},$$

where N is an integer chosen so that $N > 2R$. The first sum, which is finite, defines a meromorphic function in the disc $|s| < R$ with poles at the desired points and the correct residues. The second sum converges uniformly in that disc, hence defines a holomorphic function there, since $n > N > 2R$ and $|n+s| \geq R$ imply

$$\left|\frac{(-1)^n}{n!(n+s)}\right| \leq \frac{1}{n!R}.$$

Since R was arbitrary, we conclude that the series in (3) has the desired properties.

In particular, the relation (3) now holds on all of $\mathbb{C}$.

1.2 Further properties of Γ

The following identity reveals the symmetry of Γ about the line $\text{Re}(s) = 1/2$.

the fact that $\Gamma(1) = 1$ yields

$$\begin{aligned} e^{-A} &= \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n} \\ &= \lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)e^{-1/n} \\ &= \lim_{N\to\infty} e^{\sum_{n=1}^{N}[\log(1+1/n)-1/n]} \\ &= \lim_{N\to\infty} e^{-\left(\sum_{n=1}^{N}1/n\right)+\log N+\log(1+1/N)} \\ &= e^{-\gamma}. \end{aligned}$$

Therefore $A = \gamma + 2\pi ik$ for some integer k. Since $\Gamma(s)$ is real whenever s is real, we must have $k = 0$, and the argument is complete.

Note that the proof shows that the function $1/\Gamma$ is essentially characterized (up to two normalizing constants) as the entire function that has:

(i) simple zeros at $s = 0, -1, -2, \dots$ and vanishes nowhere else, and

(ii) order of growth ≤ 1.

Observe that $\sin\pi s$ has a similar characterization (except the zeros are now at *all* the integers). However, while $\sin\pi s$ has a stricter growth estimate of the form $\sin\pi s = O\left(e^{c|s|}\right)$, this estimate (without the logarithm in the exponent) does not hold for $1/\Gamma(s)$ as Exercise 12 demonstrates.

2 The zeta function

The Riemann **zeta function** is initially defined for real $s > 1$ by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

As in the case of the gamma function, ζ can be continued into the complex plane. There are several proofs of this fact, and we present in the next section the one that relies on the functional equation of ζ.

2.1 Functional equation and analytic continuation

In parallel to the gamma function, we first provide a simple extension of ζ to a half-plane in $\mathbb{C}$.

Proposition 2.1 *The series defining $\zeta(s)$ converges for* $\mathrm{Re}(s) > 1$, *and the function ζ is holomorphic in this half-plane.*

Proof. If $s = \sigma + it$ where σ and t are real, then

$$|n^{-s}| = |e^{-s\log n}| = e^{-\sigma\log n} = n^{-\sigma}.$$

As a consequence, if $\sigma > 1 + \delta > 1$ the series defining ζ is uniformly bounded by $\sum_{n=1}^{\infty} 1/n^{1+\delta}$, which converges. Therefore, the series $\sum 1/n^s$ converges uniformly on every half-plane $\mathrm{Re}(s) > 1 + \delta > 1$, and therefore defines a holomorphic function in $\mathrm{Re}(s) > 1$.

The analytic continuation of ζ to a meromorphic function in $\mathbb{C}$ is more subtle than in the case of the gamma function. The proof we present here relates ζ to Γ and another important function.

Consider the **theta function**, already introduced in Chapter 4, which is defined for real $t > 0$ by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

An application of the Poisson summation formula (Theorem 2.4 in Chapter 4) gave the functional equation satisfied by ϑ, namely

$$\vartheta(t) = t^{-1/2}\vartheta(1/t).$$

The growth and decay of ϑ we shall need are

$$\vartheta(t) \leq Ct^{-1/2} \quad \text{as } t \to 0,$$

and

$$|\vartheta(t) - 1| \leq Ce^{-\pi t} \quad \text{for some } C > 0, \text{ and all } t \geq 1.$$

The inequality for t tending to zero follows from the functional equation, while the behavior as t tends to infinity follows from the fact that

$$\sum_{n\geq 1} e^{-\pi n^2 t} \leq \sum_{n\geq 1} e^{-\pi n t} \leq Ce^{-\pi t}$$

for $t \geq 1$.

We are now in a position to prove an important relation among ζ, Γ and ϑ.

Theorem 2.2 *If* $\mathrm{Re}(s) > 1$, *then*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}[\vartheta(u) - 1]\, du.$$

Proof. This and further arguments are based on the observation that

$$(6) \qquad \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1}\, du = \pi^{-s/2}\Gamma(s/2)n^{-s}, \quad \text{if } n \geq 1.$$

Indeed, if we make the change of variables $u = t/\pi n^2$ in the integral, the left-hand side becomes

$$\left(\int_0^\infty e^{-t} t^{(s/2)-1}\, dt\right)(\pi n^2)^{-s/2},$$

which is precisely $\pi^{-s/2}\Gamma(s/2)n^{-s}$. Next, note that

$$\frac{\vartheta(u) - 1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 u}.$$

The estimates for ϑ given before the statement of the theorem justify an interchange of the infinite sum with the integral, and thus

$$\begin{aligned}\frac{1}{2}\int_0^\infty u^{(s/2)-1}[\vartheta(u) - 1]\, du &= \sum_{n=1}^\infty \int_0^\infty u^{(s/2)-1} e^{-\pi n^2 u}\, du \\ &= \pi^{-s/2}\Gamma(s/2)\sum_{n=1}^\infty n^{-s} \\ &= \pi^{-s/2}\Gamma(s/2)\zeta(s),\end{aligned}$$

as was to be shown.

In view of this, we now consider the modification of the ζ function called the **xi function**, which makes the former appear more symmetric. It is defined for $\mathrm{Re}(s) > 1$ by

$$(7) \qquad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Theorem 2.3 *The function* ξ *is holomorphic for* $\mathrm{Re}(s) > 1$ *and has an analytic continuation to all of* $\mathbb{C}$ *as a meromorphic function with simple poles at* $s = 0$ *and* $s = 1$. *Moreover,*

$$\xi(s) = \xi(1 - s) \quad \textit{for all } s \in \mathbb{C}.$$

Proof. The idea of the proof is to use the functional equation for ϑ, namely

$$\sum_{n=-\infty}^\infty e^{-\pi n^2 u} = u^{-1/2}\sum_{n=-\infty}^\infty e^{-\pi n^2/u}, \qquad u > 0.$$

We then could multiply both sides by $u^{(s/2)-1}$ and try to integrate in u. Disregarding the terms corresponding to $n = 0$ (which produce infinities in both sums), we would get the desired equality once we invoked formula (6), and the parallel formula obtained by making the change of variables $u \mapsto 1/u$. The actual proof requires a little more work and goes as follows.

Let $\psi(u) = [\vartheta(u) - 1]/2$. The functional equation for the theta function, namely $\vartheta(u) = u^{-1/2}\vartheta(1/u)$, implies

$$\psi(u) = u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}.$$

Now, by Theorem 2.2 for $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^\infty u^{(s/2)-1}\psi(u)\, du \\ &= \int_0^1 u^{(s/2)-1}\psi(u)\, du + \int_1^\infty u^{(s/2)-1}\psi(u)\, du \\ &= \int_0^1 u^{(s/2)-1}\left[u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}\right] du + \\ &\quad + \int_1^\infty u^{(s/2)-1}\psi(u)\, du \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{(-s/2)-1/2} + u^{(s/2)-1}\right)\psi(u)\, du\end{aligned}$$

whenever $\mathrm{Re}(s) > 1$. Therefore

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{(-s/2)-1/2} + u^{(s/2)-1}\right)\psi(u)\, du.$$

Since the function ψ has exponential decay at infinity, the integral above defines an entire function, and we conclude that ξ has an analytic continuation to all of $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Moreover, it is immediate that the integral remains unchanged if we replace s by $1-s$, and the same is true for the sum of the two terms $1/(s-1) - 1/s$. We conclude that $\xi(s) = \xi(1-s)$ as was to be shown.

From the identity we have proved for ξ we obtain the desired result for the zeta function: its analytic continuation and its functional equation.

Theorem 2.4 *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.*

Proof. A look at (7) provides the meromorphic continuation of ζ, namely

$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}.$$

Recall that $1/\Gamma(s/2)$ is entire with simple zeros at $0, -2, -4, \ldots$, so the simple pole of $\xi(s)$ at the origin is cancelled by the corresponding zero of $1/\Gamma(s/2)$. As a consequence, the only singularity of ζ is a simple pole at $s = 1$.

We shall now present a more elementary approach to the analytic continuation of the zeta function, which easily leads to its extension in the half-plane $\text{Re}(s) > 0$. This method will be useful in studying the growth properties of ζ near the line $\text{Re}(s) = 1$ (which will be needed in the next chapter). The idea behind it is to compare the sum $\sum_{n=1}^{\infty} n^{-s}$ with the integral $\int_1^{\infty} x^{-s}\,dx$.

Proposition 2.5 *There is a sequence of entire functions $\{\delta_n(s)\}_{n=1}^{\infty}$ that satisfy the estimate $|\delta_n(s)| \le |s|/n^{\sigma+1}$, where $s = \sigma + it$, and such that*

$$\sum_{1\le n<N}\frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\le n<N}\delta_n(s), \tag{8}$$

whenever N is an integer > 1.

This proposition has the following consequence.

Corollary 2.6 *For $\text{Re}(s) > 0$ we have*

$$\zeta(s) - \frac{1}{s-1} = H(s),$$

where $H(s) = \sum_{n=1}^{\infty}\delta_n(s)$ is holomorphic in the half-plane $\text{Re}(s) > 0$.

To prove the proposition we compare $\sum_{1\le n<N} n^{-s}$ with $\sum_{1\le n<N}\int_n^{n+1} x^{-s}\,dx$, and set

$$\delta_n(s) = \int_n^{n+1}\left[\frac{1}{n^s} - \frac{1}{x^s}\right]dx. \tag{9}$$

The mean-value theorem applied to $f(x) = x^{-s}$ yields

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \le \frac{|s|}{n^{\sigma+1}}, \quad \text{whenever } n \le x \le n+1.$$

Therefore $|\delta_n(s)| \le |s|/n^{\sigma+1}$, and since

$$\int_1^N \frac{dx}{x^s} = \sum_{1\le n<N}\int_n^{n+1}\frac{dx}{x^s},$$

the proposition is proved.

Turning to the corollary, we assume first that $\text{Re}(s) > 1$. We let N tend to infinity in formula (8) of the proposition, and observe that by the estimate $|\delta_n(s)| \le |s|/n^{\sigma+1}$ we have the uniform convergence of the series $\sum \delta_n(s)$ (in any half-plane $\text{Re}(s) \ge \delta$ when $\delta > 0$). Since $\text{Re}(s) > 1$, the series $\sum n^{-s}$ converges to $\zeta(s)$, and this proves the assertion when $\text{Re}(s) > 1$. The uniform convergence also shows that $\sum \delta_n(s)$ is holomorphic when $\text{Re}(s) > 0$, and thus shows that $\zeta(s)$ is extendable to that half-plane, and that the identity continues to hold there.

Remark. The idea described above can be developed step by step to yield the continuation of ζ into the entire complex plane, as shown in Problems 2 and 3. Another argument giving the full analytic continuation of ζ is outlined in Exercises 15 and 16.

As an application of the proposition we can show that the growth of $\zeta(s)$ near the line $\text{Re}(s) = 1$ is "mild." Recall that when $\text{Re}(s) > 1$, we have $|\zeta(s)| \le \sum_{n=1}^{\infty} n^{-\sigma}$, and so $\zeta(s)$ is bounded in any half-plane $\text{Re}(s) \ge 1 + \delta$, with $\delta > 0$. We shall see that on the line $\text{Re}(s) = 1$, $|\zeta(s)|$ is majorized by $|t|^{\epsilon}$, for every $\epsilon > 0$, and that the growth near the line is not much worse. The estimates below are not optimal. In fact, they are rather crude but suffice for what is needed later on.

Proposition 2.7 *Suppose $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Then for each σ_0, $0 \le \sigma_0 \le 1$, and every $\epsilon > 0$, there exists a constant c_ϵ so that*

(i) $|\zeta(s)| \le c_\epsilon |t|^{1-\sigma_0+\epsilon}$, *if $\sigma_0 \le \sigma$ and $|t| \ge 1$.*

following function:

$$\pi(x) = \text{number of primes less than or equal to } x.$$

The erratic growth of the function $\pi(x)$ gives little hope of finding a simple formula for it. Instead, one is led to study the asymptotic behavior of $\pi(x)$ as x becomes large. About 60 years after Euler's discovery, Legendre and Gauss observed after numerous calculations that it was likely that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty. \tag{1}$$

(The asymptotic relation $f(x) \sim g(x)$ as $x \to \infty$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.) Another 60 years later, shortly before Riemann's work, Tchebychev proved by elementary methods (and in particular, without the zeta function) the weaker result that

$$\pi(x) \approx \frac{x}{\log x} \quad \text{as } x \to \infty. \tag{2}$$

Here, by definition, the symbol $\approx$ means that there are positive constants $A < B$ such that

$$A\frac{x}{\log x} \leq \pi(x) \leq B\frac{x}{\log x}$$

for all sufficiently large x.

In 1896, about 40 years after Tchebychev's result, Hadamard and de la Vallée Poussin gave a proof of the validity of the relation (1). Their result is known as the prime number theorem. The original proofs of this theorem, as well as the one we give below, use complex analysis. We should remark that since then other proofs have been found, some depending on complex analysis, and others more elementary in nature.

At the heart of the proof of the prime number theorem that we give below lies the fact that $\zeta(s)$ does not vanish on the line $\text{Re}(s) = 1$. In fact, it can be shown that these two propositions are equivalent.

1 Zeros of the zeta function

We have seen in Theorem 1.10, Chapter 8 in Book I, Euler's identity, which states that for $\text{Re}(s) > 1$ the zeta function can be expressed as an infinite product

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}.$$

For the sake of completeness we provide a proof of the above identity. The key observation is that $1/(1-p^{-s})$ can be written as a convergent (geometric) power series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}} + \cdots,$$

and taking formally the product of these series over all primes p, yields the desired result. A precise argument goes as follows.

Suppose M and N are positive integers with $M > N$. Observe now that, by the fundamental theorem of arithmetic,[1] any positive integer $n \leq N$ can be written uniquely as a product of primes, and that each prime that occurs in the product must be less than or equal to N and repeated less than M times. Therefore

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p\leq N}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right)\\ &\leq \prod_{p\leq N}\left(\frac{1}{1-p^{-s}}\right)\\ &\leq \prod_{p}\left(\frac{1}{1-p^{-s}}\right).\end{aligned}$$

Letting N tend to infinity in the series now yields

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p}\left(\frac{1}{1-p^{-s}}\right).$$

For the reverse inequality, we argue as follows. Again, by the fundamental theorem of arithmetic, we find that

$$\prod_{p\leq N}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \leq \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Letting M tend to infinity gives

$$\prod_{p\leq N}\left(\frac{1}{1-p^{-s}}\right) \leq \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

[1] A proof of this elementary (but essential) fact is given in the first section of Chapter 8 in Book I.

Hence

$$\prod_p \left(\frac{1}{1-p^{-s}}\right) \le \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and the proof of the product formula for ζ is complete.

From the product formula we see, by Proposition 3.1 in Chapter 5, that $\zeta(s)$ does not vanish when $\mathrm{Re}(s) > 1$.

To obtain further information about the location of the zeros of ζ, we use the functional equation that provided the analytic continuation of ζ. We may write the fundamental relation $\xi(s) = \xi(1-s)$ in the form

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

and therefore

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

Now observe that for $\mathrm{Re}(s) < 0$ the following are true:

(i) $\zeta(1-s)$ has no zeros because $\mathrm{Re}(1-s) > 1$.

(ii) $\Gamma((1-s)/2)$ is zero free.

(iii) $1/\Gamma(s/2)$ has zeros at $s = -2, -4, -6, \ldots$.

Therefore, the only zeros of ζ in $\mathrm{Re}(s) < 0$ are located at the negative even integers $-2, -4, -6, \ldots$.

This proves the following theorem.

Theorem 1.1 *The only zeros of ζ outside the strip $0 \le \mathrm{Re}(s) \le 1$ are at the negative even integers, $-2, -4, -6, \ldots$.*

The region that remains to be studied is called the **critical strip**, $0 \le \mathrm{Re}(s) \le 1$. A key fact in the proof of the prime number theorem is that ζ has no zeros on the line $\mathrm{Re}(s) = 1$. As a simple consequence of this fact and the functional equation, it follows that ζ has no zeros on the line $\mathrm{Re}(s) = 0$.

In the seminal paper where Riemann introduced the analytic continuation of the ζ function and proved its functional equation, he applied these insights to the theory of prime numbers, and wrote down "explicit" formulas for determining the distribution of primes. While he did not succeed in fully proving and exploiting his assertions, he did initiate many important new ideas. His analysis led him to believe the truth of what has since been called the **Riemann hypothesis**:

The zeros of $\zeta(s)$ in the critical strip lie on the line $\mathrm{Re}(s) = 1/2$.

He said about this: "It would certainly be desirable to have a rigorous demonstration of this proposition; nevertheless I have for the moment set this aside, after several quick but unsuccessful attempts, because it seemed unneeded for the immediate goal of my study." Although much of the theory and numerical results point to the validity of this hypothesis, a proof or a counter-example remains to be discovered. The Riemann hypothesis is today one of mathematics' most famous unresolved problems.

In particular, it is for this reason that the zeros of ζ located outside the critical strip are sometimes called the **trivial zeros** of the zeta function. See also Exercise 5 for an argument proving that ζ has no zeros on the real segment, $0 \le \sigma \le 1$, where $s = \sigma + it$.

In the rest of this section we shall restrict ourselves to proving the following theorem, together with related estimates on ζ, which we shall use in the proof of the prime number theorem.

Theorem 1.2 *The zeta function has no zeros on the line* $\mathrm{Re}(s) = 1$.

Of course, since we know that ζ has a pole at $s = 1$, there are no zeros in a neighborhood of this point, but what we need is the deeper property that

$$\zeta(1+it) \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

The next sequence of lemmas gathers the necessary ingredients for the proof of Theorem 1.2.

Lemma 1.3 *If* $\mathrm{Re}(s) > 1$, *then*

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

for some $c_n \ge 0$.

Proof. Suppose first that $s > 1$. Taking the logarithm of the Euler product formula, and using the power series expansion for the logarithm

$$\log\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m},$$