Consider an infinite rod, which we model by the real line, and suppose that we are given an initial temperature distribution $f(x)$ on the rod at time $t = 0$. We wish now to determine the temperature $u(x,t)$ at a point x at time $t > 0$. Considerations similar to the ones given in Chapter 1 show that when u is appropriately normalized, it solves the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{8}$$

called the **heat equation**. The initial condition we impose is $u(x,0) = f(x)$.

Just as in the case of the circle, the solution is given in terms of a convolution. Indeed, define the **heat kernel** of the line by

$$\mathcal{H}_t(x) = K_\delta(x), \qquad \text{with } \delta = 4\pi t,$$

so that

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \quad \text{and} \quad \hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 t\xi^2}.$$

Taking the Fourier transform of equation (8) in the x variable (formally) leads to

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = -4\pi^2\xi^2\hat{u}(\xi,t).$$

Fixing ξ, this is an ordinary differential equation in the variable t (with unknown $\hat{u}(\xi,\cdot)$), so there exists a constant $A(\xi)$ so that

$$\hat{u}(\xi,t) = A(\xi)e^{-4\pi^2\xi^2 t}.$$

We may also take the Fourier transform of the initial condition and obtain $\hat{u}(\xi,0) = \hat{f}(\xi)$, hence $A(\xi) = \hat{f}(\xi)$. This leads to the following theorem.

Theorem 2.1 *Given $f \in \mathcal{S}(\mathbb{R})$, let*

$$u(x,t) = (f * \mathcal{H}_t)(x) \qquad \textit{for } t > 0$$

where $\mathcal{H}_t$ is the heat kernel. Then:

(i) *The function u is C^2 when $x \in \mathbb{R}$ and $t > 0$, and u solves the heat equation.*

(ii) *$u(x,t) \to f(x)$ uniformly in x as $t \to 0$. Hence if we set $u(x,0) = f(x)$, then u is continuous on the closure of the upper half-plane $\overline{\mathbb{R}^2_+} = \{(x,t) : x \in \mathbb{R},\ t \geq 0\}$.*

(iii) *$\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2\,dx \to 0$ as $t \to 0$.*

Proof. Because $u = f * \mathcal{H}_t$, taking the Fourier transform in the x-variable gives $\hat{u} = \hat{f}\hat{\mathcal{H}}_t$, and so $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-4\pi^2\xi^2 t}$. The Fourier inversion formula gives

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-4\pi^2 t\xi^2}e^{2\pi i\xi x}\,d\xi.$$

By differentiating under the integral sign, one verifies (i). In fact, one observes that u is indefinitely differentiable. Note that (ii) is an immediate consequence of Corollary 1.7. Finally, by Plancherel's formula, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2\,dx &= \int_{-\infty}^{\infty} |\hat{u}(\xi,t) - \hat{f}(\xi)|^2\,d\xi \\ &= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,|e^{-4\pi^2 t\xi^2} - 1|\,d\xi.\end{aligned}$$

To see that this last integral goes to 0 as $t \to 0$, we argue as follows: since $|e^{-4\pi^2 t\xi^2} - 1| \leq 2$ and $f \in \mathcal{S}(\mathbb{R})$, we can find N so that

$$\int_{|\xi|\geq N} |\hat{f}(\xi)|^2|e^{-4\pi^2 t\xi^2} - 1|\,d\xi < \epsilon,$$

and for all small t we have $\sup_{|\xi|\leq N} |\hat{f}(\xi)|^2|e^{-4\pi^2 t\xi^2} - 1| < \epsilon/2N$ since $\hat{f}$ is bounded. Thus

$$\int_{|\xi|\leq N} |\hat{f}(\xi)|^2\,|e^{-4\pi^2 t\xi^2} - 1|\,d\xi < \epsilon \quad \text{for all small } t.$$

This completes the proof of the theorem.

The above theorem guarantees the existence of a solution to the heat equation with initial data f. This solution is also unique, if uniqueness is formulated appropriately. In this regard, we note that $u = f * \mathcal{H}_t$, $f \in \mathcal{S}(\mathbb{R})$, satisfies the following additional property.

Corollary 2.2 *$u(\cdot,t)$ belongs to $\mathcal{S}(\mathbb{R})$ uniformly in t, in the sense that for any $T > 0$*

$$\sup_{\substack{x\in\mathbb{R} \\ 0<t<T}} |x|^k \left|\frac{\partial^\ell}{\partial x^\ell}u(x,t)\right| < \infty \qquad \textit{for each } k,\ell \geq 0. \tag{9}$$

Proof. This result is a consequence of the following estimate:

$$\begin{aligned}|u(x,t)| &\leq \int_{|y|\leq|x|/2} |f(x-y)|\mathcal{H}_t(y)\,dy + \int_{|y|\geq|x|/2} |f(x-y)|\mathcal{H}_t(y)\,dy \\ &\leq \frac{C_N}{(1+|x|)^N} + \frac{C}{\sqrt{t}}e^{-cx^2/t}.\end{aligned}$$

Indeed, since f is rapidly decreasing, we have $|f(x-y)| \leq C_N/(1+|x|)^N$ when $|y| \leq |x|/2$. Also, if $|y| \geq |x|/2$ then $\mathcal{H}_t(y) \leq Ct^{-1/2}e^{-cx^2/t}$, and we obtain the above inequality. Consequently, we see that $u(x,t)$ is rapidly decreasing uniformly for $0 < t < T$.

The same argument can be applied to the derivatives of u in the x variable since we may differentiate under the integral sign and apply the above estimate with f replaced by f', and so on.

This leads to the following uniqueness theorem.

Theorem 2.3 *Suppose $u(x,t)$ satisfies the following conditions:*

(i) *u is continuous on the closure of the upper half-plane.*

(ii) *u satisfies the heat equation for $t > 0$.*

(iii) *u satisfies the boundary condition $u(x,0) = 0$.*

(iv) *$u(\cdot,t) \in \mathcal{S}(\mathbb{R})$ uniformly in t, as in (9).*

Then, we conclude that $u = 0$.

Below we use the abbreviations $\partial_x^\ell u$ and $\partial_t u$ to denote $\partial^\ell u/\partial x^\ell$ and $\partial u/\partial t$, respectively.

Proof. We define the energy at time t of the solution $u(x,t)$ by

$$E(t) = \int_{\mathbb{R}} |u(x,t)|^2\,dx.$$

Clearly $E(t) \geq 0$. Since $E(0) = 0$ it suffices to show that E is a decreasing function, and this is achieved by proving that $dE/dt \leq 0$. The assumptions on u allow us to differentiate $E(t)$ under the integral sign

$$\frac{dE}{dt} = \int_{\mathbb{R}} [\partial_t u(x,t)\overline{u}(x,t) + u(x,t)\partial_t\overline{u}(x,t)]\,dx.$$

But u satisfies the heat equation, therefore $\partial_t u = \partial_x^2 u$ and $\partial_t \overline{u} = \partial_x^2\overline{u}$, so that after an integration by parts, where we use the fact that u and its

x derivatives decrease rapidly as $|x| \to \infty$, we find

$$\begin{aligned}\frac{dE}{dt} &= \int_{\mathbb{R}} [\partial_x^2 u(x,t)\overline{u}(x,t) + u(x,t)\partial_x^2\overline{u}(x,t)]\,dx \\ &= -\int_{\mathbb{R}} [\partial_x u(x,t)\partial_x\overline{u}(x,t) + \partial_x u(x,t)\partial_x\overline{u}(x,t)]\,dx \\ &= -2\int_{\mathbb{R}} |\partial_x u(x,t)|^2\,dx \\ &\leq 0,\end{aligned}$$

as claimed. Thus $E(t) = 0$ for all t, hence $u = 0$.

Another uniqueness theorem for the heat equation, with a less restrictive assumption than (9), can be found in Problem 6. Examples when uniqueness fails are given in Exercise 12 and Problem 4.

2.2 The steady-state heat equation in the upper half-plane

The equation we are now concerned with is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{10}$$

in the upper half-plane $\mathbb{R}_+^2 = \{(x,y) : x \in \mathbb{R},\ y > 0\}$. The boundary condition we require is $u(x,0) = f(x)$. The operator $\triangle$ is the Laplacian and the above partial differential equation describes the steady-state heat distribution in $\mathbb{R}_+^2$ subject to $u = f$ on the boundary. The kernel that solves this problem is called the **Poisson kernel** for the upper half-plane, and is given by

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \quad \text{where } x \in \mathbb{R} \text{ and } y > 0.$$

This is the analogue of the Poisson kernel for the disc discussed in Section 5.4 of Chapter 2.

Note that for each fixed y the kernel $\mathcal{P}_y$ is only of moderate decrease as a function of x, so we will use the theory of the Fourier transform appropriate for these types of functions (see Section 1.7).

We proceed as in the case of the time-dependent heat equation, by taking the Fourier transform of equation (10) (formally) in the x variable, thereby obtaining

$$-4\pi^2\xi^2\hat{u}(\xi,y) + \frac{\partial^2\hat{u}}{\partial y^2}(\xi,y) = 0$$

with the boundary condition $\hat{u}(\xi, 0) = \hat{f}(\xi)$. The general solution of this ordinary differential equation in y (with ξ fixed) takes the form

$$\hat{u}(\xi, y) = A(\xi)e^{-2\pi|\xi|y} + B(\xi)e^{2\pi|\xi|y}.$$

If we disregard the second term because of its rapid exponential increase we find, after setting $y = 0$, that

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-2\pi|\xi|y}.$$

Therefore u is given in terms of the convolution of f with a kernel whose Fourier transform is $e^{-2\pi|\xi|y}$. This is precisely the Poisson kernel given above, as we prove next.

Lemma 2.4 *The following two identities hold:*

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y}e^{2\pi i\xi x}\,d\xi = \mathcal{P}_y(x),$$

$$\int_{-\infty}^{\infty} \mathcal{P}_y(x)e^{-2\pi ix\xi}\,dx = e^{-2\pi|\xi|y}.$$

Proof. The first formula is fairly straightforward since we can split the integral from $-\infty$ to 0 and 0 to ∞. Then, since $y > 0$ we have

$$\int_0^{\infty} e^{-2\pi\xi y}e^{2\pi i\xi x}\,d\xi = \int_0^{\infty} e^{2\pi i(x+iy)\xi}\,d\xi = \left[\frac{e^{2\pi i(x+iy)\xi}}{2\pi i(x+iy)}\right]_0^{\infty} = -\frac{1}{2\pi i(x+iy)},$$

and similarly,

$$\int_{-\infty}^{0} e^{2\pi\xi y}e^{2\pi i\xi x}\,d\xi = \frac{1}{2\pi i(x-iy)}.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y}e^{2\pi i\xi x}\,d\xi = \frac{1}{2\pi i(x-iy)} - \frac{1}{2\pi i(x+iy)} = \frac{y}{\pi(x^2+y^2)}.$$

The second formula is now a consequence of the Fourier inversion theorem applied in the case when f and $\hat{f}$ are of moderate decrease.

Lemma 2.5 *The Poisson kernel is a good kernel on $\mathbb{R}$ as $y \to 0$.*

Proof. Setting $\xi = 0$ in the second formula of the lemma shows that $\int_{-\infty}^{\infty} \mathcal{P}_y(x)\,dx = 1$, and clearly $\mathcal{P}_y(x) \geq 0$, so it remains to check the last property of good kernels. Given a fixed $\delta > 0$, we may change variables $u = x/y$ so that

$$\int_{\delta}^{\infty} \frac{y}{x^2+y^2}\,dx = \int_{\delta/y}^{\infty} \frac{du}{1+u^2} = [\arctan u]_{\delta/y}^{\infty} = \pi/2 - \arctan(\delta/y),$$

and this quantity goes to 0 as $y \to 0$. Since $\mathcal{P}_y(x)$ is an even function, the proof is complete.

The following theorem establishes the existence of a solution to our problem.

Theorem 2.6 *Given $f \in \mathcal{S}(\mathbb{R})$, let $u(x,y) = (f * \mathcal{P}_y)(x)$. Then:*

(i) *$u(x,y)$ is C^2 in $\mathbb{R}^2_+$ and $\triangle u = 0$.*

(ii) *$u(x,y) \to f(x)$ uniformly as $y \to 0$.*

(iii) *$\int_{-\infty}^{\infty} |u(x,y) - f(x)|^2\,dx \to 0$ as $y \to 0$.*

(iv) *If $u(x,0) = f(x)$, then u is continuous on the closure $\overline{\mathbb{R}^2_+}$ of the upper half-plane, and vanishes at infinity in the sense that*

$$u(x,y) \to 0 \quad \text{as } |x| + y \to \infty.$$

Proof. The proofs of parts (i), (ii), and (iii) are similar to the case of the heat equation, and so are left to the reader. Part (iv) is a consequence of two easy estimates whenever f is of moderate decrease. First, we have

$$|(f * \mathcal{P}_y)(x)| \leq C\left(\frac{1}{(1+x^2)} + \frac{y}{x^2+y^2}\right)$$

which is proved (as in the case of the heat equation) by splitting the integral $\int_{-\infty}^{\infty} f(x-t)\mathcal{P}_y(t)\,dt$ into the part where $|t| \leq |x|/2$ and the part where $|t| \geq |x|/2$. Also, we have $|(f * \mathcal{P}_y)(x)| \leq C/y$, since $\sup_x \mathcal{P}_y(x) \leq c/y$.

Using the first estimate when $|x| \geq |y|$ and the second when $|x| \leq |y|$ gives the desired decrease at infinity.

We next show that the solution is essentially unique.

Theorem 2.7 *Suppose u is continuous on the closure of the upper half-plane $\overline{\mathbb{R}^2_+}$, satisfies $\triangle u = 0$ for $(x,y) \in \mathbb{R}^2_+$, $u(x,0) = 0$, and $u(x,y)$ vanishes at infinity. Then $u = 0$.*

A simple example shows that a condition concerning the decay of u at infinity is needed: take $u(x,y) = y$. Clearly u satisfies the steady-state heat equation and vanishes on the real line, yet u is not identically zero.

The proof of the theorem relies on a basic fact about harmonic functions, which are functions satisfying $\triangle u = 0$. The fact is that the value of a harmonic function at a point equals its average value around any circle centered at that point.

Lemma 2.8 (Mean-value property) *Suppose Ω is an open set in $\mathbb{R}^2$ and let u be a function of class C^2 with $\triangle u = 0$ in Ω. If the closure of the disc centered at (x,y) and of radius R is contained in Ω, then*

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta$$

for all $0 \le r \le R$.

Proof. Let $U(r,\theta) = u(x + r\cos\theta, y + r\sin\theta)$. Expressing the Laplacian in polar coordinates, the equation $\triangle u = 0$ then implies

$$0 = \frac{\partial^2 U}{\partial \theta^2} + r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right).$$

If we define $F(r) = \frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)\, d\theta$, the above gives

$$r\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = \frac{1}{2\pi}\int_0^{2\pi} -\frac{\partial^2 U}{\partial \theta^2}(r,\theta)\, d\theta.$$

The integral of $\partial^2 U/\partial\theta^2$ over the circle vanishes since $\partial U/\partial\theta$ is periodic, hence $r\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = 0$, and consequently $r\partial F/\partial r$ must be constant. Evaluating this expression at $r = 0$ we find that $\partial F/\partial r = 0$. Thus F is constant, but since $F(0) = u(x,y)$, we finally find that $F(r) = u(x,y)$ for all $0 \le r \le R$, which is the mean-value property.

Finally, note that the argument above is implicit in the proof of Theorem 5.7, Chapter 2.

To prove Theorem 2.7 we argue by contradiction. Considering separately the real and imaginary parts of u, we may suppose that u itself is real-valued, and is somewhere strictly positive, say $u(x_0, y_0) > 0$ for some $x_0 \in \mathbb{R}$ and $y_0 > 0$. We shall see that this leads to a contradiction. First, since u vanishes at infinity, we can find a large semi-disc of radius R, $D_R^+ = \{(x,y) : x^2 + y^2 \le R,\ \ y \ge 0\}$ outside of which $u(x,y) \le \frac{1}{2}u(x_0,y_0)$. Next, since u is continuous in D_R^+, it attains its maximum M there, so there exists a point $(x_1, y_1) \in D_R^+$ with $u(x_1,y_1) = M$, while

$u(x,y) \le M$ in the semi-disc; also, since $u(x,y) \le \frac{1}{2}u(x_0,y_0) \le M/2$ outside of the semi-disc, we have $u(x,y) \le M$ throughout the entire upper half-plane. Now the mean-value property for harmonic functions implies

$$u(x_1,y_1) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1 + \rho\cos\theta, y_1 + \rho\sin\theta)\, d\theta$$

whenever the circle of integration lies in the upper half-plane. In particular, this equation holds if $0 < \rho < y_1$. Since $u(x_1,y_1)$ equals the maximum value M, and $u(x_1 + \rho\cos\theta, y_1 + \rho\sin\theta) \le M$, it follows by continuity that $u(x_1 + \rho\cos\theta, y_1 + \rho\sin\theta) = M$ on the whole circle. For otherwise $u(x,y) \le M - \epsilon$, on an arc of length $\delta > 0$ on the circle, and this would give

$$\frac{1}{2\pi}\int_0^{2\pi} u(x_1 + \rho\cos\theta, y_1 + \rho\sin\theta)\, d\theta \le M - \frac{\epsilon\delta}{2\pi} < M,$$

contradicting the fact that $u(x_1,y_1) = M$. Now letting $\rho \to y_1$, and using the continuity of u again, we see that this implies $u(x_1, 0) = M > 0$, which contradicts the fact that $u(x,0) = 0$ for all x.

3 The Poisson summation formula

The definition of the Fourier transform was motivated by the desire for a continuous version of Fourier series, applicable to functions defined on the real line. We now show that there exists a further remarkable connection between the analysis of functions on the circle and related functions on $\mathbb{R}$.

Given a function $f \in \mathcal{S}(\mathbb{R})$ on the real line, we can construct a new function on the circle by the recipe

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

Since f is rapidly decreasing, the series converges absolutely and uniformly on every compact subset of $\mathbb{R}$, so F_1 is continuous. Note that $F_1(x+1) = F_1(x)$ because passage from n to $n+1$ in the above sum merely shifts the terms on the series defining $F_1(x)$. Hence F_1 is periodic with period 1. The function F_1 is called the **periodization** of f.

There is another way to arrive at a "periodic version" of f, this time by Fourier analysis. Start with the identity

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi,$$

and consider its discrete analogue, where the integral is replaced by a sum

$$F_2(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Once again, the sum converges absolutely and uniformly since $\hat{f}$ belongs to the Schwartz space, hence F_2 is continuous. Moreover, F_2 is also periodic of period 1 since this is the case for each one of the exponentials $e^{2\pi i n x}$.

The fundamental fact is that these two approaches, which produce F_1 and F_2, actually lead to the same function.

Theorem 3.1 (Poisson summation formula) *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

In particular, setting $x = 0$ we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

In other words, the Fourier coefficients of the periodization of f are given precisely by the values of the Fourier transform of f on the integers.

Proof. To check the first formula it suffices, by Theorem 2.1 in Chapter 2, to show that both sides (which are continuous) have the same Fourier coefficients (viewed as functions on the circle). Clearly, the m^{th} Fourier coefficient of the right-hand side is $\hat{f}(m)$. For the left-hand side we have

$$\begin{aligned}
\int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) \right) e^{-2\pi i m x}\, dx &= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m x}\, dx \\
&= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(y) e^{-2\pi i m y}\, dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y}\, dy \\
&= \hat{f}(m),
\end{aligned}$$

where the interchange of the sum and integral is permissible since f is rapidly decreasing. This completes the proof of the theorem.

We observe that the theorem extends to the case when we merely assume that both f and $\hat{f}$ are of moderate decrease; the proof is in fact unchanged.

It turns out that the operation of periodization is important in a number of questions, even when the Poisson summation formula does not apply. We give an example by considering the elementary function $f(x) = 1/x$, $x \neq 0$. The result is that $\sum_{n=-\infty}^{\infty} 1/(x+n)$, when summed symmetrically, gives the partial fraction decomposition of the cotangent function. In fact this sum equals $\pi \cot \pi x$, when x is not an integer. Similarly with $f(x) = 1/x^2$, we get $\sum_{n=-\infty}^{\infty} 1/(x+n)^2 = \pi^2/(\sin \pi x)^2$, whenever $x \notin \mathbb{Z}$ (see Exercise 15).

3.1 Theta and zeta functions

We define the **theta function** $\vartheta(s)$ for $s > 0$ by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}.$$

The condition on s ensures the absolute convergence of the series. A crucial fact about this special function is that it satisfies the following functional equation.

Theorem 3.2 *$s^{-1/2}\vartheta(1/s) = \vartheta(s)$ whenever $s > 0$.*

The proof of this identity consists of a simple application of the Poisson summation formula to the pair

$$f(x) = e^{-\pi s x^2} \quad \text{and} \quad \hat{f}(\xi) = s^{-1/2} e^{-\pi \xi^2/s}.$$

The theta function $\vartheta(s)$ also extends to complex values of s when $\text{Re}(s) > 0$, and the functional equation is still valid then. The theta function is intimately connected with an important function in number theory, the **zeta function** $\zeta(s)$ defined for $\text{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Later we will see that this function carries essential information about the prime numbers (see Chapter 8).

It also turns out that ζ, ϑ, and another important function Γ are related by the following identity:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} t^{s/2-1}(\vartheta(s) - 1)\, dt,$$

which is valid for $s > 1$ (Exercises 17 and 18).

Returning to the function ϑ, define the generalization $\Theta(z|\tau)$ given by

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau} e^{2\pi i n z}$$

whenever $\mathrm{Im}(\tau) > 0$ and $z \in \mathbb{C}$. Taking $z = 0$ and $\tau = is$ we get $\Theta(z|\tau) = \vartheta(s)$.

3.2 Heat kernels

Another application related to the Poisson summation formula and the theta function is the time-dependent heat equation on the circle. A solution to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to $u(x,0) = f(x)$, where f is periodic of period 1, was given in the previous chapter by

$$u(x,t) = (f * H_t)(x)$$

where $H_t(x)$ is the heat kernel on the circle, that is,

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}.$$

Note in particular that with our definition of the generalized theta function in the previous section, we have $\Theta(x|4\pi i t) = H_t(x)$. Also, recall that the heat equation on $\mathbb{R}$ gave rise to the heat kernel

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

where $\hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2\xi^2 t}$. The fundamental relation between these two objects is an immediate consequence of the Poisson summation formula:

Theorem 3.3 *The heat kernel on the circle is the periodization of the heat kernel on the real line:*

$$H_t(x) = \sum_{n=-\infty}^{\infty} \mathcal{H}_t(x+n).$$

Although the proof that $\mathcal{H}_t$ is a good kernel on $\mathbb{R}$ was fairly straightforward, we left open the harder problem that H_t is a good kernel on the circle. The above results allow us to resolve this matter.

Corollary 3.4 *The kernel $H_t(x)$ is a good kernel for $t \to 0$.*

Proof. We already observed that $\int_{|x|\le 1/2} H_t(x)\,dx = 1$. Now note that $H_t \ge 0$, which is immediate from the above formula since $\mathcal{H}_t \ge 0$. Finally, we claim that when $|x| \le 1/2$,

$$H_t(x) = \mathcal{H}_t(x) + \mathcal{E}_t(x),$$

where the error satisfies $|\mathcal{E}_t(x)| \le c_1 e^{-c_2/t}$ with $c_1, c_2 > 0$ and $0 < t \le 1$. To see this, note again that the formula in the theorem gives

$$H_t(x) = \mathcal{H}_t(x) + \sum_{|n|\ge 1} \mathcal{H}_t(x+n);$$

therefore, since $|x| \le 1/2$,

$$\mathcal{E}_t(x) = \frac{1}{\sqrt{4\pi t}} \sum_{|n|\ge 1} e^{-(x+n)^2/4t} \le C t^{-1/2} \sum_{n\ge 1} e^{-cn^2/t}.$$

Note that $n^2/t \ge n^2$ and $n^2/t \ge 1/t$ whenever $0 < t \le 1$, so $e^{-cn^2/t} \le e^{-\frac{c}{2}n^2} e^{-\frac{c}{2}\frac{1}{t}}$. Hence

$$|\mathcal{E}_t(x)| \le C t^{-1/2} e^{-\frac{c}{2}\frac{1}{t}} \sum_{n\ge 1} e^{-\frac{c}{2}n^2} \le c_1 e^{-c_2/t}.$$

The proof of the claim is complete, and as a result $\int_{|x|\le 1/2} |\mathcal{E}_t(x)|\,dx \to 0$ as $t \to 0$. It is now clear that H_t satisfies

$$\int_{\eta<|x|\le 1/2} |H_t(x)|\,dx \to 0 \quad \text{as } t \to 0,$$

because $\mathcal{H}_t$ does.

3.3 Poisson kernels

In a similar manner to the discussion above about the heat kernels, we state the relation between the Poisson kernels for the disc and the upper half-plane where

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{y^2+x^2}.$$