follows that the Fourier transform $\mathcal{F}$ is a bijective map of $\mathcal{S}(\mathbb{R}^d)$ to itself, whose inverse is

$$\mathcal{F}^*(g)(x) = \int_{\mathbb{R}^d} g(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

Step 4. Next we turn to the convolution, defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy, \qquad f, g \in \mathcal{S}.$$

We have that $f * g \in \mathcal{S}(\mathbb{R}^d)$, $f * g = g * f$, and $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. The argument is similar to that in one-dimension. The calculation of the Fourier transform of $f * g$ involves an integration of $f(y)g(x-y)e^{-2\pi i x\cdot\xi}$ (over $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$) expressed as a repeated integral.

Then, following the same argument in the previous chapter, we obtain the d-dimensional Plancherel formula, thereby concluding the proof of Theorem 2.4.

3 The wave equation in $\mathbb{R}^d \times \mathbb{R}$

Our next goal is to apply what we have learned about the Fourier transform to the study of the wave equation. Here, we once again simplify matters by restricting ourselves to functions in the Schwartz class $\mathcal{S}$. We note that in any further analysis of the wave equation it is important to allow functions that have much more general behavior, and in particular that may be discontinuous. However, what we lose in generality by only considering Schwartz functions, we gain in transparency. Our study in this restricted context will allow us to explain certain basic ideas in their simplest form.

3.1 Solution in terms of Fourier transforms

The motion of a vibrating string satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

which we referred to as the one-dimensional wave equation.

A natural generalization of this equation to d space variables is

$$\text{(1)} \qquad \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In fact, it is known that in the case $d = 3$, this equation determines the behavior of electromagnetic waves in vacuum (with $c =$ speed of light).

Also, this equation describes the propagation of sound waves. Thus (1) is called the **d-dimensional wave equation**.

Our first observation is that we may assume $c = 1$, since we can rescale the variable t if necessary. Also, if we define the **Laplacian** in d dimensions by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2},$$

then the wave equation can be rewritten as

$$\text{(2)} \qquad \triangle u = \frac{\partial^2 u}{\partial t^2}.$$

The goal of this section is to find a solution to this equation, subject to the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

where $f, g \in \mathcal{S}(\mathbb{R}^d)$. This is called the **Cauchy problem** for the wave equation.

Before solving this problem, we note that while we think of the variable t as time, we do not restrict ourselves to $t > 0$. As we will see, the solution we obtain makes sense for all $t \in \mathbb{R}$. This is a manifestation of the fact that the wave equation can be reversed in time (unlike the heat equation).

A formula for the solution of our problem is given in the next theorem. The heuristic argument which leads to this formula is important since, as we have already seen, it applies to some other boundary value problems as well.

Suppose u solves the Cauchy problem for the wave equation. The technique employed consists of taking the Fourier transform of the equation and of the initial conditions, with respect to the space variables $x_1, \ldots, x_d$. This reduces the problem to an ordinary differential equation in the time variable. Indeed, recalling that differentiation with respect to x_j becomes multiplication by $2\pi i\xi_j$, and the differentiation with respect to t commutes with the Fourier transform in the space variables, we find that (2) becomes

$$-4\pi^2|\xi|^2\hat{u}(\xi,t) = \frac{\partial^2 \hat{u}}{\partial t^2}(\xi,t).$$

For each fixed $\xi \in \mathbb{R}^d$, this is an ordinary differential equation in t whose solution is given by

$$\hat{u}(\xi,t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t),$$

where for each ξ, $A(\xi)$ and $B(\xi)$ are unknown constants to be determined by the initial conditions. In fact, taking the Fourier transform (in x) of the initial conditions yields

$$\hat{u}(\xi,0) = \hat{f}(\xi) \quad \text{and} \quad \frac{\partial \hat{u}}{\partial t}(\xi,0) = \hat{g}(\xi).$$

We may now solve for $A(\xi)$ and $B(\xi)$ to obtain

$$A(\xi) = \hat{f}(\xi) \quad \text{and} \quad 2\pi|\xi|B(\xi) = \hat{g}(\xi).$$

Therefore, we find that

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|},$$

and the solution u is given by taking the inverse Fourier transform in the ξ variables. This formal derivation then leads to a precise existence theorem for our problem.

Theorem 3.1 *A solution of the Cauchy problem for the wave equation is*

$$(3) \qquad u(x,t) = \int_{\mathbb{R}^d}\left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi i x\cdot\xi}\,d\xi.$$

Proof. We first verify that u solves the wave equation. This is straightforward once we note that we can differentiate in x and t under the integral sign (because f and g are both Schwartz functions) and therefore u is at least C^2. On the one hand we differentiate the exponential with respect to the x variables to get

$$\triangle u(x,t) = \int_{\mathbb{R}^d}\left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right](-4\pi^2|\xi|^2)e^{2\pi i x\cdot\xi}\,d\xi,$$

while on the other hand we differentiate the terms in brackets with respect to t twice to get

$$\frac{\partial^2 u}{\partial t^2}(x,t) =$$

$$\int_{\mathbb{R}^d}\left[-4\pi^2|\xi|^2\hat{f}(\xi)\cos(2\pi|\xi|t) - 4\pi^2|\xi|^2\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi i x\cdot\xi}\,d\xi.$$

This shows that u solves equation (2). Setting $t=0$ we get

$$u(x,0) = \int_{\mathbb{R}^d}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = f(x)$$

by the Fourier inversion theorem. Finally, differentiating once with respect to t, setting $t=0$, and using the Fourier inversion shows that

$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

Thus u also verifies the initial conditions, and the proof of the theorem is complete.

As the reader will note, both $\hat{f}(\xi)\cos(2\pi|\xi|t)$ and $\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$ are functions in $\mathcal{S}$, assuming as we do that f and g are in $\mathcal{S}$. This is because both $\cos u$ and $(\sin u)/u$ are even functions that are indefinitely differentiable.

Having proved the existence of a solution to the Cauchy problem for the wave equation, we raise the question of uniqueness. Are there solutions to the problem

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \quad \text{subject to} \quad u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

other than the one given by the formula in the theorem? In fact the answer is, as expected, no. The proof of this fact, which will not be given here (but see Problem 3), can be based on a conservation of energy argument. This is a local counterpart of a global conservation of energy statement which we will now present.

We observed in Exercise 10, Chapter 3, that in the one-dimensional case, the total energy of the vibrating string is conserved in time. The analogue of this fact holds in higher dimensions as well. Define the **energy** of a solution by

$$E(t) = \int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x_1}\right|^2 + \cdots + \left|\frac{\partial u}{\partial x_d}\right|^2 dx.$$

Theorem 3.2 *If u is the solution of the wave equation given by formula (3), then $E(t)$ is conserved, that is,*

$$E(t) = E(0), \quad \textit{for all } t \in \mathbb{R}.$$

The proof requires the following lemma.

Lemma 3.3 *Suppose a and b are complex numbers and α is real. Then*

$$|a\cos\alpha + b\sin\alpha|^2 + |-a\sin\alpha + b\cos\alpha|^2 = |a|^2 + |b|^2.$$

This follows directly because $e_1 = (\cos\alpha, \sin\alpha)$ and $e_2 = (-\sin\alpha, \cos\alpha)$ are a pair of orthonormal vectors, hence with $Z = (a, b) \in \mathbb{C}^2$, we have

$$|Z|^2 = |Z \cdot e_1|^2 + |Z \cdot e_2|^2,$$

where $\cdot$ represents the inner product in $\mathbb{C}^2$.

Now by Plancherel's theorem,

$$\int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}\right|^2 dx = \int_{\mathbb{R}^d} \left|-2\pi|\xi|\hat{f}(\xi)\sin(2\pi|\xi|t) + \hat{g}(\xi)\cos(2\pi|\xi|t)\right|^2 d\xi.$$

Similarly,

$$\int_{\mathbb{R}^d} \sum_{j=1}^{d} \left|\frac{\partial u}{\partial x_j}\right|^2 dx = \int_{\mathbb{R}^d} \left|2\pi|\xi|\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\sin(2\pi|\xi|t)\right|^2 d\xi.$$

We now apply the lemma with

$$a = 2\pi|\xi|\hat{f}(\xi), \qquad b = \hat{g}(\xi) \quad \text{and} \quad \alpha = 2\pi|\xi|t.$$

The result is that

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x_1}\right|^2 + \cdots + \left|\frac{\partial u}{\partial x_d}\right|^2 dx \\ &= \int_{\mathbb{R}^d} (4\pi^2|\xi|^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2)\, d\xi, \end{aligned}$$

which is clearly independent of t. Thus Theorem 3.2 is proved.

The drawback with formula (3), which does give the solution of the wave equation, is that it is quite indirect, involving the calculation of the Fourier transforms of f and g, and then a further inverse Fourier transform. However, for every dimension d there is a more explicit formula. This formula is very simple when $d = 1$ and a little less so when $d = 3$. More generally, the formula is "elementary" whenever d is odd, and more complicated when d is even (see Problems 4 and 5).

In what follows we consider the cases $d = 1$, $d = 3$, and $d = 2$, which together give a picture of the general situation. Recall that in Chapter 1, when discussing the wave equation over the interval $[0, L]$, we found that the solution is given by d'Alembert's formula

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy. \tag{4}$$

with the interpretation that both f and g are extended outside $[0, L]$ by making them *odd* in $[-L, L]$, and periodic on the real line, with period $2L$. The same formula (4) holds for the solution of the wave equation when $d = 1$ and when the initial data are functions in $\mathcal{S}(\mathbb{R})$. In fact, this follows directly from (3) if we note that

$$\cos(2\pi|\xi|t) = \frac{1}{2}(e^{2\pi i|\xi|t} + e^{-2\pi i|\xi|t})$$

and

$$\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} = \frac{1}{4\pi i|\xi|}(e^{2\pi i|\xi|t} - e^{-2\pi i|\xi|t}).$$

Finally, we note that the two terms that appear in d'Alembert's formula (4) consist of appropriate averages. Indeed, the first term is precisely the average of f over the two points that are the boundary of the interval $[x-t, x+t]$; the second term is, up to a factor of t, the mean value of g over this interval, that is, $(1/2t)\int_{x-t}^{x+t} g(y)\, dy$. This suggests a generalization to higher dimensions, where we might expect to write the solution of our problem as averages of the initial data. This is in fact the case, and we now treat in detail the particular situation $d = 3$.

3.2 The wave equation in $\mathbb{R}^3 \times \mathbb{R}$

If S^2 denotes the unit sphere in $\mathbb{R}^3$, we define the **spherical mean** of the function f over the sphere of radius t centered at x by

$$M_t(f)(x) = \frac{1}{4\pi}\int_{S^2} f(x - t\gamma)\, d\sigma(\gamma), \tag{5}$$

where $d\sigma(\gamma)$ is the element of surface area for S^2. Since 4π is the area of the unit sphere, we can interpret $M_t(f)$ as the average value of f over the sphere centered at x of radius t.

Lemma 3.4 *If $f \in \mathcal{S}(\mathbb{R}^3)$ and t is fixed, then $M_t(f) \in \mathcal{S}(\mathbb{R}^3)$. Moreover, $M_t(f)$ is indefinitely differentiable in t, and each t-derivative also belongs to $\mathcal{S}(\mathbb{R}^3)$.*

Proof. Let $F(x) = M_t(f)(x)$. To show that F is rapidly decreasing, start with the inequality $|f(x)| \le A_N/(1 + |x|^N)$ which holds for every fixed $N \ge 0$. As a simple consequence, whenever t is fixed, we have

$$|f(x - \gamma t)| \le A'_N/(1 + |x|^N) \quad \text{for all } \gamma \in S^2.$$

To see this consider separately the cases when $|x| \leq 2|t|$, and $|x| > 2|t|$. Therefore, by integration

$$|F(x)| \leq A'_N/(1+|x|^N),$$

and since this holds for every N, the function F is rapidly decreasing. One next observes that F is indefinitely differentiable, and

$$\left(\frac{\partial}{\partial x}\right)^\alpha F(x) = M_t(f^{(\alpha)})(x) \tag{6}$$

where $f^{(\alpha)}(x) = (\partial/\partial x)^\alpha f$. It suffices to prove this when $(\partial/\partial x)^\alpha = \partial/\partial x_k$, and then proceed by induction to get the general case. Furthermore, it is enough to take $k = 1$. Now

$$\frac{F(x_1+h, x_2, x_3) - F(x_1, x_2, x_3)}{h} = \frac{1}{4\pi}\int_{S^2} g_h(\gamma)\, d\sigma(\gamma)$$

where

$$g_h(\gamma) = \frac{f(x + e_1 h - \gamma t) - f(x - \gamma t)}{h},$$

and $e_1 = (1, 0, 0)$. Now, it suffices to observe that $g_h \to \frac{\partial}{\partial x_1} f(x - \gamma t)$ as $h \to 0$ uniformly in γ. As a result, we find that (6) holds, and by the first argument, it follows that $\left(\frac{\partial}{\partial x}\right)^\alpha F(x)$ is also rapidly decreasing, hence $F \in \mathcal{S}$. The same argument applies to each t-derivative of $M_t(f)$.

The basic fact about integration on spheres that we shall need is the following Fourier transform formula.

Lemma 3.5 $\displaystyle\frac{1}{4\pi}\int_{S^2} e^{-2\pi i \xi\cdot\gamma}\, d\sigma(\gamma) = \frac{\sin(2\pi|\xi|)}{2\pi|\xi|}.$

This formula, as we shall see in the following section, is connected to the fact that the Fourier transform of a radial function is radial.

Proof. Note that the integral on the left is radial in ξ. Indeed, if R is a rotation then

$$\int_{S^2} e^{-2\pi i R(\xi)\cdot\gamma}\, d\sigma(\gamma) = \int_{S^2} e^{-2\pi i \xi\cdot R^{-1}(\gamma)}\, d\sigma(\gamma) = \int_{S^2} e^{-2\pi i \xi\cdot\gamma}\, d\sigma(\gamma)$$

because we may change variables $\gamma \to R^{-1}(\gamma)$. (For this, see formula (4) in the appendix.) So if $|\xi| = \rho$, it suffices to prove the lemma with

$\xi = (0, 0, \rho)$. If $\rho = 0$, the lemma is obvious. If $\rho > 0$, we choose spherical coordinates to find that the left-hand side is equal to

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-2\pi i \rho\cos\theta}\sin\theta\, d\theta\, d\varphi.$$

The change of variables $u = -\cos\theta$ gives

$$\begin{aligned}\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-2\pi i \rho\cos\theta}\sin\theta\, d\theta\, d\varphi &= \frac{1}{2}\int_0^{\pi} e^{-2\pi i\rho\cos\theta}\sin\theta\, d\theta\\ &= \frac{1}{2}\int_{-1}^{1} e^{2\pi i\rho u}\, du\\ &= \frac{1}{4\pi i\rho}\left[e^{2\pi i\rho u}\right]_{-1}^{1}\\ &= \frac{\sin(2\pi\rho)}{2\pi\rho},\end{aligned}$$

and the formula is proved.

By the defining formula (5) we may interpret $M_t(f)$ as a convolution of the function f with the element $d\sigma$, and since the Fourier transform interchanges convolutions with products, we are led to believe that $\widehat{M_t(f)}$ is the product of the corresponding Fourier transforms. Indeed, we have the identity

$$\widehat{M_t(f)}(\xi) = \hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}. \tag{7}$$

To see this, write

$$\widehat{M_t(f)}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi}\left(\frac{1}{4\pi}\int_{S^2} f(x - \gamma t)\, d\sigma(\gamma)\right) dx,$$

and note that we may interchange the order of integration and make a simple change of variables to achieve the desired identity.

As a result, we find that the solution of our problem may be expressed by using the spherical means of the initial data.

Theorem 3.6 *The solution when $d = 3$ of the Cauchy problem for the wave equation*

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \quad \textit{subject to} \quad u(x,0) = f(x) \quad \textit{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

is given by

$$u(x,t) = \frac{\partial}{\partial t}(tM_t(f)(x)) + tM_t(g)(x).$$

Proof. Consider first the problem

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \quad \text{subject to} \quad u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Then by Theorem 3.1, we know that its solution u_1 is given by

$$\begin{aligned} u_1(x,t) &= \int_{\mathbb{R}^3} \left[\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right] e^{2\pi i x\cdot\xi}\, d\xi \\ &= t\int_{\mathbb{R}^3} \left[\hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}\right] e^{2\pi i x\cdot\xi}\, d\xi \\ &= tM_t(g)(x), \end{aligned}$$

where we have used (7) applied to g, and the Fourier inversion formula.

According to Theorem 3.1 again, the solution to the problem

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \quad \text{subject to} \quad u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0$$

is given by

$$\begin{aligned} u_2(x,t) &= \int_{\mathbb{R}^3} \left[\hat{f}(\xi)\cos(2\pi|\xi|t)\right] e^{2\pi i x\cdot\xi}\, d\xi \\ &= \frac{\partial}{\partial t}\left(t\int_{\mathbb{R}^3} \left[\hat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|t}\right] e^{2\pi i x\cdot\xi}\, d\xi\right) \\ &= \frac{\partial}{\partial t}(tM_t(f)(x)). \end{aligned}$$

We may now superpose these two solutions to obtain $u = u_1 + u_2$ as the solution of our original problem.

Huygens principle

The solutions to the wave equation in one and three dimensions are given, respectively, by

$$u(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy$$

and

$$u(x,t) = \frac{\partial}{\partial t}(tM_t(f)(x)) + tM_t(g)(x).$$

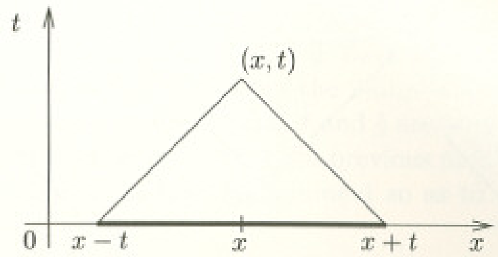

Figure 1. Huygens principle, $d = 1$

We observe that in the one-dimensional problem, the value of the solution at (x,t) depends only on the values of f and g in the interval centered at x of length $2t$, as shown in Figure 1.

If in addition $g = 0$, then the solution depends only on the data at the two boundary points of this interval. In three dimensions, this boundary dependence always holds. More precisely, the solution $u(x,t)$ depends only on the values of f and g in an immediate neighborhood of the sphere centered at x and of radius t. This situation is depicted in Figure 2, where we have drawn the cone originating at (x,t) and with its base the ball centered at x of radius t. This cone is called the **backward light cone** originating at (x,t).

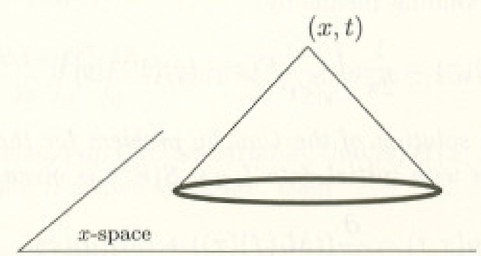

Figure 2. Backward light cone originating at (x,t)

Alternatively, the data at a point x_0 in the plane $t = 0$ influences the solution only on the boundary of a cone originating at x_0, called the **forward light cone** and depicted in Figure 3.

This phenomenon, known as the **Huygens principle**, is immediate from the formulas for u given above.

Another important aspect of the wave equation connected with these

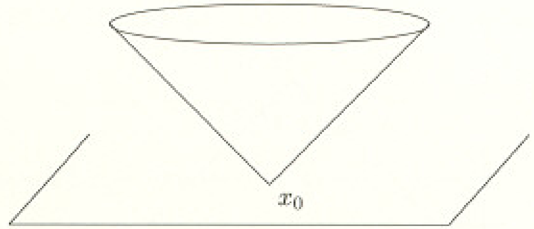

Figure 3. The forward light cone originating at x_0

considerations is that of the **finite speed of propagation**. (In the case where $c = 1$, the speed is 1.) This means that if we have an initial disturbance localized at $x = x_0$, then after a finite time t, its effects will have propagated only inside the ball centered at x_0 of radius $|t|$. To state this precisely, suppose the initial conditions f and g are supported in the ball of radius δ, centered at x_0 (think of δ as small). Then $u(x,t)$ is supported in the ball of radius $|t| + \delta$ centered at x_0. This assertion is clear from the above discussion.

3.3 The wave equation in $\mathbb{R}^2 \times \mathbb{R}$: descent

It is a remarkable fact that the solution of the wave equation in three dimensions leads to a solution of the wave equation in two dimensions. Define the corresponding means by

$$\widetilde{M}_t(F)(x) = \frac{1}{2\pi}\int_{|y|\le 1} F(x - ty)(1 - |y|^2)^{-1/2}\,dy.$$

Theorem 3.7 *A solution of the Cauchy problem for the wave equation in two dimensions with initial data $f, g \in \mathcal{S}(\mathbb{R}^2)$ is given by*

$$u(x,t) = \frac{\partial}{\partial t}(t\widetilde{M}_t(f)(x)) + t\widetilde{M}_t(g)(x). \tag{8}$$

Notice the difference between this case and the case $d = 3$. Here, u at (x,t) depends on f and g in the whole disc (of radius $|t|$ centered at x), and not just on the values of the initial data near the boundary of that disc.

Formally, the identity in the theorem arises as follows. If we start with an initial pair of functions f and g in $\mathcal{S}(\mathbb{R}^2)$, we may consider the corresponding functions $\tilde{f}$ and $\tilde{g}$ on $\mathbb{R}^3$ that are merely extensions of f and g that are constant in the x_3 variable, that is,

$$\tilde{f}(x_1, x_2, x_3) = f(x_1, x_2) \quad \text{and} \quad \tilde{g}(x_1, x_2, x_3) = g(x_1, x_2).$$

Now, if $\tilde{u}$ is the solution (given in the previous section) of the 3-dimensional wave equation with initial data $\tilde{f}$ and $\tilde{g}$, then one can expect that $\tilde{u}$ is also constant in x_3 so that $\tilde{u}$ satisfies the 2-dimensional wave equation. A difficulty with this argument is that $\tilde{f}$ and $\tilde{g}$ are not rapidly decreasing since they are constant in x_3, so that our previous methods do not apply. However, it is easy to modify the argument so as to obtain a proof of Theorem 3.7.

We fix $T > 0$ and consider a function $\eta(x_3)$ that is in $\mathcal{S}(\mathbb{R})$, such that $\eta(x_3) = 1$ if $|x_3| \le 3T$. The trick is to truncate $\tilde{f}$ and $\tilde{g}$ in the x_3-variable, and consider instead

$$\tilde{f}^\flat(x_1, x_2, x_3) = f(x_1, x_2)\eta(x_3) \quad \text{and} \quad \tilde{g}^\flat(x_1, x_2, x_3) = g(x_1, x_2)\eta(x_3).$$

Now both $\tilde{f}^\flat$ and $\tilde{g}^\flat$ are in $\mathcal{S}(\mathbb{R}^3)$, so Theorem 3.6 provides a solution $\tilde{u}^\flat$ of the wave equation with initial data $\tilde{f}^\flat$ and $\tilde{g}^\flat$. It is easy to see from the formula that $\tilde{u}^\flat(x,t)$ is independent of x_3, whenever $|x_3| \le T$ and $|t| \le T$. In particular, if we define $u(x_1, x_2, t) = \tilde{u}^\flat(x_1, x_2, 0, t)$, then u satisfies the 2-dimensional wave equation when $|t| \le T$. Since T is arbitrary, u is a solution to our problem, and it remains to see why u has the desired form.

By definition of the spherical coordinates, we recall that the integral of a function H over the sphere S^2 is given by

$$\frac{1}{4\pi}\int_{S^2} H(\gamma)\,d\sigma(\gamma) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} H(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\sin\theta\,d\theta\,d\varphi.$$

If H does not depend on the last variable, that is, $H(x_1, x_2, x_3) = h(x_1, x_2)$ for some function h of two variables, then

$$M_t(H)(x_1, x_2, 0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(x_1 - t\sin\theta\cos\varphi, x_2 - t\sin\theta\sin\varphi)\sin\theta\,d\theta\,d\varphi.$$

To calculate this last integral, we split the θ-integral from 0 to $\pi/2$ and then $\pi/2$ to π. By making the change of variables $r = \sin\theta$, we find, after a final change to polar coordinates, that

$$\begin{aligned} M_t(H)(x_1, x_2, 0) &= \frac{1}{2\pi}\int_{|y|\le 1} h(x - ty)(1 - |y|^2)^{-1/2}\,dy \\ &= \widetilde{M}_t(h)(x_1, x_2). \end{aligned}$$

Applying this to $H = \tilde{f}^\flat$, $h = f$, and $H = \tilde{g}^\flat$, $h = g$, we find that u is given by the formula (8), and the proof of Theorem 3.7 is complete.

Remark. In the case of general d, the solution of the wave equation shares many of the properties we have discussed in the special cases $d = 1, 2$, and 3.

- At a given time t, the initial data at a point x only affects the solution u in a specific region. When $d > 1$ is odd, the data influences only the points on the boundary of the forward light cone originating at x, while when $d = 1$ or d is even, it affects all points of the forward light cone. Alternatively, the solution at a point (x, t) depends only on the data at the base of the backward light cone originating at (x, t). In fact, when $d > 1$ is odd, only the data in an immediate neighborhood of the boundary of the base will influence $u(x, t)$.

- Waves propagate with finite speed: if the initial data is supported in a bounded set, then the support of the solution u spreads with velocity 1 (or more generally c, if the wave equation is not normalized).

We can illustrate some of these facts by the following observation about the different behavior of the propagation of waves in three and two dimensions. Since the propagation of light is governed by the three-dimensional wave equation, if at $t = 0$ a light flashes at the origin, the following happens: any observer will see the flash (after a finite amount of time) only for an instant. In contrast, consider what happens in two dimensions. If we drop a stone in a lake, any point on the surface will begin (after some time) to undulate; although the amplitude of the oscillations will decrease over time, the undulations will continue (in principle) indefinitely.

The difference in character of the formulas for the solutions of the wave equation when $d = 1$ and $d = 3$ on the one hand, and $d = 2$ on the other hand, illustrates a general principle in d-dimensional Fourier analysis: a significant number of formulas that arise are simpler in the case of odd dimensions, compared to the corresponding situations in even dimensions. We will see several further examples of this below.

4 Radial symmetry and Bessel functions

We observed earlier that the Fourier transform of a radial function in $\mathbb{R}^d$ is also radial. In other words, if $f(x) = f_0(|x|)$ for some f_0, then

$\hat{f}(\xi) = F_0(|\xi|)$ for some F_0. A natural problem is to determine a relation between f_0 and F_0.

This problem has a simple answer in dimensions one and three. If $d = 1$ the relation we seek is

$$F_0(\rho) = 2\int_0^\infty \cos(2\pi\rho r) f_0(r)\, dr. \tag{9}$$

If we recall that $\mathbb{R}$ has only two rotations, the identity and multiplication by -1, we find that a function is radial precisely when it is even. Having made this observation it is easy to see that if f is radial, and $|\xi| = \rho$, then

$$\begin{aligned} F_0(\rho) = \hat{f}(|\xi|) &= \int_{-\infty}^\infty f(x) e^{-2\pi i x|\xi|}\, dx \\ &= \int_0^\infty f_0(r)(e^{-2\pi i r|\xi|} + e^{2\pi i r|\xi|})\, dr \\ &= 2\int_0^\infty \cos(2\pi\rho r) f_0(r)\, dr. \end{aligned}$$

In the case $d = 3$, the relation between f_0 and F_0 is also quite simple and given by the formula

$$F_0(\rho) = 2\rho^{-1}\int_0^\infty \sin(2\pi\rho r) f_0(r) r\, dr. \tag{10}$$

The proof of this identity is based on the formula for the Fourier transform of the surface element $d\sigma$ given in Lemma 3.5:

$$\begin{aligned} F_0(\rho) = \hat{f}(\xi) &= \int_{\mathbb{R}^3} f(x) e^{-2\pi i x\cdot\xi}\, dx \\ &= \int_0^\infty f_0(r)\int_{S^2} e^{-2\pi i r\gamma\cdot\xi} d\sigma(\gamma) r^2\, dr \\ &= \int_0^\infty f_0(r)\frac{2\sin(2\pi\rho r)}{\rho r} r^2\, dr \\ &= 2\rho^{-1}\int_0^\infty \sin(2\pi\rho r) f_0(r) r\, dr. \end{aligned}$$

More generally, the relation between f_0 and F_0 has a nice description in terms of a family of special functions that arise naturally in problems that exhibit radial symmetry.

The **Bessel function** of order $n \in \mathbb{Z}$, denoted $J_n(\rho)$, is defined as the n^{th} Fourier coefficient of the function $e^{i\rho\sin\theta}$. So

$$J_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\rho\sin\theta} e^{-in\theta}\, d\theta,$$