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Quantum 3-Manifold Invariants

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Introduction

The idea to derive topological invariants of smooth manifolds from partition functions of certain action functionals was suggested by A Schwarz (1978) and highlighted by E Witten (1988). Witten interpreted the Jones polynomial of links in the 3-sphere S^3 as a partition function of the Chern–Simons field theory. Witten conjectured the existence of mathematically defined topological invariants of 3-manifolds, generalizing the Jones polynomial (or rather its values in complex roots of unity) to links in arbitrary closed oriented 3-manifolds. A rigorous construction of such invariants was given by N Reshetikhin and V Turaev (1989) using the theory of quantum groups. The Witten–Reshetikhin–Turaev invariants of 3-manifolds, also called the "quantum invariants," extend to a topological quantum field theory (TQFT) in dimension 3.

Ribbon and Modular Categories

The Reshetikhin–Turaev approach begins with fixing suitable algebraic data, which are best described in terms of monoidal categories. Let $\mathcal{C}$ be a monoidal category (i.e., a category with an associative tensor product and unit object $\mathbb{1}$). A "braiding" in $\mathcal{C}$ assigns to any objects $V, W \in \mathcal{C}$ an invertible morphism $c_{V,W} : V \otimes W \to W \otimes V$ such that, for any $U, V, W \in \mathcal{C}$,

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$$
$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$$

A "twist" in $\mathcal{C}$ assigns to any object $V \in \mathcal{C}$ an invertible morphism $\theta_V : V \to V$ such that, for any $V, W \in \mathcal{C}$,

$$\theta_{V\otimes W} = c_{W,V}\, c_{V,W}(\theta_V \otimes \theta_W)$$

A "duality" in $\mathcal{C}$ assigns to any object $V \in \mathcal{C}$ a "dual" object $V^* \in \mathcal{C}$, and evaluation and co-evaluation morphisms $d_V : V^* \otimes V \to \mathbb{1}$, $b_V : \mathbb{1} \to V \otimes V^*$ such that

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$
$$(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

The category $\mathcal{C}$ with duality, braiding, and twist is ribbon, if for any $V \in \mathcal{C}$,

$$(\theta_V \otimes \mathrm{id}_{V^*})b_V = (\mathrm{id}_V \otimes \theta_{V^*})b_V$$

For an endomorphism $f : V \to V$ of an object $V \in \mathcal{C}$, its trace "$\mathrm{tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$" is defined as

$$\mathrm{tr}(f) = d_V c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*})b_V : \mathbb{1} \to \mathbb{1}$$

This trace shares a number of properties of the standard trace of matrices, in particular, $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ and $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\mathrm{tr}(g)$. For an object $V \in \mathcal{C}$, set

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) = d_V c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*})b_V$$

Ribbon categories nicely fit the theory of knots and links in S^3. A link $L \subset S^3$ is a closed one-dimensional submanifold of S^3. (A manifold is closed if it is compact and has no boundary.) A link is oriented (resp. framed) if all its components are oriented (resp. provided with a homotopy class of nonsingular normal vector fields). Given a framed oriented link $L \subset S^3$ whose components are labeled with objects of a ribbon category $\mathcal{C}$, one defines a tensor $\langle L \rangle \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. To compute $\langle L \rangle$, present L by a plane diagram with only double transversal crossings such that the framing of L is orthogonal to the plane. Each double point of the diagram is an intersection of two branches of L, going over and under, respectively. Associate with such a crossing the tensor $(c_{V,W})^{\pm 1}$ where $V, W \in \mathcal{C}$ are the labels of these two branches and ± 1 is the sign of the crossing determined by the orientation of L. We also associate certain tensors with the points of the diagram where the tangent line is parallel to a fixed axis on the plane. These tensors are derived from the evaluation and co-evaluation morphisms and the twists. Finally, all these tensors are contracted into a single element $\langle L \rangle \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. It does not depend on the intermediate choices and is preserved under isotopy of L in S^3. For the trivial knot $O(V)$ with framing 0 and label $V \in \mathcal{C}$, we have $\langle O(V) \rangle = \dim(V)$.

Further constructions need the notion of a tangle. An (oriented) tangle is a compact (oriented) one-dimensional submanifold of $\boldsymbol{R}^2 \times [0, 1]$ with endpoints on $\boldsymbol{R} \times 0 \times \{0, 1\}$. Near each of its endpoints, an oriented tangle T is directed either down or up, and thus acquires a sign ± 1. One can view T as a morphism from the sequence of ± 1's associated with its bottom ends to the sequence of ± 1's associated with its top ends. Tangles can be composed by putting one on top of the other. This defines a category of tangles $\mathcal{T}$ whose objects are finite sequences of ± 1's and whose morphisms are isotopy classes of framed oriented tangles. Given a ribbon category $\mathcal{C}$, we can consider $\mathcal{C}$-labeled tangles, that is, (framed oriented) tangles whose components are labeled with objects of $\mathcal{C}$. They form a category $\mathcal{T}_{\mathcal{C}}$. Links appear here as tangles without endpoints, that is, as morphisms $\emptyset \to \emptyset$. The link invariant $\langle L \rangle$ generalizes to a functor $\langle \cdot \rangle : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$.

To define 3-manifold invariants, we need modular categories (Turaev 1994). Let $\boldsymbol{k}$ be a field. A monoidal category $\mathcal{C}$ is $\boldsymbol{k}$-additive if its Hom sets are $\boldsymbol{k}$-vector spaces, the composition and tensor product of the morphisms are bilinear, and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \boldsymbol{k}$. An object $V \in \mathcal{C}$ is simple if $\mathrm{End}_{\mathcal{C}}(V) = \boldsymbol{k}$. A modular category is a $\boldsymbol{k}$-additive ribbon category $\mathcal{C}$ with a finite family of simple objects $\{V_\lambda\}_\lambda$ such that (1) for any object $V \in \mathcal{C}$ there is a finite expansion $\mathrm{id}_V = \sum_i f_i g_i$ for certain morphisms $g_i : V \to V_{\lambda_i}$, $f_i : V_{\lambda_i} \to V$ and (2) the S-matrix $(S_{\lambda,\mu})$ is invertible over $\boldsymbol{k}$ where $S_{\lambda,\mu} = \mathrm{tr}(c_{V_\lambda, V_\mu} c_{V_\mu, V_\lambda})$. Note that $S_{\lambda,\mu} = \langle H(\lambda, \mu) \rangle$ where $H(\lambda, \mu)$ is the oriented Hopf link with framing 0, linking number +1, and labels V_λ, V_μ.

Axiom (1) implies that every simple object in $\mathcal{C}$ is isomorphic to exactly one of V_λ. In most interesting cases (when there is a well-defined direct summation in $\mathcal{C}$), this axiom may be rephrased by saying that $\mathcal{C}$ is finite semisimple, that is, $\mathcal{C}$ has a finite set of isomorphism classes of simple objects and all objects of $\mathcal{C}$ are direct sums of simple objects. A weaker version of the axiom (2) yields premodular categories.

The invariant $\langle \cdot \rangle$ of links and tangles extends by linearity to the case where labels are finite linear combinations of objects of $\mathcal{C}$ with coefficients in $\boldsymbol{k}$. Such a linear combination $\Omega = \sum_\lambda \dim(V_\lambda) V_\lambda$ is called the Kirby color. It has the following sliding property: for any object $V \in \mathcal{C}$, the two tangles in Figure 1 yield the same morphism $V \to V$. Here, the dashed line represents an arc on the closed component labeled by Ω. This arc can be knotted or linked with other components of the tangle (not shown in the figure).

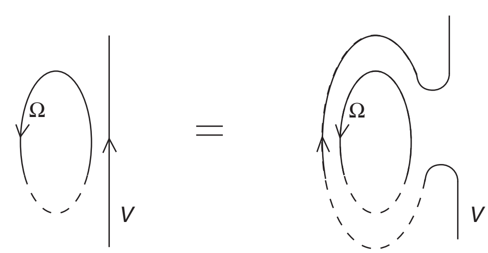

Figure 1 Sliding property.

Invariants of Closed 3-Manifolds

Given an embedded solid torus $g: S^1 \times D^2 \hookrightarrow S^3$, where D^2 is a 2-disk and $S^1 = \partial D^2$, a 3-manifold can be built as follows. Remove from S^3 the interior of $g(S^1 \times D^2)$ and glue back the solid torus $D^2 \times S^1$ along $g|_{S^1 \times S^1}$. This process is known as "surgery." The resulting 3-manifold depends only on the isotopy class of the framed knot represented by g. More generally, a surgery on a framed link $L = \cup_{i=1}^m L_i$ in S^3 with m components yields a closed oriented 3-manifold M_L. A theorem of W Lickorish and A Wallace asserts that any closed connected oriented 3-manifold is homeomorphic to M_L for some L. R Kirby proved that two framed links give rise to homeomorphic 3-manifolds if and only if these links are related by isotopy and a finite sequence of geometric transformations called Kirby moves. There are two Kirby moves: adjoining a distant unknot O^ε with framing $\varepsilon = \pm 1$, and sliding a link component over another one as in Figure 1.

Let $L = \cup_{i=1}^m L_i \subset S^3$ be a framed link and let $(b_{i,j})_{i,j=1,\ldots,m}$ be its linking matrix: for $i \neq j, b_{i,j}$ is the linking number of L_i, L_j, and $b_{i,i}$ is the framing number of L_i. Denote by e_+ (resp. e_-) the number of positive (resp. negative) eigenvalues of this matrix. The sliding property of modular categories implies the following theorem. In its statement, a knot K with label Ω is denoted by $K(\Omega)$.

Theorem 1 *Let $\mathcal{C}$ be a modular category with Kirby color Ω. Then $\langle \mathrm{O}^1(\Omega)\rangle \neq 0, \langle \mathrm{O}^{-1}(\Omega)\rangle \neq 0$ and the expression*

$$\tau_{\mathcal{C}}(M_L) = \langle \mathrm{O}^1(\Omega)\rangle^{-e_+} \langle \mathrm{O}^{-1}(\Omega)\rangle^{-e_-} \langle L_1(\Omega), \ldots, L_m(\Omega)\rangle$$

is invariant under the Kirby moves on L. This expression yields, therefore, a well-defined topological invariant $\tau_{\mathcal{C}}$ of closed connected oriented 3-manifolds.

Several competing normalizations of $\tau_{\mathcal{C}}$ exist in the literature. Here, the normalization used is such that $\tau_{\mathcal{C}}(S^3) = 1$ and $\tau_{\mathcal{C}}(S^1 \times S^2) = \sum_\lambda (\dim(V_\lambda))^2$. The invariant $\tau_{\mathcal{C}}$ extends to 3-manifolds with a framed oriented $\mathcal{C}$-labeled link K inside by

$$\begin{aligned}&\tau_{\mathcal{C}}(M_L, K)\\ &\quad = \langle \mathrm{O}^1(\Omega)\rangle^{-e_+} \langle \mathrm{O}^{-1}(\Omega)\rangle^{-e_-} \langle L_1(\Omega), \ldots, L_m(\Omega), K\rangle\end{aligned}$$

Three-Dimensional TQFTs

A three-dimensional TQFT $\boldsymbol{V}$ assigns to every closed oriented surface X a finite-dimensional vector space $\boldsymbol{V}(X)$ over a field $\boldsymbol{k}$ and assigns to every cobordism (M, X, Y) a linear map $\boldsymbol{V}(M) = \boldsymbol{V}(M, X, Y): \boldsymbol{V}(X) \to \boldsymbol{V}(Y)$. Here, a "cobordism" (M, X, Y) between surfaces X and Y is a compact oriented 3-manifold M with $\partial M = (-X) \amalg Y$ (the minus sign indicates the orientation reversal). A TQFT has to satisfy axioms which can be expressed by saying that $\boldsymbol{V}$ is a monoidal functor from the category of surfaces and cobordisms to the category of vector spaces over $\boldsymbol{k}$. Homeomorphisms of surfaces should induce isomorphisms of the corresponding vector spaces compatible with the action of cobordisms. From the definition, $\boldsymbol{V}(\emptyset) = \boldsymbol{k}$. Every compact oriented 3-manifold M is a cobordism between $\emptyset$ and ∂M so that $\boldsymbol{V}$ yields a "vacuum" vector $\boldsymbol{V}(M) \in \mathrm{Hom}(\boldsymbol{V}(\emptyset), \boldsymbol{V}(\partial M)) = \boldsymbol{V}(\partial M)$. If $\partial M = \emptyset$, then this gives a numerical invariant $\boldsymbol{V}(M) \in \boldsymbol{V}(\emptyset) = \boldsymbol{k}$.

Interestingly, TQFTs are often defined for surfaces and 3-cobordisms with additional structure. The surfaces X are normally endowed with Lagrangians, that is, with maximal isotropic subspaces in $H_1(X; \boldsymbol{R})$. For 3-cobordisms, several additional structures are considered in the literature: for example, 2-framings, p_1-structures, and numerical weights. All these choices are equivalent. The TQFTs requiring such additional structures are said to be "projective" since they provide projective linear representations of the mapping class groups of surfaces.

Every modular category $\mathcal{C}$ with ground field $\boldsymbol{k}$ and simple objects $\{V_\lambda\}_\lambda$ gives rise to a projective three-dimensional TQFT $\boldsymbol{V}_{\mathcal{C}}$. It depends on the choice of a square root $\mathcal{D}$ of $\sum_\lambda (\dim(V_\lambda))^2 \in \boldsymbol{k}$. For a connected surface X of genus g,

$$\boldsymbol{V}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}\left(\mathbb{1}, \bigoplus_{\lambda_1,\ldots,\lambda_g} \bigotimes_{r=1}^{g} (V_{\lambda_r} \otimes V_{\lambda_r}^*)\right)$$

The dimension of this vector space enters the Verlinde formula

$$\dim_{\boldsymbol{k}}(\boldsymbol{V}_{\mathcal{C}}(X)) \cdot 1_{\boldsymbol{k}} = \mathcal{D}^{2g-2} \sum_\lambda (\dim(V_\lambda))^{2-2g}$$

where $1_{\boldsymbol{k}} \in \boldsymbol{k}$ is the unit of the field $\boldsymbol{k}$. If $\mathrm{char}(\boldsymbol{k}) = 0$, then this formula computes $\dim_{\boldsymbol{k}}(\boldsymbol{V}_{\mathcal{C}}(X))$. For a closed connected oriented 3-manifold M with numerical weight zero, $\boldsymbol{V}_{\mathcal{C}}(M) = \mathcal{D}^{-b_1(M)-1}\tau_{\mathcal{C}}(M)$, where $b_1(M)$ is the first Betti number of M.

The TQFT $\boldsymbol{V}_{\mathcal{C}}$ extends to a vaster class of surfaces and cobordisms. Surfaces may be enriched with a finite set of marked points, each labeled with an object of $\mathcal{C}$ and endowed with a tangent direction. Cobordisms may be enriched with ribbon (or fat) graphs whose edges are labeled with objects of $\mathcal{C}$ and whose vertices are labeled with appropriate intertwiners. The resulting TQFT, also denoted $\boldsymbol{V}_{\mathcal{C}}$, is nondegenerate in the sense that, for any surface X, the vacuum vectors in $\boldsymbol{V}(X)$ determined by all M

with $\partial M = X$ span $\boldsymbol{V}(X)$. A detailed construction of $\boldsymbol{V}_{\mathcal{C}}$ is given in Turaev (1994).

The two-dimensional part of $\boldsymbol{V}_{\mathcal{C}}$ determines a "modular functor" in the sense of G Segal, G Moore, and N Seiberg.

Constructions of Modular Categories

The universal enveloping algebra $U\mathfrak{g}$ of a (finite-dimensional complex) simple Lie algebra $\mathfrak{g}$ admits a deformation $U_q\mathfrak{g}$, which is a quasitriangular Hopf algebra. The representation category $\mathrm{Rep}(U_q\mathfrak{g})$ is $\boldsymbol{C}$-linear and ribbon. For generic $q \in \boldsymbol{C}$, this category is semisimple. (The irreducible representations of $\mathfrak{g}$ can be deformed to irreducible representations of $U_q\mathfrak{g}$.) For q, an appropriate root of unity, a certain subquotient of $\mathrm{Rep}(U_q\mathfrak{g})$ is a modular category with ground field $\boldsymbol{k} = \boldsymbol{C}$. For $\mathfrak{g} = \mathrm{sl}_2(\boldsymbol{C})$, it was pointed out by Reshetikhin and Turaev; the general case involves the theory of tilting modules. The corresponding 3-manifold invariant τ is denoted $\tau_q^{\mathfrak{g}}$. For example, if $\mathfrak{g} = \mathrm{sl}_2(\boldsymbol{C})$ and M is the Poincaré homology sphere (obtained by surgery on a left-hand trefoil with framing –1), then (Le 2003)

$$\tau_q^{\mathfrak{g}}(M) = (1-q)^{-1} \sum_{n \geq 0} q^n (1 - q^{n+1}) \times (1 - q^{n+2}) \cdots (1 - q^{2n+1})$$

The sum here is finite since q is a root of unity.

There is another construction (Le 2003) of a modular category associated with a simple Lie algebra $\mathfrak{g}$ and certain roots of unity q. The corresponding quantum invariant of 3-manifolds is denoted $\tau_q^{P\mathfrak{g}}$. (Here, it is normalized so that $\tau_q^{P\mathfrak{g}}(S^3) = 1$.) Under mild assumptions on the order of q, we have $\tau_q^{\mathfrak{g}}(M) = \tau_q^{\mathfrak{g}}(M)\tau'(M)$ for all M, where $\tau'(M)$ is a certain Gauss sum determined by $\mathfrak{g}$, the homology group $H = H_1(M)$ and the linking form $\mathrm{Tors}\, H \times \mathrm{Tors}\, H \to \boldsymbol{Q}/\boldsymbol{Z}$.

A different construction derives modular categories from the category of framed oriented tangles $\mathcal{T}$. Given a ring K, a bigger category $K[\mathcal{T}]$ can be considered whose morphisms are linear combinations of tangles with coefficients in K. Both $\mathcal{T}$ and $K[\mathcal{T}]$ have a natural structure of a ribbon monoidal category.

The skein method builds ribbon categories by quotienting $K[\mathcal{T}]$ using local "skein" relations, which appear in the theory of knot polynomials (the Alexander–Conway polynomial, the Homfly polynomial, and the Kauffman polynomial). In order to obtain a semisimple category, one completes the quotient category with idempotents as objects (the Karoubi completion). Choosing appropriate skein relations, one can recover the modular categories derived from quantum groups of series A, B, C, D. In particular, the categories determined by the series A arise from the Homfly skein relation shown in Figure 2 where $a, s \in K$. The categories determined by the series B, C, D arise from the Kauffman skein relation.

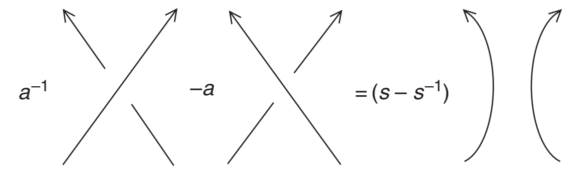

Figure 2 The Homfly relation.

The quantum invariants of 3-manifolds and the TQFTs associated with sl_N can be directly described in terms of the Homfly skein theory, avoiding the language of ribbon categories (W Lickorish, C Blanchet, N Habegger, G Masbaum, P Vogel for sl_2 and Y Yokota for all sl_N).

Unitarity

From both physical and topological viewpoints, one is mainly interested in Hermitian and unitary TQFTs (over $\boldsymbol{k} = \boldsymbol{C}$). A TQFT $\boldsymbol{V}$ is Hermitian if the vector space $\boldsymbol{V}(X)$ is endowed with a nondegenerate Hermitian form $\langle .,. \rangle_X : \boldsymbol{V}(X) \otimes_{\mathbb{C}} \boldsymbol{V}(X) \to \mathbb{C}$ such that:

1. the form $\langle .,. \rangle_X$ is natural with respect to homeomorphisms and multiplicative with respect to disjoint union and
2. for any cobordism (M, X, Y) and any $x \in \boldsymbol{V}(X)$, $y \in \boldsymbol{V}(Y)$,

$$\langle \boldsymbol{V}(M, X, Y)(x), y \rangle_Y = \langle x, \boldsymbol{V}(-M, Y, X)(y) \rangle_X$$

If $\langle .,. \rangle_X$ is positive definite for every X, then the Hermitian TQFT is "unitary." Note two features of Hermitian TQFTs. If $\partial M = \emptyset$, then $\boldsymbol{V}(-M) = \overline{\boldsymbol{V}(M)}$. The group of self-homeomorphisms of any X acts in $\boldsymbol{V}(X)$ preserving the form $\langle .,. \rangle_X$. For a unitary TQFT, this gives an action by unitary matrices.

The three-dimensional TQFT derived from a modular category $\mathcal{V}$ is Hermitian (resp. unitary) under additional assumptions on $\mathcal{V}$ which are discussed briefly. A "conjugation" in $\mathcal{V}$ assigns to each morphism $f : V \to W$ in $\mathcal{V}$ a morphism $\bar{f} : W \to V$ so that

$$\begin{aligned} &\bar{\bar{f}} = f, \quad \overline{f+g} = \bar{f} + \bar{g} && \text{for any } f, g : V \to W \\ &\overline{f \otimes g} = \bar{f} \otimes \bar{g} && \text{for any morphisms } f, g \text{ in } \mathcal{C} \\ &\overline{f \circ g} = \bar{g} \circ \bar{f} && \text{for any morphisms} \\ & && f : V \to W,\ g : W \to V \end{aligned}$$

One calls $\mathcal{V}$ Hermitian if it is endowed with conjugation such that

$$\begin{aligned}\overline{\theta_V} &= (\theta_V)^{-1}, \quad \overline{c_{V,W}} = (c_{V,W})^{-1}\\ \overline{b_V} &= d_V c_{V,V^*}(\theta_V \otimes \mathbf{1}_{V^*})\\ \overline{d_V} &= (\mathbf{1}_{V^*} \otimes \theta_V^{-1}) c_{V^*,V}^{-1} b_V\end{aligned}$$

for any objects V, W of $\mathcal{V}$. A Hermitian modular category $\mathcal{V}$ is unitary if $\mathrm{tr}(f\bar{f}) \geq 0$ for any morphism f in $\mathcal{V}$. The three-dimensional TQFT, derived from a Hermitian (resp. unitary) modular category, has a natural structure of a Hermitian (resp. unitary) TQFT.

The modular category derived from a simple Lie algebra $\mathfrak{g}$ and a root of unity q is always Hermitian. It may be unitary for some q. For simply laced $\mathfrak{g}$, there are always such roots of unity q of any given sufficiently big order. For non-simply-laced $\mathfrak{g}$, this holds under certain divisibility conditions on the order of q.

Integral Structures in TQFTs

The quantum invariants of 3-manifolds have one fundamental property: up to an appropriate rescaling, they are algebraic integers. This was first observed by H Murakami, who proved that $\tau_q^{\mathrm{sl}_2}(M)$ is an algebraic integer, provided the order of q is an odd prime and M is a homology sphere. This extends to an arbitrary closed connected oriented 3-manifold M and an arbitrary simple Lie algebra $\mathfrak{g}$ as follows (Le 2003): for any sufficiently big prime integer r and any primitive rth root of unity q,

$$\tau_q^{P\mathfrak{g}}(M) \in \mathbf{Z}[q] = \mathbf{Z}[\exp(2\pi \mathrm{i}/r)] \qquad [1]$$

This inclusion allows one to expand $\tau_q^{P\mathfrak{g}}(M)$ as a polynomial in q. A study of its coefficients leads to the Ohtsuki invariants of rational homology spheres and further to perturbative invariants of 3-manifolds due to T Le, J Murakami, and T Ohtsuki (see Ohtsuki (2002)). Conjecturally, the inclusion [1] holds for nonprime (sufficiently big) r as well. Connections with the algebraic number theory (specifically modular forms) were studied by D Zagier and R Lawrence.

It is important to obtain similar integrality results for TQFTs. Following P Gilmer, fix a Dedekind domain $D \subset \mathbf{C}$ and call a TQFT $\boldsymbol{V}$ almost D-integral if it is nondegenerate and there is $d \in \mathbf{C}$ such that $d\boldsymbol{V}(M) \in D$ for all M with $\partial M = \emptyset$. Given an almost-integral TQFT $\boldsymbol{V}$ and a surface X, we define $S(\hat{X})$ to be the D-submodule of $\boldsymbol{V}(X)$, generated by all vacuum vectors for X. This module is preserved under the action of self-homeomorphisms of X. It turns out that $S(X)$ is a finitely generated projective D-module and $\boldsymbol{V}(X) = S(X) \otimes_D \mathbf{C}$. A cobordism (M, X, Y) is targeted if all its connected components meet Y along a nonempty set. In this case, $\boldsymbol{V}(M)(S(X)) \subset S(Y)$. Thus, applying S to surfaces and restricting τ to targetet cobordisms, we obtain an "integral version" of $\boldsymbol{V}$. In many interesting cases, the D-module $S(X)$ is free and its basis may be described explicitly. A simple Lie algebra $\mathfrak{g}$ and a primitive rth (in some cases $4r$th) root of unity q with sufficiently big prime r give rise to an almost D-integral TQFT for $D = \mathbf{Z}[q]$.

State-Sum Invariants

Another approach to three-dimensional TQFTs is based on the theory of $6j$-symbols and state sums on triangulations of 3-manifolds. This approach introduced by V Turaev and O Viro is a quantum deformation of the Ponzano–Regge model for the three-dimensional lattice gravity. The quantum $6j$-symbols derived from representations of $U_q(\mathrm{sl}_2\mathbf{C})$ are $\mathbf{C}$-valued rational functions of the variable $q_0 = q^{1/2}$

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \qquad [2]$$

numerated by 6-tuples of non-negative integers i, j, k, l, m, n. One can think of these integers as labels sitting on the edges of a tetrahedron (see Figure 3). The $6j$-symbol admits various equivalent normalizations and we choose the one which has full tetrahedral symmetry. Now, let $q_0 \in \mathbf{C}$ be a primitive $2r$th root of unity with $r \geq 2$. Set $I = \{0, 1, \ldots, r-2\}$. Given a labeled tetrahedron T as in Figure 3 with $i, j, k, l, m, n \in I$, the $6j$-symbol [2] can be evaluated at q_0 and we can obtain a complex number denoted $|T|$. Consider a closed three-dimensional manifold M with triangulation t. (Note that all 3-manifolds can be triangulated.) A coloring of M is a mapping φ from the set $\mathrm{Edg}(t)$ of the edges of t to I. Set

$$|M| = (\sqrt{2r}/(q_0 - q_0^{-1}))^{-2a} \sum_{\varphi} \prod_{e \in \mathrm{Edg}(t)} \langle \varphi(e) \rangle \prod_T |T^\varphi|$$

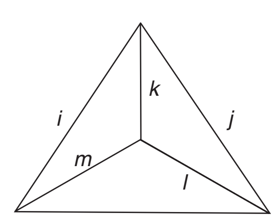

Figure 3 Labeled tetrahedron.

where a is the number of vertices of t, $\langle n \rangle = (-1)^n (q_0^n - q_0^{-n})/(q_0 - q_0^{-1})$ for any integer n, T runs over all tetrahedra of t, and T^φ is T with the labeling induced by φ. It is important to note that $|M|$ *does not depend on the choice of t and thus yields a topological invariant of M*.

The invariant $|M|$ is closely related to the quantum invariant $\tau_q^{\mathfrak{g}}(M)$ for $\mathfrak{g} = \mathrm{sl}_2(\mathbf{C})$. Namely, $|M|$ is the square of the absolute value of $\tau_q^{\mathfrak{g}}(M)$, that is, $|M| = |\tau_q^{\mathfrak{g}}(M)|^2$. This computes $|\tau_q^{\mathfrak{g}}(M)|$ inside M without appeal to surgery. No such computation of the phase of $\tau_q^{\mathfrak{g}}(M)$ is known.

These constructions generalize in two directions. First, they extend to manifolds with boundary. Second, instead of the representation category of $U_q(\mathrm{sl}_2\mathbf{C})$, one can use an arbitrary modular category $\mathcal{C}$. This yields a three-dimensional TQFT, which associates to a surface X a vector space $|X|_{\mathcal{C}}$, and to a 3-cobordism (M, X, Y) a homomorphism $|M|_{\mathcal{C}} : |X|_{\mathcal{C}} \to |Y|_{\mathcal{C}}$, (see Turaev (1994)). When $X = Y = \emptyset$, this homomorphism is multiplication $\mathbf{C} \to \mathbf{C}$ by a topological invariant $|M|_{\mathcal{C}} \in \mathbf{C}$. The latter is computed as a state sum on a triangulation of M involving the $6j$-symbols associated with $\mathcal{C}$. In general, these $6j$-symbols are not numbers but tensors so that, instead of their product, one should use an appropriate contraction of tensors. The vectors in $V(X)$ are geometrically represented by trivalent graphs on X such that every edge is labeled with a simple object of $\mathcal{C}$ and every vertex is labeled with an intertwiner between the three objects labeling the incident edges. The TQFT $|\cdot|_{\mathcal{C}}$ is related to the TQFT $\boldsymbol{V} = \boldsymbol{V}_{\mathcal{C}}$ by $|M|_{\mathcal{C}} = |\boldsymbol{V}(M)|^2$. Moreover, for any closed oriented surface X,

$$
\begin{aligned}
|X|_{\mathcal{C}} &= \mathrm{End}(\boldsymbol{V}(X)) = \boldsymbol{V}(X) \otimes (\boldsymbol{V}(X))^* \\
&= \boldsymbol{V}(X) \otimes \boldsymbol{V}(-X)
\end{aligned}
$$

and for any three-dimensional cobordism (M, X, Y),

$$
\begin{aligned}
|M|_{\mathcal{C}} = \boldsymbol{V}(M) \otimes \boldsymbol{V}(-M) &: \boldsymbol{V}(X) \otimes \boldsymbol{V}(-X) \\
&\to \boldsymbol{V}(Y) \otimes \boldsymbol{V}(-Y)
\end{aligned}
$$

J Barrett and B Westbury introduced a generalization of $|M|_{\mathcal{C}}$ derived from the so-called spherical monoidal categories (which are assumed to be semisimple with a finite set of isomorphism classes of simple objects). This class includes modular categories and a most interesting family of (unitary monoidal) categories arising in the theory of subfactors (see Evans and Kawahigashi (1998) and Kodiyalam and Sunder (2001)). Every spherical category $\mathcal{C}$ gives rise to a topological invariant $|M|_{\mathcal{C}}$ of a closed oriented 3-manifold M. (It seems that this approach has not yet been extended to cobordisms.)

Every monoidal category $\mathcal{C}$ gives rise to a double (or a center) $Z(\mathcal{C})$, which is a braided monoidal category (see Majid (1995)). If $\mathcal{C}$ is spherical, then $Z(\mathcal{C})$ is modular. Conjecturally, $|M|_{\mathcal{C}} = \tau_{Z(\mathcal{C})}(M)$. In the case where $\mathcal{C}$ arises from a subfactor, this has been recently proved by Y Kawahigashi, N Sato, and M Wakui.

The state sum invariants above are closely related to spin networks, spin foam models, and other models of quantum gravity in dimension $2+1$ (see Baez (2000) and Carlip (1998)).

See also: Axiomatic Approach to Topological Quantum Field Theory; Braided and Modular Tensor Categories; Chern–Simons Models: Rigorous Results; Finite-type Invariants of 3-Manifolds; Large-*N* and Topological Strings; Schwarz-Type Topological Quantum Field Theory; Topological Quantum Field Theory: Overview; von Neumann Algebras: Subfactor Theory.

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