CHAPTER 1

Directed graphs and Cuntz-Krieger families

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge e, $s(e)$ is the *source* of e and $r(e)$ the *range* of e; if $s(e) = v$ and $r(e) = w$, then we also say that v *emits* e and that w *receives* e, or that e is an edge from v to w. All the graphs in these notes are directed, so we sometimes get lazy and call them graphs. If there is more than one graph around, we might write r_E and s_E to emphasise that we are talking about the range and source maps for E.

We usually draw a graph by placing the vertices in a plane, and drawing a directed line from $s(e)$ to $r(e)$ for each edge $e \in E^1$. If necessary, we label the edge by its name.

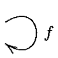
EXAMPLE 1.1. If $E^0 = \{v, w\}$, $E^1 = \{e, f\}$, $r(e) = s(e) = v$, $s(f) = w$ and $r(f) = v$, then we could draw

(1.1)

An edge which begins and ends at the same vertex v, like the edge e in Example 1.1, is called a *loop based at* v[1]. A vertex which does not receive any edges, like the vertex w in Example 1.1, is called a *source*. (Using the word "source" in two ways doesn't seem to cause confusion.) A vertex which emits no edges is called a *sink*.

Conversely, every drawing like (1.1) determines a graph.

EXAMPLE 1.2. The drawing



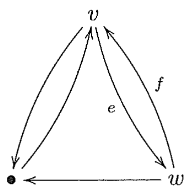
represents a graph E in which $E^0 = \{v\}$, $E^1 = \{e, f\}$ and e and f are both loops based at v. Notice that we are allowing multiple edges between the same pair of vertices; graph theorists often don't allow this.

Drawings are a useful aid when trying to follow arguments about graphs. However, there are many ways to draw the same graph, so it is important to remember that two directed graphs E and F are the same (formally, *isomorphic*) if and only if there are bijections $\phi^0 : E^0 \to F^0$ and $\phi^1 : E^1 \to F^1$ such that $r_F \circ \phi^1 = \phi^0 \circ r_E$ and $s_F \circ \phi^1 = \phi^0 \circ s_E$.

When it doesn't matter what an edge is called, we don't bother to label it in a drawing; when it doesn't matter what a vertex is called, we denote it by a $\bullet$.

[1] This is standard graph-theory terminology. Unfortunately the word "loop" is used in the graph-algebra literature to mean a closed path.

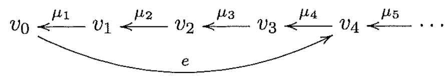
EXAMPLE 1.3. The drawing



represents a graph E with three vertices, two of which are called v and w, and five edges, two of which are called e and f. We do this to simplify notation when we are only going to refer to e and f.

These first three examples are all *finite graphs* in which both E^0 and E^1 are finite sets. In general, we allow either or both to be infinite.

EXAMPLE 1.4. The drawing



represents a graph E in which $E^0 = \{v_n : n \geq 0\}$ is infinite, and E^1 is the union of a singleton set $\{e\}$ and an infinite set $\{\mu_i : i \geq 1\}$.

EXAMPLE 1.5. The drawing

$$\cdots \longrightarrow v_3 \longrightarrow v_2 \longrightarrow v_1 \overset{(\infty)}{\longrightarrow} v_0 \longrightarrow v_{-1} \longrightarrow v_{-2} \longrightarrow \cdots$$

represents a graph E with $E^0 = \{v_i : i \in \mathbb{Z}\}$, one edge from v_{i+1} to v_i for each $i \neq 0$, and infinitely many edges from v_1 to v_0.

For reasons which we will discuss in Chapter 5, graphs in which some vertices receive infinitely many edges pose extra problems for us. So we shall consider mainly the *row-finite graphs* in which each vertex receives at most finitely many edges, that is, in which $r^{-1}(v)$ is a finite set for every $v \in E^0$. The graph in Example 1.4 is row-finite, but that in Example 1.5 is not because v_0 receives infinitely many edges.

REMARK 1.6. The word "row-finite" refers to the corresponding property of the *vertex matrix* A_E of the graph E, which is the $E^0 \times E^0$ matrix defined by

$$A_E(v, w) = \#\{e \in E^1 : r(e) = v,\ s(e) = w\}.$$

(A_E is sometimes called the *adjacency matrix* of E.) The graph E is row-finite if and only if each row $\{A_E(v, w) : w \in E^0\}$ of A_E has finite sum.

We now seek to represent a directed graph by operators on Hilbert space: the vertices will be represented by orthogonal projections and the edges by partial isometries. Formally, let E be a row-finite directed graph and $\mathcal{H}$ a Hilbert space. A *Cuntz-Krieger E-family* $\{S, P\}$ *on* $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$, such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK2) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ whenever v is not a source.

Conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations*, and Condition (CK2) in particular is often called the *Cuntz-Krieger relation at v*.

Saying that the projections P_v are mutually orthogonal means that the ranges $P_v\mathcal{H}$ are mutually orthogonal subspaces of $\mathcal{H}$. The relation (CK1) says that S_e is a partial isometry with initial space $P_{s(e)}\mathcal{H}$ (Proposition A.4); relation (CK2) implies that the range projection $S_e S_e^*$ of S_e is dominated by $P_{r(e)}$, and hence that $S_e\mathcal{H} \subset P_{r(e)}\mathcal{H}$ (Proposition A.1). Thus S_e is an isometry of $P_{s(e)}\mathcal{H}$ onto a closed subspace of $P_{r(e)}\mathcal{H}$; expressing this algebraically gives the relation

$$S_e = P_{r(e)} S_e = S_e P_{s(e)}, \tag{1.2}$$

which is used all the time in manipulations with Cuntz-Krieger families. Relation (CK2) also implies that the partial isometries S_e associated to the edges e with $r(e) = v$ have mutually orthogonal ranges (see Corollary A.3) with span $P_v\mathcal{H}$, so

$$P_v\mathcal{H} = \bigoplus_{\{e \in E^1 : r(e) = v\}} S_e\mathcal{H},$$

in the sense that the map $(h_e) \mapsto \sum_e h_e$ is an isomorphism of the direct sum onto $P_v\mathcal{H}$.

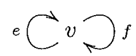
REMARK 1.7. Since the initial and range projections of the S_e are all contained in $\mathcal{H}_e := \overline{\operatorname{span}}\{\bigcup_{v \in E^0} P_v\mathcal{H}\}$, we may as well assume that $\mathcal{H} = \mathcal{H}_e$, in which case we say the family is *non-degenerate*. If $\{S, P\}$ is a non-degenerate Cuntz-Krieger E-family, then the mutual orthogonality of the P_v implies that $\mathcal{H} = \bigoplus_{v \in E^0} P_v\mathcal{H}$, in the sense that the obvious map $(h_v) \mapsto \sum_v h_v$ of

$$\bigoplus_{v \in E^0} P_v\mathcal{H} := \{(h_v) \in \prod_{v \in E^0} P_v\mathcal{H} : \sum_v \|h_v\|^2 < \infty\}$$

into $\mathcal{H}$ is an isomorphism (that $\sum_v \|h_v\|^2 < \infty$ implies that the sum $\sum_v h_v$ converges in norm in $\mathcal{H}$).

REMARK 1.8. Since the orthogonal projections on closed subspaces of $\mathcal{H}$ are the bounded operators P satisfying $P^2 = P = P^*$ and the partial isometries are the bounded operators S satisfying $S = SS^*S$, we can talk about projections and partial isometries in any C^*-algebra B (see Appendix A.1). A Cuntz-Krieger E-family in B then consists of projections $\{P_v \in B : v \in E^0\}$ satisfying $P_v P_w = 0$ for $v \neq w$ (so that $\{P_v\}$ is a mutually orthogonal family of projections) and partial isometries $\{S_e \in B : e \in E^1\}$ satisfying (CK1) and (CK2).

EXAMPLE 1.9. Consider the directed graph:



We have $S_e^* S_e = P_v = S_f^* S_f$, $P_v = S_e S_e^* + S_f S_f^*$. Take $\mathcal{H} = \ell^2(\mathbb{N}) = \overline{\operatorname{span}}\{e_n : n \geq 0\}$, P_v to be the identity operator 1, $S_e(e_n) = e_{2n}$ and $S_f(e_n) = e_{2n+1}$. Then $\{S, P\}$ is a Cuntz-Krieger family for this graph.

In any Cuntz-Krieger family $\{T, Q\}$ for this graph with Q_v non-zero, $Q_v\mathcal{H}$ must be infinite-dimensional. To see this, note that T_e is an isometry of $Q_v\mathcal{H}$ onto $T_e\mathcal{H}$, so $\dim Q_v\mathcal{H} = \dim T_e\mathcal{H}$. Similarly, $\dim Q_v\mathcal{H} = \dim T_f\mathcal{H}$. Thus $Q_v\mathcal{H} = T_e\mathcal{H} \oplus T_f\mathcal{H}$ implies

$$\dim Q_v\mathcal{H} = \dim T_e\mathcal{H} + \dim T_f\mathcal{H} = 2\dim Q_v\mathcal{H},$$

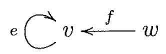
so $\dim Q_v\mathcal{H}$ can only be 0 or ∞.

In general there is no problem finding Cuntz-Krieger E-families with every P_v and every S_e non-zero: take $\mathcal{H}_v$ to be a separable infinite-dimensional Hilbert space for each $v \in E^0$, set $\mathcal{H} = \bigoplus_v \mathcal{H}_v$, take P_v to be the projection of $\mathcal{H}$ on $\mathcal{H}_v$, decompose $\mathcal{H}_v$ as a direct sum $\mathcal{H}_v = \bigoplus_{r(e)=v} \mathcal{H}_{v,e}$ of infinite-dimensional subspaces, and take S_e to be a unitary isomorphism of $\mathcal{H}_{s(e)}$ onto $\mathcal{H}_{r(e),e}$, viewed as a partial isometry on $\mathcal{H}$ with initial space $\mathcal{H}_{s(e)}$.

Example 1.9 shows why we need to take the spaces $\mathcal{H}_v$ to be infinite-dimensional. This is not always necessary, though:

EXAMPLE 1.10. Consider the graph E which consists of a single vertex v and a single loop e based at v. Then the Cuntz-Krieger relations say that $S_e^*S_e = P_v = S_eS_e^*$, so that S_e is a unitary operator on $P_v\mathcal{H}$ (and is 0 on $(P_v\mathcal{H})^\perp$). There is no other restriction on S_e: if U is a unitary operator on $\mathcal{H}$, then we can take $P_v = 1$ and $S_e = U$. So there is no restriction on $\dim \mathcal{H}$; we could even take $\mathcal{H} = \mathbb{C}$ and U to be multiplication by $e^{i\theta}$, for example.

EXAMPLE 1.11. For the graph



we can define a Cuntz-Krieger family on $\mathcal{H} = \ell^2$ by

$$P_v(x_0, x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \quad P_w(x_0, x_1, x_2, \cdots) = (x_0, 0, 0, \cdots),$$
$$S_f(x_0, x_1, x_2, \cdots) = (0, x_0, 0, \cdots), \quad \text{and} \quad S_e(x_0, x_1, x_2, \cdots) = (0, 0, x_1, x_2, \cdots).$$

It is important here that $P_v\mathcal{H}$ is infinite-dimensional: in any Cuntz-Krieger family for this graph, the Cuntz-Krieger relation at v implies that

$$\dim(P_v\mathcal{H}) = \dim(S_f\mathcal{H}) + \dim(S_e\mathcal{H}) = \dim(P_w\mathcal{H}) + \dim(P_v\mathcal{H}),$$

so if P_w and P_v are both non-zero, $P_v\mathcal{H}$ must be infinite-dimensional. It is worth observing now that the crucial factor in this argument is the presence of the edge f entering the loop.

We will be interested in the C^*-algebras $C^*(S,P)$ generated by Cuntz-Krieger families $\{S,P\}$, so we now investigate the $*$-algebraic consequences of the Cuntz-Krieger relations.

PROPOSITION 1.12. *Suppose that E is a row-finite graph and $\{S,P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Then*

(a) *the projections $\{S_eS_e^* : e \in E^1\}$ are mutually orthogonal;*
(b) $S_e^*S_f \neq 0 \Longrightarrow e = f$;
(c) $S_eS_f \neq 0 \Longrightarrow s(e) = r(f)$;
(d) $S_eS_f^* \neq 0 \Longrightarrow s(e) = s(f)$.

PROOF. For part (a), suppose first that $r(e) = r(f)$. Then the Cuntz-Krieger relation at $r(e)$ implies that $P_{r(e)}$ is the sum of $S_eS_e^*$, $S_fS_f^*$ and other projections, which because $P_{r(e)}$ is a projection implies that $S_eS_e^*$ and $S_fS_f^*$ are mutually orthogonal (see Corollary A.3). On the other hand, if $r(e) \neq r(f)$, then (1.2) implies that

$$(S_eS_e^*)(S_fS_f^*) = (S_eS_e^*P_{r(e)})(P_{r(f)}S_fS_f^*) = (S_eS_e^*)0(S_fS_f^*) = 0.$$

Part (b) follows from (a), since $S_e^*S_f = S_e^*(S_eS_e^*)(S_fS_f^*)S_f = 0$ when $e \neq f$. For (c), we just note that part (a) implies that $S_eS_f = (S_eP_{s(e)})(P_{r(f)}S_f)$ vanishes unless $s(e) = r(f)$, and a similar argument gives (d). □

Part (c) of Proposition 1.12 is particularly crucial: it says that S_eS_f is zero unless the pair ef is a path of length 2 in the graph E. More generally, a *path of length n* in a directed graph E is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$. We write $|\mu| := n$ for the length of μ, and regard vertices as paths of length 0; we denote by E^n the set of paths of length n, and write $E^* := \bigcup_{n\geq 0} E^n$. (Now our notation for the sets of vertices and edges should make more sense.) We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ and ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1\cdots\mu_{|\mu|}\nu_1\cdots\nu_{|\nu|}$.

REMARK 1.13. It may seem slightly odd to define the source of $\mu = \mu_1\mu_2\cdots\mu_n$ to be $s(\mu_n)$ rather than $s(\mu_1)$. However, this is forced on us by Proposition 1.12(c): the conventions of composition, which is the multiplication in $B(\mathcal{H})$, say that in the product RT we perform T first. This means that if we want juxtaposition of edges in a path μ to be consistent with juxtaposition of the corresponding partial isometries S_{μ_i} in $B(\mathcal{H})$, then we need to traverse μ_{i+1} before μ_i.

For $\mu \in \prod_{i=1}^n E^1$, we define $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$, and for $v \in E^0$, we define $S_v := P_v$. Proposition 1.12(c) says that $S_\mu = 0$ unless μ is a path; if μ is a path, then

$$\begin{aligned}
S_\mu^*S_\mu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^*S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*(S_{\mu_1}^*S_{\mu_1})S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{s(\mu_1)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{r(\mu_2)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_3}(S_{\mu_2}^*S_{\mu_2})S_{\mu_3}^*\cdots S_{\mu_n} \\
&\ \ \vdots \\
&= P_{s(\mu_n)} = P_{s(\mu)}.
\end{aligned} \tag{1.3}$$

Thus for $\mu \in E^*$, S_μ is a partial isometry with initial projection $P_{s(\mu)}$, and since $P_{r(\mu)}S_\mu S_\mu^* = S_\mu S_\mu^*$, the range of S_μ is a subspace of $P_{r(\mu)}\mathcal{H}$.

Proposition 1.12 extends to the partial isometries S_μ as follows:

COROLLARY 1.14. *Suppose that E is a row-finite graph and $\{S,P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Let $\mu, \nu \in E^*$. Then*

(a) *if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$;*
(b) $S_\mu^*S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise;} \end{cases}$
(c) *if $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu = S_{\mu\nu}$;*
(d) *if $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.*

PROOF. For (a), let i be the smallest integer such that $\mu_i \neq \nu_i$. Then, applying (1.3) to $\mu_1\mu_2\cdots\mu_{i-1}$ gives

$$\begin{aligned} S_\mu^* S_\nu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^* S_{\nu_1}S_{\nu_2}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^*(S_{\mu_{i-1}}^*\cdots S_{\mu_1}^*)(S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{i-1}})S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{s(\mu_{i-1})}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{r(\mu_i)}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* S_{\nu_i}\cdots S_{\nu_n}, \end{aligned}$$

which vanishes by Proposition 1.12(b), giving (a).

For part (b), assume first that $n := |\mu| \le |\nu|$, and factor $\nu = \alpha\nu'$ with $|\alpha| = n$. Then

$$S_\mu^* S_\nu = S_\mu^*(S_\alpha S_{\nu'}) = (S_\mu^* S_\alpha)S_{\nu'}.$$

If $\mu = \alpha$, then (1.3) implies that

$$S_\mu^* S_\nu = P_{s(\mu)}S_{\nu'} = P_{r(\nu')}S_{\nu'} = S_{\nu'}.$$

If $\mu \neq \alpha$, then part (a) implies that $S_\mu^* S_\nu = (S_\mu^* S_\alpha)S_{\nu'} = 0$. This gives (b) when $|\mu| \le |\nu|$. When $|\mu| > |\nu|$, we can either run a similar argument factoring $\mu = \beta\mu'$, or take adjoints and apply what we have just proved.

Parts (c) and (d) follow from the corresponding parts of Proposition 1.12. □

COROLLARY 1.15. *Suppose that E is a row-finite graph and $\{S,P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. For μ, ν, α, $\beta \in E^*$, we have*

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'}S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

In particular, it follows that every non-zero finite product of the partial isometries S_e and S_f^* has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.

PROOF. The formula follows from part (b) of Corollary 1.14. To see the last statement, we suppose that W is a non-zero word — that is, a product of finitely many S_e and S_f^*. Any adjacent S_e's can be combined into a single term S_μ, and since W is non-zero, μ must be a path. Similarly, any adjacent S_f^*'s can be combined into an S_ν^* for some $\nu \in E^*$. Thus W is a product of terms of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$. (Since $E^0 \subset E^*$, we can write S_ν^*, for example, as $S_{s(\nu)}S_\nu^* = P_{s(\nu)}S_\nu^*$.) The formula (1.4) implies that we can combine this product into one term of the same form. □

COROLLARY 1.16. *If $\{S,P\}$ is a Cuntz-Krieger E-family for a row-finite graph E, then*

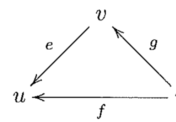
$$C^*(S,P) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}.$$

PROOF. The formula (1.4) implies that

$$\operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$$

is a subalgebra of $C^*(S,P)$, and since $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$, it is a $*$-subalgebra. Thus its closure is a C^*-subalgebra of $C^*(S,P)$, and since it contains the generators $S_e = S_e S_{s(e)}^*$ and $P_v = S_v S_v^*$, it is all of $C^*(S,P)$. □

EXAMPLE 1.17. Let $\{S,P\}$ be a Cuntz-Krieger family for the following directed graph E:



When $s(\mu) = s(\nu)$ we have $S_\mu S_\nu^* = S_\mu P_{s(\mu)} S_\nu^*$; unless $s(\mu) = w$, we can apply the Cuntz-Krieger relation at $s(\mu)$, and keep doing this until the paths begin at w. For example,

$$\begin{aligned} P_u &= S_e S_e^* + S_f S_f^* = S_e P_v S_e^* + S_f S_f^* \\ &= S_e\left(S_g S_g^*\right)S_e^* + S_f S_f^* \\ &= S_{eg}S_{eg}^* + S_f S_f^*. \end{aligned}$$

Thus

$$\begin{aligned} C^*(S,P) &= \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu,\nu \in E^*,\ s(\mu) = s(\nu) = w\} \\ &= \operatorname{span}\{S_\mu S_\nu^* : \mu,\nu \in \{w,f,g,eg\}\}. \end{aligned}$$

Since w is a source, two paths μ, ν with $s(\mu) = w = s(\nu)$ cannot satisfy $\nu = \mu\nu'$ unless $\mu = \nu$. Hence

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_\mu S_\beta^* & \text{if } \alpha = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{S_\mu S_\nu^* : \mu,\nu \in \{w,f,g,eg\}\}$ is a set of matrix units which spans $C^*(S,P)$. So we have by Proposition A.5 that if one is non-zero, they all are, and $C^*(S,P)$ is isomorphic to $M_4(\mathbb{C})$.

A path μ in a directed graph E is a *cycle*[2] if $|\mu| \ge 1$, $r(\mu) = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. In the graph-theory literature, people sometimes insist that $|\mu| > 1$, but this would make our statements much more complicated. The crucial feature of the graph in Example 1.17 is that there are no cycles.

PROPOSITION 1.18. *Suppose E is a finite directed graph with no cycles, and $w_1, \cdots, w_n$ are the sources in E. Then for every Cuntz-Krieger E-family $\{S,P\}$ in which each P_v is non-zero we have*

$$C^*(S,P) \cong \textstyle\bigoplus_{i=1}^n M_{|s^{-1}(w_i)|}(\mathbb{C}),$$

where $s^{-1}(w_i) = \{\mu \in E^* : s(\mu) = w_i\}$.

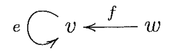
PROOF. As in Example 1.17, finitely many applications of the Cuntz-Krieger relations show that

$$C^*(S,P) = \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i \text{ for some } i\}$$

and $A_i := \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i\}$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$. When $\mu \in s^{-1}(w_i)$ and $\alpha \in s^{-1}(w_j)$ for some $j \neq i$, μ cannot extend α and vice versa. Thus $A_iA_j = 0$, and $C^*(S,P) \cong \bigoplus_{i=1}^n A_i$ by Proposition A.7. □

[2]In the graph-algebra literature, cycles are called *simple loops*.

EXAMPLE 1.19. Consider a Cuntz-Krieger family $\{S, P\}$ for the following directed graph E:



The Cuntz-Krieger relations say that $S_e^*S_e = P_{s(e)} = P_v$, $S_f^*S_f = P_w$ and $P_v = S_eS_e^* + S_fS_f^*$. The element $P_v + P_w$ is an identity for $C^*(S, P)$. The element $S_e + S_f$ satisfies

$$(S_e + S_f)^*(S_e + S_f) = S_e^*S_e + S_f^*S_e + S_e^*S_f + S_f^*S_f = P_v + 0 + 0 + P_w,$$

and hence is an isometry in $C^*(S, P)$. Since

$$(S_e + S_f)(S_e + S_f)^* = S_eS_e^* + S_fS_e^* + S_eS_f^* + S_fS_f^* = P_v,$$

we can recover P_v, $P_w = (S_e+S_f)^*(S_e+S_f) - P_v$, $S_e = (S_e+S_f)P_v$ and $S_f = (S_e+S_f)P_w$ from the single element S_e+S_f. Thus $C^*(S, P)$ is generated by the isometry S_e+S_f. Conversely, if V is an isometry, then $P_w = 1-VV^*$, $P_v = VV^*$, $S_e = VP_v$, $S_f = VP_w$ defines a Cuntz-Krieger E-family such that $C^*(S, P) = C^*(V)$.

Coburn's theorem [**93**, Theorem 3.5.18] says that all C^*-algebras generated by one non-unitary isometry are isomorphic, and in particular isomorphic to the *Toeplitz algebra* $\mathcal{T}$ generated by the unilateral shift. The isometry $S_e + S_f$ is non-unitary precisely when $P_w \neq 0$, so we deduce that all Cuntz-Krieger E-families with $P_w \neq 0$ generate C^*-algebras isomorphic to $\mathcal{T}$.

Proposition 1.18 and Example 1.19 suggest that, provided two Cuntz-Krieger families are non-trivial in the sense that appropriate vertex projections P_v are non-zero, the Cuntz-Krieger families generate isomorphic C^*-algebras. This is indeed a general phenomenon. To study it, we introduce a C^*-algebra which is universal for C^*-algebras generated by Cuntz-Krieger E-families, and analyse the representations of this C^*-algebra.

To build the universal C^*-algebra generated by a Cuntz-Krieger E-family, we mimic the behaviour of the spanning set $\{S_\mu S_\nu^*\}$. In the next proposition, the symbols $d_{\mu,\nu}$ are purely formal, all but finitely many coefficients $z_{\mu,\nu}$ in each sum are 0, and the vector space operations on the formal sums are defined by

$$a\big(\sum w_{\mu,\nu}d_{\mu,\nu}\big) + b\big(\sum z_{\mu,\nu}d_{\mu,\nu}\big) = \sum(aw_{\mu,\nu} + bz_{\mu,\nu})d_{\mu,\nu}.$$

The elements $d_{\alpha,\beta}$ obtained by setting $z_{\alpha,\beta} = 1$ and $z_{\mu,\nu} = 0$ otherwise then form a basis for V.

PROPOSITION 1.20. *Let E be a row-finite directed graph. Then the vector space V of formal linear combinations*

$$V = \big\{\sum z_{\mu,\nu}d_{\mu,\nu} : \mu, \nu \in E^*, s(\mu) = s(\nu)\big\}$$

is a $*$-algebra with $(d_{\mu,\nu})^* = d_{\nu,\mu}$ and

$$d_{\mu,\nu}d_{\alpha,\beta} = \begin{cases} d_{\mu\alpha',\beta} & \text{if } \alpha = \nu\alpha' \\ d_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, one has to check that the product is associative and compatible with the $*$-operation. This is tedious but routine.

For every Cuntz-Krieger family $\{S, P\}$ on $\mathcal{H}$, the operators $\{S_\mu S_\nu^*\}$ satisfy the relations imposed on the $\{d_{\mu,\nu}\}$, and hence there is a $*$-representation $\pi_{S,P}$ of V on $\mathcal{H}$ such that $\pi_{S,P}(d_{\mu,\nu}) = S_\mu S_\nu^*$. Since the norm of a projection P satisfies $\|P\|^2 = \|P^*P\| = \|P\|$, every non-zero projection has norm 1, and thus for every non-zero partial isometry W, we have $\|W\|^2 = \|W^*W\| = 1$. Thus

$$\big\|\pi_{S,P}\big(\sum z_{\mu,\nu}d_{\mu,\nu}\big)\big\| \le \sum |z_{\mu,\nu}|\,\|S_\mu S_\nu^*\| \le \sum |z_{\mu,\nu}|.$$

It follows that

$$\|a\|_1 := \sup\{\|\pi_{S,P}(a)\| : \{S, P\} \text{ is a Cuntz-Krieger } E\text{-family}\}$$

is finite for every v in V, and $\|\cdot\|_1$ is an algebra seminorm satisfying $\|a^*a\|_1 = \|a\|_1^2$. Let I be the $*$-ideal $\{u \in V : \|u\|_1 = 0\}$. Then $V_0 = V/I$ is a $*$-algebra, and the quotient norm $\|\cdot\|_0$ defined by $\|v+I\|_0 = \inf\{\|u+j\|_1 : j \in I\}$ is a C^*-norm, so the completion $\overline{V_0}$ is a C^*-algebra. Each $\pi_{S,P}$ is $\|\cdot\|_0$-continuous, and hence extends uniquely to a representation of $\overline{V_0}$.

We have now outlined the main steps in the proof of:

PROPOSITION 1.21. *For any row-finite directed graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra B, there is a homomorphism $\pi_{T,Q}$ of $C^*(E)$ into B satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in E^0$.*

PROOF. Take $C^*(E) = \overline{V_0}$, and check that $s_e := d_{e,s(e)}$, $p_v := d_{v,v}$ form a Cuntz-Krieger E-family which generates V_0. To get $\pi_{T,Q}$, choose a faithful representation $\rho : B \to B(\mathcal{H})$, and take $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)}$. □

The C^*-algebra $C^*(E)$ is called the *C^*-algebra of the graph E* or the *Cuntz-Krieger algebra of E*, and is generically described as a *graph algebra*. In these notes, $\{s, p\}$ will always be the universal family which generates $C^*(E)$; in general, we will try to use lower-case letters for a Cuntz-Krieger family only when we think the family has a universal property.

Those whose native languages have definite and indefinite articles may have noticed that we have been making implicit uniqueness assertions about $C^*(E)$ and $\{s, p\}$. We take the view that the next corollary justifies this, and that it is okay to talk about "the" graph algebra $C^*(E)$ and "the" generating family $\{s, p\}$, provided we remember when it matters that they are only unique up to isomorphism in the following precise sense.

COROLLARY 1.22. *Suppose E is a row-finite directed graph, and C is a C^*-algebra generated by a Cuntz-Krieger E-family $\{w, r\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra B, there is a homomorphism $\rho_{T,Q}$ of C into B satisfying $\rho_{T,Q}(w_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(r_v) = Q_v$ for every $v \in E^0$. Then there is an isomorphism ϕ of $C^*(E)$ onto C such that $\phi(s_e) = w_e$ for every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

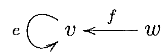
PROOF. We take $\phi := \pi_{w,r}$. It is onto because the range of $\pi_{w,r}$ is a C^*-algebra containing $\{w_e, r_v\}$, hence is all of C. Since $\rho_{s,p} \circ \pi_{w,r}$ is the identity on $\{s, p\}$, it is the identity on all of $C^*(E)$. Thus

$$\phi(a) = 0 \Longrightarrow \pi_{w,r}(a) = 0 \Longrightarrow a = \rho_{s,p}(\pi_{w,r}(a)) = 0,$$

and ϕ is injective. □

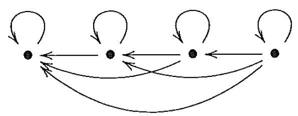
EXAMPLE 1.23. For the graph E which consists of a single loop at a single vertex v, Cuntz-Krieger E-families $\{S, P\}$ are determined by the single operator S_e, which is a unitary operator of $P_v\mathcal{H}$ onto $P_v\mathcal{H}$. The operator P_v is an identity for $C^*(S,P)$, and S_e is a unitary element of $C^*(S,P)$. So $(C^*(E), s_e)$ is universal for C^*-algebras generated by a unitary element: if U is a unitary element of a C^*-algebra B, then there is a homomorphism $\pi_U : C^*(E) \to B$ such that $\pi(s_e) = U$. We know from spectral theory that if $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\iota : \mathbb{T} \to \mathbb{C}$ is the function $\iota(z) = z$, then $(C(\mathbb{T}), \iota)$ has this same universal property. So Corollary 1.22 gives an isomorphism ϕ of $C^*(E)$ onto $C(\mathbb{T})$ such that $\phi(s_e) = \iota$.

EXAMPLE 1.24. In Example 1.19 we considered Cuntz-Krieger families $\{S, P\}$ for the following directed graph E:



We saw there that $S_e + S_f$ is an isometry which generates $C^*(S,P)$, and that every isometry on Hilbert space gives a Cuntz-Krieger E-family. Thus $(C^*(E), s_e + s_f)$ is the universal C^*-algebra (A, a) generated by an isometry a. In the representation-theoretic analysis of (A, a), Coburn's theorem becomes the assertion that, if π is a representation of A and $\pi(a)$ is non-unitary, then π is faithful on A (see [**1**], for example).

EXAMPLE 1.25. For a more exotic example, consider the following graph E:



Hong and Szymański prove in [**56**, Theorem 4.4] that $C^*(E)$ is isomorphic to the non-commutative sphere $C(S_q^7)$ of Vaksman and Soibelman, by checking that $C(S_q^7)$ has the universal property which characterises $C^*(E)$ and applying Corollary 1.22. In [**56**] and [**58**], they show that a broad range of non-commutative spheres, projective spaces and lens spaces are isomorphic to the C^*-algebras of suitable directed graphs.

REMARK 1.26. We have insisted that our graphs are countable, and hence all our graph algebras are separable. However, we have not really used this hypothesis, and one can talk about graph algebras of uncountable graphs. Katsura has recently shown that there is a graph E with uncountably many vertices whose C^*-algebra is prime but not primitive [**77**, Proposition 13.4]. Of course one can take this two ways: as evidence that uncountable graphs are interesting, or as evidence that they should be avoided.

CHAPTER 2

Uniqueness theorems for graph algebras

Since the C^*-algebra of a graph E has a universal property, we can prove that a C^*-algebra B is isomorphic to $C^*(E)$ by finding a Cuntz-Krieger E-family $\{T, Q\}$ which generates B and has the universal property (see Corollary 1.22). The first two big theorems of the subject say that it is often not necessary to check that $\{T, Q\}$ has the universal property. In this chapter we discuss these theorems and their implications, and in the next chapter we will prove them.

The first of these uniqueness theorems says that $C^*(E)$ is characterised by the existence of a special action of the circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, which is a compact topological group under multiplication. In general, an *action* of a locally compact group G on a C^*-algebra A is a homomorphism $s \mapsto \alpha_s$ of G into the group $\operatorname{Aut} A$ of automorphisms of A such that $s \mapsto \alpha_s(a)$ is continuous for each fixed $a \in A$. The particular action γ constructed in the next proposition is called the *gauge action* of $\mathbb{T}$ on $C^*(E)$.

PROPOSITION 2.1. *Let E be a row-finite directed graph. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$ such that $\gamma_z(s_e) = zs_e$ for every $e \in E^1$ and $\gamma_z(p_v) = p_v$ for every $v \in E^0$.*

PROOF. Fix $z \in \mathbb{T}$. Then $\{zs, p\} = \{zs_e, p_v\}$ is a Cuntz-Krieger E-family which generates $C^*(E)$. If $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then so is $\{\overline{z}T, Q\}$, and

$$\pi_{\overline{z}T,Q}(zs_e) = z\pi_{\overline{z}T,Q}(s_e) = z(\overline{z}T_e) = T_e.$$

Thus with $\rho_{T,Q} := \pi_{\overline{z}T,Q}$, the pair $(C^*(E), \{zs, p\})$ has the property described in Corollary 1.22, and hence there is an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. For $w \in \mathbb{T}$, the isomorphisms $\gamma_z \circ \gamma_w$ and γ_{zw} agree on generators, and hence on all of $C^*(E)$. So γ is a homomorphism of $\mathbb{T}$ into $\operatorname{Aut} C^*(E)$.

To check continuity, fix $z \in \mathbb{T}$, $a \in C^*(E)$ and $\epsilon > 0$. Choose $c := \sum \lambda_{\mu,\nu} s_\mu s_\nu^*$ such that $\|a - c\| < \epsilon/3$. Notice that $\gamma_z(s_\mu) = z^{|\mu|} s_\mu$. Thus, since scalar multiplication is continuous, so is

$$w \mapsto \gamma_w(c) = \sum \lambda_{\mu,\nu} w^{|\mu|-|\nu|} s_\mu s_\nu^*,$$

and there exists $\delta > 0$ such that $|w - z| < \delta \implies \|\gamma_w(c) - \gamma_z(c)\| < \epsilon/3$. Since automorphisms of C^*-algebras preserve the norm, we have $\|\gamma_z(a - c)\| < \epsilon/3$. Thus for $|w - z| < \delta$ we have

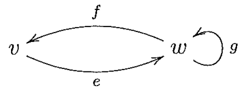
$$\|\gamma_w(a) - \gamma_z(a)\| \leq \|\gamma_w(a - c)\| + \|\gamma_w(c) - \gamma_z(c)\| + \|\gamma_z(a - c)\| < 3(\epsilon/3) = \epsilon,$$

as required. □

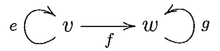
THEOREM 2.2 (The gauge-invariant uniqueness theorem). *Let E be a row-finite directed graph, and suppose that $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z(T_e) = zT_e$ for every $e \in E^1$ and $\beta_z(Q_v) = Q_v$ for every $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Notice that the gauge-invariant uniqueness theorem has no hypotheses on the graph, and hence it is very useful for proving general statements about graph algebras. For a wide class of graphs, though, we can do even better. Recall that a *cycle* in a directed graph E is a path $\mu = \mu_1 \cdots \mu_n$ with $n \geq 1$, $s(\mu_n) = r(\mu_1)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. An edge e is an *entry* to the cycle μ if there exists i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$. Our second uniqueness theorem says that, provided every cycle has an entry, all non-trivial Cuntz-Krieger families generate isomorphic C^*-algebras. Notice that this hypothesis may be trivially satisfied if, for example, E has no cycles.

EXAMPLE 2.3. In the following directed graph E



ef, fe and g are cycles; fge and $efef$ are closed paths but are not cycles, because they visit w twice. Every cycle has an entry; for example, g is an entry to ef. However, in the graph F



the cycle e has no entry.

THEOREM 2.4 (The Cuntz-Krieger uniqueness theorem). *Suppose E is a row-finite directed graph in which every cycle has an entry, and $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

As the name suggests, this theorem is essentially due to Cuntz and Krieger **[16]** (see Remark 2.17 below). The name is often used to refer to the following consequence of Theorem 2.4, which looks more like their original uniqueness theorem.

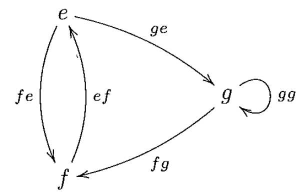
COROLLARY 2.5 (The Cuntz-Krieger uniqueness theorem). *Suppose E is a row-finite directed graph in which every cycle has an entry. If $\{S, P\}$ and $\{T, Q\}$ are two Cuntz-Krieger E-families on Hilbert space such that $P_v \neq 0$ and $Q_v \neq 0$ for all $v \in E^0$, then then there is an isomorphism ϕ of $C^*(S, P)$ onto $C^*(T, Q)$ such that $\phi(S_e) = T_e$ for every $e \in E^1$ and $\phi(P_v) = Q_v$ for every $v \in E^0$.*

PROOF. Theorem 2.4 implies that both $\pi_{S,P}$ and $\pi_{T,Q}$ are faithful representations of $C^*(E)$. Then $\phi := \pi_{T,Q} \circ \pi_{S,P}^{-1}$ has the required properties. □

We will prove these uniqueness theorems in the next chapter, but first we want to illustrate how they are used. We begin with two applications of the gauge-invariant uniqueness theorem. As we will see in Remark 2.8, the first has historical interest.

COROLLARY 2.6. *Suppose E is a row-finite directed graph with no sources, and define the* dual graph $\widehat{E}$ *by $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$, $r_{\widehat{E}}(ef) = e$ and $s_{\widehat{E}}(ef) = f$. Then $\widehat{E}$ is row-finite and $C^*(\widehat{E}) \cong C^*(E)$.*

EXAMPLE 2.7. For the graph E of Example 2.3, the dual graph $\widehat{E}$ looks like



PROOF OF COROLLARY 2.6. For $e \in \widehat{E}^0 = E^1$,

$$\#r_{\widehat{E}}^{-1}(e) = \#\{ef \in E^2 = \widehat{E}^1\} = \#\{f : s_E(e) = r_E(f)\} = \#r_E^{-1}(s_E(e))$$

is finite because E is row-finite. Thus $\widehat{E}$ is row-finite.

Let $\{s, p\}$ be the universal Cuntz-Krieger family generating $C^*(E)$, and define $Q_e := s_e s_e^*$, $T_{fe} := s_f s_e s_e^*$. The projections Q_e are mutually orthogonal because the range projections in any Cuntz-Krieger family are mutually orthogonal. For $fe \in \widehat{E}^1$ we have

$$\begin{aligned} T_{fe}^* T_{fe} &= (s_f s_e s_e^*)^*(s_f s_e s_e^*) = s_e s_e^* s_f^* s_f s_e s_e^* \\ &= s_e s_e^* p_{s(f)} s_e s_e^* = s_e s_e^* p_{r(e)} s_e s_e^* \\ &= s_e s_e^* = Q_e = Q_{s(fe)}; \end{aligned}$$

since Q_e is a projection, this also implies that T_{fe} is a partial isometry. To verify the Cuntz-Krieger $\widehat{E}$-relation at $f \in \widehat{E}^0$, we compute

$$\begin{aligned} Q_f &= s_f s_f^* = s_f p_{s(f)} s_f^* \\ &= s_f \Big(\sum_{r(e)=s(f)} s_e s_e^* \Big) s_f^* = s_f \Big(\sum_{r(e)=s(f)} (s_e s_e^*)(s_e s_e^*) \Big) s_f^* \\ &= \sum_{r(e)=s(f)} s_f (s_e s_e^*)(s_e s_e^*) s_f^* = \sum_{r(fe)=f} T_{fe} T_{fe}^*. \end{aligned}$$

Thus $\{T, Q\}$ is a Cuntz-Krieger $\widehat{E}$-family, and the universal property of $C^*(\widehat{E})$ gives a homomorphism $\pi_{T,Q} : C^*(\widehat{E}) \to C^*(E)$ which carries the canonical generating Cuntz-Krieger family $\{t, q\}$ into $\{T, Q\}$. We can check on generators that the homomorphism $\pi_{T,Q}$ intertwines the gauge actions, and since all the s_e are non-zero, so are the projections Q_e. Thus the gauge-invariant uniqueness theorem implies that $\pi_{T,Q}$ is an isomorphism of $C^*(\widehat{E})$ onto $C^*(T, Q)$. Since the generators $p_v = \sum_{r(e)=v} Q_e$ and $s_f = \sum_{s(f)=r(e)} T_{fe}$ of $C^*(E)$ lie in the range of $\pi_{T,Q}$ and the range of $\pi_{T,Q}$ is a C^*-algebra, $\pi_{T,Q}$ is surjective. (It is at this last step that we need to know that E has no sources: if v is a source, we cannot recover p_v from the Q_e.) □

REMARK 2.8. In their pioneering paper **[16]**, Cuntz and Krieger considered an $n \times n$ matrix $A = (a_{ij})$ with each entry a_{ij} either 0 or 1 and no zero rows or columns. Nowadays, we define the *Cuntz-Krieger algebra* $\mathcal{O}_A$ to be the universal

C^*-algebra generated by partial isometries s_i satisfying $s_i^* s_i = \sum_j a_{ij} s_j s_j^*$. (For the original definition, see Remark 2.17 below.) We can define a graph E_A by $E_A^0 = \{1, 2, \cdots, n\}$, $E_A^1 = \{ij : a_{ij} = 1\}$, $s(ij) = j$ and $r(ij) = i$, and recover A as the vertex matrix of E_A. The argument of Corollary 2.6 shows that $\mathcal{O}_A$ is isomorphic to $C^*(E_A)$.

The vertex matrix A_E of a finite directed graph E does not necessarily have entries in $\{0, 1\}$, but the *edge matrix* B_E defined by

$$B_E(e, f) = \begin{cases} 1 & \text{if } s(e) = r(f) \\ 0 & \text{otherwise} \end{cases}$$

does. The graph E_{B_E} is the dual graph $\widehat{E}$, so Corollary 2.6 implies that $C^*(E) \cong C^*(\widehat{E}) \cong \mathcal{O}_{B_E}$. Thus the Cuntz-Krieger algebras are the C^*-algebras of finite graphs with no sinks or sources. These connections were made by Enomoto and Watatani [**36**] very soon after [**16**] appeared.

REMARK 2.9. In [**6**], Bates extended Corollary 2.6 to the higher-order duals $E_{m,n}$ in which $m < n$, $E_{m,n}^0 = E^m$, $E_{m,n}^1 = E^n$, $s(\mu) = \mu_{n-m+1} \cdots \mu_n$ and $r(\mu) = \mu_1 \cdots \mu_m$. The main result of [**6**] says that $C^*(E_{m,n})$ is naturally isomorphic to $C^*(E_{p,q})$ when $n - m = q - p$, and not in general otherwise. The graphs $E_{0,2}$ were used by Hong and Szymański in their description of the non-commutative real projective spaces (see [**56**, Proposition 1.1]).

Our second application of the gauge-invariant uniqueness theorem is a technique for handling problems involving graphs with sinks and sources. Suppose E is a row-finite directed graph. We form a new graph F by *adding a head*

$$v \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

to each source v, and *adding a tail*

$$\cdots \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow w$$

to each sink w. The idea is that for many purposes, replacing E by F does not change the graph algebra in a substantial way. To make this precise, we need some C^*-algebra background and a lemma.

If A is a C^*-algebra which does not have an identity, such as $C^*(E)$ when E^0 is infinite, there are several ways of embedding it in a larger C^*-algebra with an identity. The *multiplier algebra* $M(A)$ consists of pairs (L, R) of maps from A to itself which satisfy the relation $aL(b) = R(a)b$. The maps L and R in such a pair are automatically bounded and linear with the same operator norm, and $M(A)$ is a C^*-algebra with $\|(L, R)\| = \|L\| = \|R\|$, $(L_1, R_1)(L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1)$ and $(L, R)^* = (R^\sharp, L^\sharp)$, where $R^\sharp(a) = R(a^*)^*$ (see [**93**, Theorem 2.1.5], for example). For $a \in A$, the pair (L_a, R_a) defined by $L_a(b) = ab$ and $R_a(b) = ba$ is a multiplier of A, and the map $a \mapsto (L_a, R_a)$ is an isomorphism of A onto an ideal A in $M(A)$. In general we think of the maps L_m, R_m defining a multiplier $m \in M(A)$ as left and right multiplication by m, and write ma for $L_m(a)$ and am for $R_m(a)$. Motivating examples are $A = \mathcal{K}(\mathcal{H})$, when $T \mapsto (L_T, R_T)$ is an isomorphism of $B(\mathcal{H})$ onto $M(\mathcal{K}(\mathcal{H}))$, and $A = C_0(X)$ for X locally compact, when pointwise multiplication of functions gives an isomorphism of the C^*-algebra $C_b(X)$ of bounded continuous functions onto $M(C_0(X))$. (These are proved in [**114**, Proposition 2.55]; a different (equivalent) definition of $M(A)$ is used in [**114**], but the ideas of the proof work with both definitions.)

For us, multiplier algebras arise because we want to make sense of infinite sums of vertex projections in $C^*(E)$. An infinite sum $\sum_{n=1}^\infty p_n$ of mutually orthogonal projections cannot converge in norm, because each difference $\sum_{n=M+1}^N p_n$ between partial sums is a projection and hence has norm 1. However, we can safely think of the projection p_V in the following lemma as $\sum_{v \in V} p_v$.

LEMMA 2.10. *Let E be a row-finite graph, and let V be an infinite set of vertices in E. Then there is a projection p_V in $M(C^*(E))$ such that*

$$p_V s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } r(\mu) \in V \\ 0 & \text{if } r(\mu) \notin V. \end{cases} \tag{2.1}$$

PROOF. Since the elements $s_\mu s_\nu^*$ span a dense subspace of $C^*(E)$ and left multiplication by a multiplier is linear and continuous, there is at most one multiplier p_V satisfying (2.1). So while we choose to list V as $V = \{v_1, v_2, \cdots\}$, the multiplier we define will not depend on this choice of listing.

For $N \in \mathbb{N}$, let $p_N = \sum_{n=1}^N p_{v_n}$. Then

$$p_N s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } r(\mu) = v_n \text{ for some } n \le N \\ 0 & \text{otherwise,} \end{cases}$$

and hence for $a \in A_0 := \operatorname{span}\{s_\mu s_\nu^* : s(\mu) = s(\nu)\}$, the sequence $\{p_N a : N \in \mathbb{N}\}$ is eventually constant. We define $L, R : A_0 \to A_0$ by $L(a) = \lim_{N\to\infty} p_N a$ and $R(a) = \lim_{N\to\infty} a p_N$. Since each p_N is a projection and hence has norm 1, both L and R are continuous, and extend to linear maps of $C^*(E) = \overline{A_0}$ into itself; an $\epsilon/3$ argument shows that we still have $L(b) = \lim_{N\to\infty} p_N b$ for all $b \in C^*(E)$. Thus for $a, b \in C^*(E)$, we have

$$aL(b) = a\big(\lim_{N\to\infty} p_N b\big) = \lim_{N\to\infty} a(p_N b) = \lim_{N\to\infty}(a p_N) b = R(a) b,$$

and $p_V := (L, R)$ is a multiplier of $C^*(E)$. For each $s_\mu s_\nu^*$, $r(\mu) \in V$ if and only if $r(\mu) = v_n$ for some n, and hence if and only if $p_N s_\mu s_\nu^* = s_\mu s_\nu^*$ for large N. Thus p_V satisfies (2.1). Since $p_M p_N = p_M$ for $N \ge M$, we have

$$p_V(p_V b) = \lim_M p_M(\lim_N p_N b) = \lim_M(\lim_N p_M p_N b) = \lim_M p_M b = p_V b,$$

so $p_V^2 = p_V$. Since it is easy to check that $p_V^* = p_V$, it follows that p_V is a projection. □

If p is a projection in the multiplier algebra $M(A)$ of a C^*-algebra A, then the set $pAp := \{pap : a \in A\}$ is a C^*-subalgebra of A, which is called a *corner* of A. The corner is *full* if it is not contained in any proper ideal of A, or in other words if the set ApA spans a dense subspace of A. The motivating example is the projection $p = e_{11}$ in $A = M_n(\mathbb{C})$, when $e_{11} M_n(\mathbb{C}) e_{11}$ is the algebra of matrices which are non-zero only in the top left-hand corner. As we shall see later, full corners inherit many important properties from the ambient C^*-algebra A.

We can now state our second application of the gauge-invariant uniqueness theorem.

COROLLARY 2.11. *Suppose that E is a row-finite directed graph and F is the directed graph obtained by adding a head to every source of E and a tail to every sink. Denote by $\{s, p\}$ and $\{t, q\}$ the canonical Cuntz-Krieger families generating $C^*(E)$ and $C^*(F)$, and let q_E be the projection in $M(C^*(F))$ obtained by applying Lemma 2.10 to the subset E^0 of F^0. Then $q_E C^*(F) q_E$ is a full corner in $C^*(F)$,* *and there is an isomorphism* ϕ *of* $C^*(E)$ *onto* $q_E C^*(F) q_E$ *such that* $\phi(s_e) = t_e$ *for* $e \in E^1$ *and* $\phi(p_v) = q_v$ *for* $v \in E^0$.

PROOF. Because there is no Cuntz-Krieger relation at a source in E, and because all the edges entering a sink in E are in E^1, the elements $\{t_e : e \in E^1\}$ and $\{q_v : v \in E^0\}$ form a Cuntz-Krieger E-family in $C^*(F)$. The resulting homomorphism $\phi := \pi_{t,q} : C^*(E) \to C^*(F)$ intertwines the gauge actions, and hence the gauge-invariant uniqueness theorem implies that ϕ is injective.

For $\mu, \nu \in F^*$ with $s(\mu) = s(\nu)$, we have

$$q_E t_\mu t_\nu^* q_E = \begin{cases} t_\mu t_\nu^* & \text{if } r(\mu) \in E^0 \text{ and } r(\nu) \in E^0 \\ 0 & \text{otherwise.} \end{cases}$$

If $r(\mu), r(\nu) \in E^0$, then either $s(\mu) = s(\nu) \in E^0$ or $s(\mu) = s(\nu)$ lies on one of the heads we have added at a source w, in which case there is a final segment α of the head such that $\mu = \mu'\alpha$ and $\nu = \nu'\alpha$, and

$$t_\mu t_\nu^* = t_{\mu'} t_\alpha t_\alpha^* t_{\nu'}^* = t_{\mu'} p_w t_{\nu'}^* = t_{\mu'} t_{\nu'}^*.$$

Thus every non-zero element of the form $q_E t_\mu t_\nu^* q_E$ is equal to one of the form $t_{\mu'} t_{\nu'}^*$ with $r(\mu')$, $r(\nu')$ and $s(\mu') = s(\nu')$ all in E^0. Since $a \mapsto q_E a q_E$ is continuous and linear, we deduce that

$$\begin{aligned} q_E C^*(F) q_E &= \overline{\operatorname{span}}\{t_\mu t_\nu^* : r(\mu) \in E^0,\ r(\nu) \in E^0 \text{ and } s(\mu) = s(\nu) \in E^0\} \\ &= \pi_{t,q}(C^*(E)). \end{aligned}$$

To see that $q_E C^*(F) q_E$ is full, suppose first that x is a vertex on a head. Then there is a unique path α from x to a source w of E, and $t_\alpha = q_w t_\alpha$, so t_α and $q_x = t_\alpha^* t_\alpha$ belong to the ideal generated by $q_E C^*(F) q_E$. It follows that every $t_f = q_{r(f)} t_f$ associated to an edge f on a head is also in this ideal. A similar argument works on tails, so the entire generating family $\{t_f, q_v\}$ lies in the ideal generated by $q_E C^*(F) q_E$. Thus this ideal is all of $C^*(F)$, and q_E is full. □

Corners arise frequently in this subject, and we digress briefly to discuss another situation where they arise. We say that a C^*-algebra A is *approximately finite-dimensional*, or that A is an *AF-algebra*, if there is a sequence of finite-dimensional C^*-subalgebras A_n such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^\infty A_n}$. We will prove a theorem of Drinen which says that every AF-algebra is isomorphic to a corner in a graph algebra. For clarity, we assume here that A is unital; the details in the non-unital case are in [**28**] and [**145**].

A *Bratteli diagram* is a locally finite directed graph E with no sources and just one sink v_0, in which E^0 is partitioned as a disjoint union $E^0 = \bigcup_{n=0}^\infty V_n$ of finite sets V_n such that $V_0 = \{v_0\}$, and such that for each edge e there exists $n \in \mathbb{N}$ such that $s(e) \in V_n$ and $r(e) \in V_{n-1}$. (Warning: we have had to reverse the usual convention to fit our definition of Cuntz-Krieger family.) For each $v \in E^0$, we denote by n_v the number of paths $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = v_0$. A Bratteli diagram E is a *Bratteli diagram of an AF-algebra* A if there is an increasing sequence of finite-dimensional unital C^*-subalgebras A_n isomorphic to $\bigoplus_{v \in V_n} M_{n_v}(\mathbb{C})$ such that $A = \overline{\bigcup_{n=1}^\infty A_n}$, and such that the inclusion of A_n in A_{n+1} maps $M_{n_v}(\mathbb{C})$ into $M_{n_w}(\mathbb{C})$ with multiplicity

$$A_E(v, w) := \#\{e \in E^1 : r(e) = v, s(e) = w\},$$

so that the dimensions of the $M_{n_v}(\mathbb{C})$ satisfy $n_w = \sum_{v \in V_n} A_E(v, w) n_v$. Bratteli investigated the relationship between the structure of AF-algebras and the associated Bratteli diagrams [**11**], and in particular proved that two unital AF-algebras with the same diagram are isomorphic (see [**18**, Proposition III.2.7]).

PROPOSITION 2.12. *Let* A *be a unital AF-algebra and let* $(E, \{V_n\}, v_0)$ *be a Bratteli diagram for* A. *Then there is an isomorphism of* A *onto* $p_{v_0} C^*(E) p_{v_0}$, *and this is a full corner of* $C^*(E)$.

PROOF. For each n and $v \in V_n$, we let

$$B_n(v) := \operatorname{span}\{s_\mu s_\nu^* : r(\mu) = r(\nu) = v_0, s(\mu) = s(\nu) = v\}.$$

The elements $s_\mu s_\nu^*$ are matrix units which span $B_n(v)$; since there are n_v paths μ with $s(\mu) = v$ and $r(\mu) = v_0$, it follows from Proposition A.5 that $B_n(v)$ is isomorphic to $M_{n_v}(\mathbb{C})$. If w is another vertex in V_n, then $B_n(v) B_n(w) = 0$, and hence it follows from Proposition A.7 that

$$B_n := \operatorname{span}\{s_\mu s_\nu^* : r(\mu) = r(\nu) = v_0, s(\mu) = s(\nu) \in V_n\}$$

is the C^*-algebraic direct sum $\bigoplus_{v \in V_n} B_n(v)$. The Cuntz-Krieger relations tell us how B_n embeds in B_{n+1}: if $r(\mu) = r(\nu) = v_0$ and $v = s(\mu) = s(\nu) \in V_n$, then

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = \sum_{r(e) = s(\mu)} s_\mu s_e s_e^* s_\nu^* = \sum_{r(e) = s(\mu)} s_{\mu e} s_{\nu e}^*$$

contains exactly $A_E(v, w)$ terms in which $s(\mu e) = w = s(\nu e)$. Thus $B := \overline{\bigcup_{n=1}^\infty B_n}$ is an AF-algebra with Bratteli diagram E.

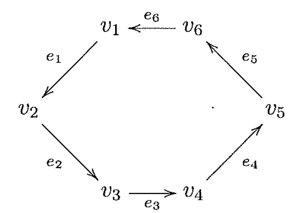
As in the proof of Corollary 2.11, compressing with p_{v_0} shows that

$$p_{v_0} C^*(E) p_{v_0} = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu) \text{ and } r(\mu) = r(\nu) = v_0\},$$

which is B. So the theorem of Bratteli we quoted above implies that A is isomorphic to $p_{v_0} C^*(E) p_{v_0}$. To see that the corner is full, let $v \in E^0$, and choose a path μ with $s(\mu) = v$ and $r(\mu) = v_0$. Then $s_\mu = p_{v_0} s_\mu$ belongs to the ideal generated by $p_{v_0} C^*(E) p_{v_0}$, and hence so does $p_v = s_\mu^* s_\mu$. Thus every vertex projection is in the ideal, and so is every $s_e = s_e p_{s(e)}$. □

REMARK 2.13. The graph algebra $C^*(E)$ of the Bratteli diagram E is itself AF, though it is not unital. Indeed, it is shown in [**82**, Theorem 2.4] that the C^*-algebra of a row-finite graph E is AF if and only if the graph E contains no cycles. This certainly applies to any Bratteli diagram, but it is easy to draw infinite graphs without cycles which are not Bratteli diagrams.

EXAMPLE 2.14. To see how there can fail to be uniqueness when there is no gauge action, consider the graph C_n which consists of a single cycle with n edges. Label the vertices as $\{v_1, v_2, \cdots, v_n\}$ and the edges as $\{e_1, e_2, \cdots, e_n\}$ in such a way that $s(e_i) = v_i$. For example, C_6 looks like



Let P_{v_i} be the $n \times n$ matrix unit e_{ii}, let S_{e_i} be the matrix unit $e_{(i+1)i}$ for $i < n$, and let $S_{e_n} = e_{1n}$. Then $\{S, P\}$ is a Cuntz-Krieger C_n-family which generates $M_n(\mathbb{C})$. However, $\pi_{S,P}$ is definitely not an isomorphism. The problem is that this family satisfies extra relations like $P_{v_1} = S_\mu := S_{e_n} \cdots S_{e_2} S_{e_1}$, whereas the Cuntz-Krieger relations demand only that S_μ has P_{v_1} as its initial and final projection, or in other words that S_μ is a unitary on the range of P_{v_1}. This suggests the fix: the C^*-algebra generated by the unitary could be any quotient of $C(\mathbb{T})$, so we need to insert a copy of $\mathbb{T}$.

Consider the C^*-algebra $C(\mathbb{T}, M_n(\mathbb{C}))$ with operations defined pointwise and norm defined in terms of the usual norm on $M_n(\mathbb{C})$ by $\|f\| := \sup_z \|f(z)\|$. We claim that if we set $P_{v_i}(z) = e_{ii}$, $S_{e_i}(z) = e_{(i+1)i}$ for $i < n$, and $S_{e_n}(z) = z e_{1n}$, then $\pi_{S,P}$ is an isomorphism of $C^*(C_n)$ onto $C(\mathbb{T}, M_n(\mathbb{C}))$. (Notice that the unitary S_μ is then the function $z \in C(\mathbb{T}) = e_{11} C(\mathbb{T}, M_n(\mathbb{C})) e_{11}$ which we know to be universal for unitary elements of C^*-algebras.)

It is easy to check by manipulating matrix units that $\{S, P\}$ is a Cuntz-Krieger family. Next, observe that since every e_{ij} can be factored as a product involving arbitrarily many copies of the product $e_{1n} e_{n(n-1)} \cdots e_{21}$, every function of the form $z \mapsto z^m e_{ij}$ for $m \geq 0$ is in the $*$-algebra generated by $\{S, P\}$; taking adjoints shows that this is also true for $m < 0$. Thus the range of $\pi_{S,P}$ contains every function of the form $z \mapsto \sum_{i,j=1}^n f_{ij}(z) e_{ij}$ in which each f_{ij} is a trigonometric polynomial. Since such functions are dense in $C(\mathbb{T}, M_n(\mathbb{C}))$, the range of $\pi_{S,P}$ must be all of $C(\mathbb{T}, M_n(\mathbb{C}))$. Next we need to find a candidate for the gauge action. For fixed $w \in \mathbb{T}$, let U_w be the diagonal unitary matrix $\sum_{j=1}^n w^j e_{jj}$, and define $\beta_w \in \operatorname{Aut} C(\mathbb{T}, M_n(\mathbb{C}))$ by

$$\beta_w(f)(z) = U_w f(w^n z) U_w^*.$$

The functions $P_{v_i} : z \mapsto e_{ii}$ are constant, and the diagonal matrix U_w commutes with e_{ii}, so $\beta_w(P_{v_i}) = P_{v_i}$; similarly, for $i < n$ we have

$$\begin{aligned}\beta_w(S_{e_i})(z) &= U_w e_{(i+1)i} U_w^* = \Big(\sum_{j=1}^n w^j e_{jj} e_{(i+1)i}\Big) U_w^* \\ &= w^{i+1} e_{(i+1)i} \Big(\sum_{k=1}^n w^{-k} e_{kk}\Big) \\ &= w^{i+1} w^{-i} e_{(i+1)i} = w S_{e_i}(z).\end{aligned}$$

When we compute $\beta_w(S_{e_n})$, however, the w^n in the variable comes into play:

$$\begin{aligned}\beta_w(S_{e_i})(z) &= U_w e_{1n} (w^n z) U_w^* = U_w (w^n z) e_{1n} U_w^* \\ &= (w^n z)(w^1 w^{-n} e_{1n}) = w z e_{1n} = w S_{e_n}(z).\end{aligned}$$

Thus β is an action of $\mathbb{T}$ on $C(\mathbb{T}, M_n(\mathbb{C}))$ such that $\pi_{S,P} \circ \gamma_w = \beta_w \circ \pi_{S,P}$ for every $w \in \mathbb{T}$, and the gauge-invariant uniqueness theorem implies that $\pi_{S,P}$ is an isomorphism.

The vertex matrix of the graph C_n is a permutation matrix E_n. Though Cuntz and Krieger did not explicitly define $\mathcal{O}_{E_n}$, they comment in [**16**] that every quotient of $C(\mathbb{T}, M_2(\mathbb{C}))$ is generated by a Cuntz-Krieger family, and Evans confirmed that a spatial construction of $\mathcal{O}_A$ yields $\mathcal{O}_{E_n} = C(\mathbb{T}, M_n(\mathbb{C}))$ [**38**, Theorem 2.2]. The above argument is from [**60**].

The Cuntz-Krieger uniqueness theorem does not apply to every graph, but when it does apply, it gives sharper results. Here we discuss a couple of examples, and we will see further applications in Theorem 4.9 and Lemma 7.10, for example.

EXAMPLE 2.15. In Example 1.24, we saw that when E is the graph

$$e \circlearrowleft v \xleftarrow{\ f\ } w,$$

$(C^*(E), s_e + s_f)$ is universal for C^*-algebras generated by an isometry. The only cycle e in E has f as an entry, so the Cuntz-Krieger uniqueness theorem applies. If $\{S, P\}$ is a Cuntz-Krieger E-family such that P_w is non-zero, then S_f is non-zero, and so is $P_v \geq S_f S_f^*$. Since P_w is the projection $1 - (S_e + S_f)(S_e + S_f)^*$ onto the kernel of the adjoint $(S_e + S_f)^*$, we deduce from Corollary 2.5 that any two isometries whose adjoints have non-zero kernels (in other words, any two non-unitary isometries) generate isomorphic C^*-algebras. Thus for this graph the Cuntz-Krieger uniqueness theorem reduces to Coburn's theorem.

EXAMPLE 2.16. Fix $n > 1$, and consider the graph E consisting of a single vertex v and n loops. The cycles in E are the loops, and each loop is an entry for the others, so the Cuntz-Krieger uniqueness theorem applies. Since p_v is an identity for $C^*(E)$, the uniqueness theorem says that any two families $\{S_i\}$ of isometries such that $\sum_{i=1}^n S_i S_i^* = 1$ generate isomorphic C^*-algebras. In particular, it follows that the C^*-algebra $C^*(S_i)$ generated by such a family is *simple* in the sense that it has no non-zero ideals. This essentially unique C^*-algebra is called the *Cuntz algebra*, and is denoted by $\mathcal{O}_n$; this uniqueness theorem was first proved in [**14**].

REMARK 2.17. The first uniqueness theorem for Cuntz-Krieger algebras was proved by Cuntz and Krieger in [**16**]. Their theorem said that if A is a $\{0,1\}$-matrix satisfying a certain condition (I), then any two Cuntz-Krieger A-families generate isomorphic C^*-algebras; they then called the essentially unique C^*-algebra generated by such a family $\mathcal{O}_A$. When we translate their result into a theorem about finite graphs, as in Remark 2.8, we recover Corollary 2.5 for finite E. This was generalised to row-finite graphs in [**96**] and [**82**], and the condition "every cycle has an entry" was introduced in [**82**], where it was called Condition (L). Theorem 2.4 is slightly more general than [**82**, Theorem 3.7], and Corollary 2.5 is [**9**, Theorem 3.1]. The universal Cuntz-Krieger algebras of finite $\{0,1\}$-matrices were introduced in [**60**], and a gauge-invariant uniqueness theorem for these algebras was proved in [**60**, Theorem 2.3]; the version we have stated here is [**9**, Theorem 2.1].

CHAPTER 3

Proofs of the uniqueness theorems

In this chapter we shall prove the gauge-invariant and Cuntz-Krieger uniqueness theorems for a row-finite graph E. The first step in both proofs is to analyse the *fixed-point algebra*

$$C^*(E)^\gamma := \{a \in C^*(E) : \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$$

for the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$, which is usually called the *core* of $C^*(E)$. When we understand the structure of the core, we can prove quite easily that if $\{T, Q\}$ is a Cuntz-Krieger E-family with each Q_v non-zero, then $\pi_{T,Q}$ is faithful on $C^*(E)^\gamma$. From here, gauge-invariant uniqueness follows easily, but there is still some work to do to get Cuntz-Krieger uniqueness.

The core is a $*$-subalgebra of $C^*(E)$, and the continuity of γ implies that it is a C^*-subalgebra. Since $\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*$, a word $s_\mu s_\nu^*$ is fixed by the gauge action if and only if $|\mu| = |\nu|$; thus

$$\overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\} \tag{3.1}$$

is contained in $C^*(E)^\gamma$. We want to prove that $C^*(E)^\gamma$ is precisely (3.1). To do this, we need a way of projecting elements of $C^*(E)$ into $C^*(E)^\gamma$, and this is done by averaging over γ. To make this precise, we need to be able to integrate functions with values in a C^*-algebra.

For a continuous function $f : \mathbb{T} \to \mathbb{C}$, we write

$$\int_{\mathbb{T}} f(z)\,dz := \int_0^1 f(e^{2\pi i t})\,dt.$$

LEMMA 3.1. *Suppose that A is a C^*-algebra and $f : \mathbb{T} \to A$ is a continuous function. Then there is a unique element $\int_{\mathbb{T}} f(z)\,dz$ of A such that, for every representation π of A on $\mathcal{H}$ and $h, k \in \mathcal{H}$, we have*

$$\Big(\pi\Big(\int_{\mathbb{T}} f(z)\,dz\Big)h \,\Big|\, k\Big) = \int_{\mathbb{T}} \big(\pi(f(z))h \,|\, k\big)\,dz. \tag{3.2}$$

We then have

(a) $b\big(\int_{\mathbb{T}} f(z)\,dz\big) = \int_{\mathbb{T}} bf(z)\,dz$ *for* $b \in A$;
(b) $\big\|\int_{\mathbb{T}} f(z)\,dz\big\| \le \int_{\mathbb{T}} \|f(z)\|\,dz$;
(c) $\phi\big(\int_{\mathbb{T}} f(z)\,dz\big) = \int_{\mathbb{T}} \phi(f(z))\,dz$ *for every homomorphism* $\phi : A \to B$;
(d) *for* $w \in \mathbb{T}$, $\int_{\mathbb{T}} f(wz)\,dz = \int_{\mathbb{T}} f(z)\,dz$.

PROOF. We begin by choosing a faithful representation $\rho : A \to B(\mathcal{H})$. Then $(h, k) \mapsto \int_{\mathbb{T}} (\rho(f(z))h \,|\, k)\,dz$ is a bounded sesquilinear form on $\mathcal{H}$, and hence there is a bounded operator T on $\mathcal{H}$ such that

$$(Th \,|\, k) = \int_{\mathbb{T}} \big(\rho(f(z))h \,|\, k\big)\,dz \quad \text{for all } h, k \in \mathcal{H}.$$

For $\epsilon > 0$, we can use a partition of unity to see that there exist finitely many $f_i \in C(\mathbb{T})$ and $a_i \in A$ such that $\|\sum_i f_i(z)a_i - f(z)\| < \epsilon$ for all $z \in \mathbb{T}$, and then

$$\Big\|T - \sum_i \Big(\int_{\mathbb{T}} f_i(z)\,dz\Big)\rho(a_i)\Big\| \leq \epsilon$$

(the details are in [**114**, page 275]). This proves, first, that T belongs to the C^*-subalgebra $\rho(A)$ of $B(\mathcal{H})$, so that we can define $\int_{\mathbb{T}} f(z)\,dz$ to be $\rho^{-1}(T)$, and, second, that we then have

$$\Big\|\int_{\mathbb{T}} f(z)\,dz - \sum_i \Big(\int_{\mathbb{T}} f_i(z)\,dz\Big)a_i\Big\| \leq \epsilon,$$

which implies that if π is any other representation of A, then

$$(3.3)\quad \Big|\Big(\pi\Big(\int_{\mathbb{T}} f(z)\,dz\Big)h \,\Big|\, k\Big) - \int_{\mathbb{T}} (\pi(f(z)h\,|\,k)\,dz\Big| \leq 2\epsilon\|h\|\,\|k\| \quad \text{for all } h, k \in \mathcal{H}.$$

Since the ϵ in (3.3) is arbitrary, this implies (3.2).

Properties (a) and (b) can be verified by applying a faithful representation π to each side and using (3.2). For (c), take a faithful representation ρ of B and apply (3.2) with $\pi = \rho \circ \phi$. To verify (d), use (3.2) to reduce to the integrals of functions $g : \mathbb{T} \to \mathbb{C}$, and write $w = e^{2\pi i\theta}$. Then

$$\int_{\mathbb{T}} g(wz)\,dz = \int_0^1 g(e^{2\pi i(\theta+t)})\,dt = \int_\theta^{1+\theta} g(e^{2\pi it})\,dt,$$

which equals $\int_0^1$ by periodicity. □

PROPOSITION 3.2. *Let α be an action of $\mathbb{T}$ on a C^*-algebra A, and define $\Phi : A \to A$ by*

$$\Phi(a) = \int_{\mathbb{T}} \alpha_z(a)\,dz.$$

Then $\Phi(a) \in A^\alpha$ for every $a \in A$, and $\Phi(a) = a$ for every $a \in A^\alpha$. The map $\Phi : A \to A^\alpha$ is linear and norm-decreasing, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

PROOF. For $a \in A$ and $w \in \mathbb{T}$, parts (c) and (d) of Lemma 3.1 imply that

$$\alpha_w(\Phi(a)) = \int_{\mathbb{T}} \alpha_w(\alpha_z(a))\,dz = \int_{\mathbb{T}} \alpha_{wz}(a)\,dz = \int_{\mathbb{T}} \alpha_z(a)\,dz = \Phi(a),$$

so $\phi(a) \in A^\alpha$. If $a \in A^\alpha$, then part (a) of Lemma 3.1 implies that $\Phi(a) = (\int_{\mathbb{T}} 1\ dz)a = a$. The linearity of Φ follows from an application of (3.2). Since automorphisms of C^*-algebras are norm-preserving, part (b) of Lemma 3.1 gives

$$\|\Phi(a)\| = \Big\|\int_{\mathbb{T}} \alpha_z(a)\,dz\Big\| \leq \int_{\mathbb{T}} \|\alpha_z(a)\|\,dz = \int_{\mathbb{T}} \|a\|\,dz = \|a\|.$$

To prove the last assertion, suppose $\Phi(a^*a) = 0$, and choose a faithful representation π of A on $\mathcal{H}$. Then for $h \in \mathcal{H}$, we have

$$(3.4)\quad 0 = (\pi(\Phi(a^*a))h\,|\,h) = \int_{\mathbb{T}} \big(\pi(\alpha_z(a^*a))h\,|\,h\big)\,dz = \int_{\mathbb{T}} \|\pi(\alpha_z(a))h\|^2\,dz.$$

Since $z \mapsto \|\pi(\alpha_z(a))h\|^2$ is a non-negative continuous function, (3.4) implies that the function is identically zero. In particular, it is zero when $z = 1$, and we deduce that $\pi(a)h = 0$ for all h. Since π is faithful, this implies that $a = 0$. □

For the rest of this chapter, Φ will denote the linear map of $C^*(E)$ onto $C^*(E)^\gamma$ obtained by applying Proposition 3.2 to the gauge action γ.

COROLLARY 3.3. *For every finite subset F of E^* and every choice of scalars $c_{\mu,\nu}$, we have*

$$(3.5)\quad \Phi\Big(\sum_{\mu,\nu\in F} c_{\mu,\nu}s_\mu s_\nu^*\Big) = \sum_{\{\mu,\nu\in F:\,|\mu|=|\nu|\}} c_{\mu,\nu}s_\mu s_\nu^*,$$

and

$$(3.6)\quad C^*(E)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\}.$$

PROOF. We have already observed that the right-hand side of (3.6) is contained in $C^*(E)^\gamma$. For $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$, part (a) of Lemma 3.1 implies that

$$\Phi(s_\mu s_\nu^*) = \int_{\mathbb{T}} z^{|\mu|-|\nu|}s_\mu s_\nu^*\,dz = \Big(\int_T z^{|\mu|-|\nu|}\,dz\Big)s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } |\mu| = |\nu| \\ 0 & \text{otherwise,}\end{cases}$$

which because Φ is linear gives (3.5). Since Φ is continuous, (3.5) implies that

$$\Phi(C^*(E)) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\}.$$

Since we know from Proposition 3.2 that $\Phi(a) = a$ for $a \in C^*(E)^\gamma$, this implies that the right-hand side of (3.6) contains $C^*(E)^\gamma$. □

We now use Corollary 3.3 to analyse the structure of the core $C^*(E)^\gamma$. For $k \in \mathbb{N}$, let

$$\mathcal{F}_k := \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| = |\nu| = k, s(\mu) = s(\nu)\}.$$

If μ, ν, α and β are all paths of length k, then μ and α cannot extend each other without being equal, and Corollary 1.15 gives

$$(3.7)\quad (s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_\mu s_\beta^* & \text{if } \nu = \alpha \\ 0 & \text{otherwise.}\end{cases}$$

This implies, first, that for each $v \in E^0$, $\{s_\mu s_\nu^* : |\mu| = |\nu| = k, s(\mu) = s(\nu) = v\}$ is a family of matrix units, and hence

$$\mathcal{F}_k(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| = |\nu| = k, s(\mu) = s(\nu) = v\}$$

is isomorphic to the C^*-algebra $\mathcal{K}(\ell^2(E^k \cap s^{-1}(v)))$ of compact operators on the ℓ^2-space of the countable set

$$E^k \cap s^{-1}(v) = \{\mu \in E^k : s(\mu) = v\}$$

(see Corollary A.9 and the following Remark). The formula (3.7) also implies that $\mathcal{F}_k(v)\mathcal{F}_k(w) = 0$ when $v \neq w$, and Corollary A.11 implies that

$$(3.8)\quad \mathcal{F}_k \cong \textstyle\bigoplus_{v\in E^0} \mathcal{F}_k(v) \cong \bigoplus_{v\in E^0} \mathcal{K}(\ell^2(E^k \cap s^{-1}(v))),$$

where the direct sum is the C^*-algebraic direct sum consisting of elements $\{a_v : v \in E^0\}$ such that $v \mapsto \|a_v\|$ vanishes at infinity.

When the graph E does not contain sources, and $\mu, \nu \in E^k \cap s^{-1}(v)$, then the Cuntz-Krieger relation at v implies that

$$s_\mu s_\nu^* = s_\mu p_v s_\nu^* = \sum_{r(e)=v} s_\mu s_e s_e^* s_\nu^* = \sum_{r(e)=v} s_{\mu e}s_{\nu e}^*,$$

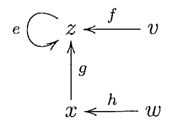
so $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, and we can deduce from Corollary 3.3 that

$$C^*(E)^\gamma = \overline{\bigcup_k \mathcal{F}_k} = \overline{\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_k(v)\big)}.$$

For general row-finite E, we use a device introduced by Yeend in the context of higher-rank graphs (see [**111**]). For $k \in \mathbb{N}$, let

$$E^{\leq k} := \{\mu \in E^* : |\mu| = k, \text{ or } |\mu| < k \text{ and } s(\mu) \text{ is a source}\}.$$

EXAMPLE 3.4. For the following directed graph E



we have $E^2 = \{ef, ee, eg, gh\}$ and $E^{\leq 2} = \{ef, ee, eg, gh, f, h, v, w\}$.

If ν and α are paths in $E^{\leq k}$ and ν is shorter than α, then $s(\nu)$ is a source and α cannot extend ν. Thus the formula (3.7) still holds for $\mu, \nu, \alpha, \beta \in E^{\leq k}$. This implies, first, that

$$\mathcal{F}_{\leq k}(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k},\ s(\mu) = s(\nu) = v\}$$

is isomorphic to $\mathcal{K}(\ell^2(E^{\leq k} \cap s^{-1}(v)))$, and, second, that

$$\mathcal{F}_{\leq k} := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k}\}$$

is the direct sum of the $\mathcal{F}_{\leq k}(v)$. If v is not a source, then the Cuntz-Krieger relation at v shows that $\mathcal{F}_{\leq k}(v) = \mathcal{F}_k(v) \subset \mathcal{F}_{k+1} \subset \mathcal{F}_{\leq k+1}$; if v is a source, then $E^{\leq k} \cap s^{-1}(v) \subset E^{\leq k+1} \cap s^{-1}(v)$, and $\mathcal{F}_{\leq k}(v) \subset \mathcal{F}_{\leq k+1}(v)$. So $\mathcal{F}_{\leq k} \subset \mathcal{F}_{\leq k+1}$, and

$$C^*(E)^\gamma \subset \overline{\bigcup_k \mathcal{F}_{\leq k}} = \overline{\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_{\leq k}(v)\big)}. \tag{3.9}$$

LEMMA 3.5. *Suppose $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for all $v \in E^0$. Then $\pi_{T,Q}$ is isometric on $C^*(E)^\gamma$.*

PROOF. For every $\mu \in E^{\leq k}$, we have $T_\mu^* T_\mu = Q_{s(\mu)}$, so every matrix unit $s_\mu s_\nu^*$ in every $\mathcal{F}_{\leq k}(v)$ has non-zero image $T_\mu T_\nu^*$ under $\pi_{T,Q}$. Thus $\pi_{T,Q}$ is injective on each $\mathcal{F}_{\leq k}(v)$, and hence also on the direct sum $\mathcal{F}_{\leq k} = \bigoplus_v \mathcal{F}_{\leq k}$. Since $\mathcal{F}_{\leq k}$ is a C^*-algebra, every injective homomorphism on $\mathcal{F}_{\leq k}$ is isometric. Thus $\pi_{T,Q}$ is isometric on $\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_{\leq k}(v)\big)$, and hence by (3.9) on $C^*(E)^\gamma$. □

REMARK 3.6. If E has sources, the inclusion in (3.9) will be proper, because $\mathcal{F}_{\leq k}(v)$ will include elements $s_\mu s_\nu^*$ where $|\mu| \neq |\nu|$. If we want to describe the structure of the core, we need to introduce the subalgebras

$$\mathcal{F}_{k,l}(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k},\ |\mu| = |\nu| = l,\ s(\mu) = s(\nu) = v\}.$$

Then $G_k(v) := \mathcal{F}_{\leq k}(v) \cap C^*(E)^\gamma$ is the direct sum of the $\mathcal{F}_{k,l}(v)$ for $l \leq k$, and $C^*(E)^\gamma$ is the closure of $\bigcup_{k=1}^\infty (\bigoplus_v G_k(v))$.

At this stage we have the tools to prove the gauge-invariant uniqueness theorem.

PROOF OF THEOREM 2.2. From parts (c) and (b) of Lemma 3.1 we have

$$\|\pi_{T,Q}(\Phi(a))\| \leq \int_{\mathbb{T}} \|\pi_{T,Q}(\gamma_z(a))\|\, dz = \int_{\mathbb{T}} \|\beta_z(\pi_{T,Q}(a))\|\, dz,$$

which, since automorphisms of C^*-algebras are norm-preserving, implies that

$$\|\pi_{T,Q}(\Phi(a))\| \leq \int_{\mathbb{T}} \|\pi_{T,Q}(a)\|\, dz = \|\pi_{T,Q}(a)\|. \tag{3.10}$$

Now putting the bits together gives

$$\begin{aligned} \pi_{T,Q}(a) = 0 &\Longleftrightarrow \pi_{T,Q}(a^*a) = 0 \\ &\Longrightarrow \pi_{T,Q}(\Phi(a^*a)) = 0 \qquad \text{(by (3.10))} \\ &\Longrightarrow \Phi(a^*a) = 0 \qquad \text{(because } \pi_{T,Q} \text{ is faithful on } C^*(E)^\gamma) \\ &\Longrightarrow a^*a = 0 \qquad \text{(by Lemma 3.2)} \\ &\Longrightarrow a = 0, \end{aligned} \tag{3.11}$$

so that $\pi_{T,Q}$ is injective. Since the range of any homomorphism between C^*-algebras is a C^*-algebra, and $\pi_{T,Q}(C^*(E))$ is generated by $\{\pi_{T,Q}(s), \pi_{T,Q}(p)\} = \{T, Q\}$, the range of $\pi_{T,Q}$ is $C^*(T, Q)$. □

The action β was only used to get the estimate (3.10). Thus to prove the Cuntz-Krieger uniqueness theorem, it suffices to prove a similar estimate. As a first step towards this, we look at a consequence of the hypothesis that every cycle has an entry.

LEMMA 3.7. *Suppose that E has no sources, and that every cycle in E has an entry. Then for each $v \in E^0$ and each $n \in \mathbb{N}$, there is a path $\lambda \in E^*$ such that $r(\lambda) = v$, $|\lambda| \geq n$ and $\lambda_k \neq \lambda_{|\lambda|}$ for $k < |\lambda|$ (we say that the path λ is non-returning).*

PROOF. If there is a path $\lambda \in E^n$ with $r(\lambda) = v$ and no repeated vertices, this will suffice. Otherwise, every path of length n which ends at v contains a return path. Choose a shortest path α such that $r(\alpha) = v$ and there is a cycle β based at $s(\alpha)$. Then β has an entry e, and for sufficently many repetitions of β, $\lambda = \alpha\beta\beta \cdots \beta\beta' e$ has the required properties, where β' is the segment of β from $r(e)$ to $s(\beta)$. □

PROOF OF THEOREM 2.4. By representing B on Hilbert space, we may assume that $\{T, Q\}$ is a Cuntz-Krieger E-family of operators on a Hilbert space $\mathcal{H}$. To avoid complicating the notation, it is convenient to first reduce to the case where E has no sources. Let E_+ be the graph obtained by adding a head to every source of E, as in Corollary 2.11. Then by enlarging the Hilbert space, we can find a Cuntz-Krieger E_+-family $\{S, P\}$ such that $S_e = T_e$ for $e \in E^1$ and $P_v = Q_v$ for $v \in E^0$. (For each source w, add the direct sum $\bigoplus_{i=1}^\infty \mathcal{H}_{i,w}$ of copies $\mathcal{H}_{i,w}$ of $P_w\mathcal{H}$, and take the partial isometry S_{e_i} associated to the edge e_i on the head to be the identity map of $\mathcal{H}_{i,w}$ onto $\mathcal{H}_{i-1,w}$, where by $\mathcal{H}_{0,w}$ we mean the original space $P_w\mathcal{H}$.) Then each P_v is non-zero. So if we know the theorem holds for E_+, then $\pi_{S,P}$ is faithful on $C^*(E_+)$, and so is $\pi_{T,Q}$, which is composition of $\pi_{S,P}$ with the injection ϕ of Corollary 2.11.

So we may suppose that E has no sources. As we observed after the proof of the gauge-invariant uniqueness theorem, it suffices to prove that $\|\pi_{T,Q}(\Phi(a))\| \leq \|\pi_{T,Q}(a)\|$ for all $a \in C^*(E)$; by continuity it suffices to do this for a of the form $a = \sum_{(\mu,\nu)\in F} c_{\mu,\nu} s_\mu s_\nu^*$, where F is a finite set of pairs (μ, ν) with $s(\mu) = s(\nu)$. The strategy is to find a projection Q which satisfies

$$\|Q\pi_{T,Q}(\Phi(a))Q\| = \|\pi_{T,Q}(\Phi(a))\|, \quad \text{and} \tag{3.12}$$

$$QT_\mu T_\nu^* Q = 0 \quad \text{when } (\mu,\nu) \in F \text{ and } |\mu| \neq |\nu|. \tag{3.13}$$

(Skip ahead to (3.15) if you want to see how this will help.)

Let $k = \max\{|\mu|, |\nu| : (\mu,\nu) \in F\}$. Because the graph E has no sources, we may suppose by applying the Cuntz-Krieger relations and changing F that $k = \min\{|\mu|, |\nu|\}$ for every pair (μ,ν) with $c_{\mu,\nu} \neq 0$. In particular, if $c_{\mu,\nu} \neq 0$ and $|\mu| = |\nu|$, then $|\mu| = |\nu| = k$. Since we know from (3.5) that

$$\Phi(a) = \sum_{\{(\mu,\nu)\in F:\, |\mu|=|\nu|\}} c_{\mu,\nu} s_\mu s_\nu^*,$$

we deduce that $\Phi(a)$ belongs to $\mathcal{F}_k = \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| = |\nu| = k\}$. Since $\mathcal{F}_k$ is the C^*-algebraic direct sum $\bigoplus_{v\in E^0} \mathcal{F}_k(v)$, there exists $v \in E^0$ such that

$$\|\Phi(a)\| = \Big\| \sum_{\{(\mu,\nu)\in F:\, |\mu|=|\nu|, s(\mu)=s(\nu)=v\}} c_{\mu,\nu} s_\mu s_\nu^* \Big\|. \tag{3.14}$$

We write

$$b_v := \sum_{\{(\mu,\nu)\in F:\, |\mu|=|\nu|, s(\mu)=s(\nu)=v\}} c_{\mu,\nu} s_\mu s_\nu^*.$$

Let G be the set of paths which arise either as μ or ν for some $(\mu,\nu) \in F$ satisfying $|\mu| = |\nu|$ and $s(\mu) = s(\nu) = v$. Notice that $\operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in G\}$ is a finite-dimensional matrix algebra containing b_v.

For the vertex v satisfying (3.14), and $n > \max\{|\mu|, |\nu| : (\mu,\nu) \in F\}$, we choose $\lambda \in E^*$ as in Lemma 3.7. We claim that

$$Q := \sum_{\tau\in G} T_{\tau\lambda} T_{\tau\lambda}^*$$

satisfies (3.12) and (3.13)

Suppose $(\mu,\nu) \in F$ satisfies $|\mu| = |\nu|$ and $\tau \in G$. Then $T_{\tau\lambda}^* T_\mu$ is non-zero if and only if $\tau = \mu$, and hence

$$QT_\mu T_\nu^* Q = \begin{cases} (T_{\mu\lambda} T_{\mu\lambda}^* T_\mu)(T_\nu^* T_{\nu\lambda} T_{\nu\lambda}^*) & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} T_{\mu\lambda}(T_\lambda^* T_\lambda) T_{\nu\lambda}^* & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} T_{\mu\lambda} T_{\nu\lambda}^* & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise.} \end{cases}$$

For $\tau \in G$, the initial projection $Q_{s(\tau\lambda)}$ of $T_{\tau\lambda}$ is non-zero, and hence the final projection $T_{\tau\lambda} T_{\tau\lambda}^*$ is also non-zero. Thus $\{QT_\mu T_\nu^* Q : \mu,\nu \in G\}$ is a collection of non-zero matrix units, and the map $b \mapsto Q\pi_{T,Q}(b)Q$ is a faithful representation of the matrix algebra $\operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in G\}$. Since faithful representations of C^*-algebras are isometric, this implies in particular that

$$\|\pi_{T,Q}(\Phi(a))\| = \|\Phi(a)\| = \|b_v\| = \|Q\pi_{T,Q}(b_v)Q\| = \|Q\pi_{T,Q}(\Phi(a))Q\|,$$

and we have verified (3.12).

Now suppose that $(\mu,\nu) \in F$ satisfies $|\mu| \neq |\nu|$. Either μ or ν has length k, say $|\mu| = k$. As before, $T_{\tau\lambda}^* T_\mu$ is non-zero if and only if $\tau = \mu$. Thus

$$QT_\mu T_\nu^* Q = \sum_{\tau\in G} T_{\mu\lambda} T_{\mu\lambda}^* T_\mu T_\nu^* T_{\tau\lambda} T_{\tau\lambda}^* = \sum_{\tau\in G} T_{\mu\lambda}(T_{\nu\lambda}^* T_{\tau\lambda}) T_{\tau\lambda}^*.$$

For $T_{\nu\lambda}^* T_{\tau\lambda}$ to be non-zero, $\nu\lambda$ must extend $\tau\lambda$, which is impossible because $0 < |\nu| - |\tau| < |\lambda|$ and λ is non-returning. Thus $QT_\mu T_\nu^* Q = 0$, and we have verified (3.13).

We now estimate, using (3.12) and (3.13):

$$\begin{aligned} \|\pi_{T,Q}(\Phi(a))\| &= \|Q\pi_{T,Q}(\Phi(a))Q\| \\ &= \Big\| Q\Big(\sum_{\{(\mu,\nu)\in F:\, |\mu|=|\nu|\}} c_{\mu,\nu} T_\mu T_\nu^* \Big) Q \Big\| \\ &= \Big\| Q\Big(\sum_{(\mu,\nu)\in F} c_{\mu,\nu} T_\mu T_\nu^* \Big) Q \Big\| \\ &\leq \Big\| \sum_{(\mu,\nu)\in F} c_{\mu,\nu} T_\mu T_\nu^* \Big\| \\ &= \|\pi_{T,Q}(a)\|. \end{aligned} \tag{3.15}$$

This estimate allows us to run the argument of (3.11), and this completes the proof. □

REMARK 3.8. This proof of Theorem 2.4 is based on the arguments of Cuntz and Krieger [**16**], as adapted to the C^*-algebras of row-finite graphs in [**9**], with Lemma 3.7 borrowed from [**73**]. The original proof in [**82**] used the groupoid model for $C^*(E)$ constructed in [**83**].

CHAPTER 4

Simplicity and ideal structure

In this chapter we investigate the ideal structure of the C^*-algebra of a row-finite graph E, using the uniqueness theorems which we proved in the previous chapter. Our first result gives a condition on E which ensures that $C^*(E)$ is simple. We will later prove that this condition is also necessary (see Theorem 4.14).

For vertices v, w in a directed graph E, we write $w \leq v$ to mean that there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. This relation $\leq$ is transitive in the sense that $w \leq v$ and $v \leq x$ imply $w \leq x$, and is reflexive because we can take μ to be the path v of length 0. The relation $\leq$ is not a partial order, because distinct vertices v, w on a cycle satisfy $v \leq w \leq v$; it is what is known as a *preorder*.

We denote by E^∞ the set of infinite paths $\lambda = \lambda_1\lambda_2\cdots$, and by $E^{\leq\infty}$ the set obtained by adding to E^∞ the finite paths which begin at sources. We say that the graph E is *cofinal* if for every $\mu \in E^{\leq\infty}$ and $v \in E^0$ there exists a vertex w on μ such that $v \leq w$.

EXAMPLE 4.1. The graph

$$e \circlearrowright v \xrightarrow{\ f\ } w$$

is cofinal, whereas

$$g \circlearrowright x \xleftarrow{\ h\ } z$$

is not: one cannot reach z from any point on the infinite path $ggg\cdots$.

PROPOSITION 4.2. *Suppose E is a row-finite graph in which every cycle has an entry. If E is cofinal, then $C^*(E)$ is simple.*

A directed graph E is called *transitive* if for every pair v, w of vertices, we have $v \leq w$ and $w \leq v$. If E is transitive and E is not one of the graphs C_n consisting of a single cycle, then E satisfies the hypotheses of Proposition 4.2. So there are lots of graphs E for which $C^*(E)$ is simple.

PROOF OF PROPOSITION 4.2. Since every ideal in a C^*-algebra is the kernel of a representation, it suffices to prove that every non-zero representation $\pi_{S,P}$ of $C^*(E)$ is faithful. Suppose $\{S, P\}$ is a Cuntz-Krieger E-family such that $\pi_{S,P}$ is non-zero. If all the vertex projections P_v were 0, then the relation $S_e^*S_e = P_{s(e)}$ would force $S_e = 0$ for all e, and $\pi_{S,P}$ would be identically 0. So at least one P_v is non-zero. We want to prove that all the vertex projections are non-zero.

Let $w \in E^0$. If the vertex v for which $P_v \neq 0$ is not a source, the Cuntz-Krieger relation at v implies that there exists $e \in E^1$ such that $r(e) = v$ and $S_eS_e^* \neq 0$. Then $P_{s(e)} = S_e^*S_e \neq 0$, and if $s(e)$ is not a source, we can repeat this argument at $s(e)$. This process either ends at a source or continues to yield an infinite path;

either way we obtain a path $\mu \in E^{\leq\infty}$ such that $r(\mu) = v$ and $P_x \neq 0$ for every vertex x on μ. By cofinality, there exists $\alpha \in E^*$ with $r(\alpha) = w$ and $s(\alpha)$ a vertex on μ. But now $S_\alpha^* S_\alpha = P_{s(\alpha)} \neq 0$, so $S_\alpha S_\alpha^* \neq 0$. Since $P_w S_\alpha S_\alpha^* = S_\alpha S_\alpha^*$, it follows that P_w is non-zero.

Thus all the vertex projections P_w are non-zero, and the Cuntz-Krieger uniqueness theorem implies that $\pi_{S,P}$ is faithful, as required. □

REMARK 4.3. There has recently been a great deal of interest in the classification of simple C^*-algebras, and it is natural to ask when these simple graph algebras satisfy the other hypotheses required in the classification program. All graph algebras are nuclear: one way to see this is to prove that $C^*(E) \rtimes_\gamma \mathbb{T}$ is AF, as we will do in Chapter 7, and then the Takesaki-Takai duality theorem implies that $C^*(E)$ is stably isomorphic to $(C^*(E) \rtimes_\gamma \mathbb{T}) \rtimes_{\hat{\gamma}} \mathbb{Z}$. Indeed, this argument shows that $C^*(E)$ belongs to the *bootstrap class* $\mathcal{N}$ of Rosenberg and Schochet (see [**125**] or [**123**, §2.4]).

A simple graph algebra $C^*(E)$ is *purely infinite* in the sense of [**123**, §4.1] if and only if for every $v \in E^0$ there is a cycle μ with $r(\mu) \geq v$ (see [**82**, Theorem 3.9] and [**9**, Proposition 5.3]). It follows from this and Remark 2.13 that there is a dichotomy for simple graph algebras: they are either AF or purely infinite [**82**, Corollary 3.11].

A graph algebra $C^*(E)$ has an identity if and only if E^0 is finite: if E^0 is finite, $\sum_{v\in E^0} p_v$ is an identity, and if $C^*(E)$ has an identity, the sums discussed in Lemma 2.10 must converge in norm, which is only possible if the sums are finite, or in other words if E^0 is finite. Criteria for the stability of $C^*(E)$ are given in [**143**], which improves earlier results in [**54**]. In particular, the C^*-algebras of infinite transitive graphs are always stable.

The simple C^*-algebras are the ones with trivial ideal structure. Our next goal is to describe the ideals of $C^*(E)$ when $C^*(E)$ is not simple. By convention, when we talk about ideals in C^*-algebras, we mean closed two-sided ideals unless otherwise stated.

Suppose I is an ideal in $C^*(E)$, and consider

$$H_I := \{v \in E^0 : p_v \in I\}.$$

The idea is that the set H_I determines the ideal I. To see why, consider the quotient map $q : C^*(E) \to C^*(E)/I$. The projections $\{q(p_v) : v \notin H_I\}$ are all non-zero. If $s(e) \notin H_I$, then $q(s_e)$ has non-zero initial projection, and $q(p_{r(e)}) \geq q(s_e)q(s_e)^*$ is also non-zero, so that $r(e)$ does not belong to H_I either. Thus

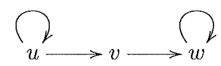
$$E\backslash H_I := \{E^0\backslash H_I, s^{-1}(E^0\backslash H_I), r, s\}$$

is a graph, and $\{q(s_e), q(p_v) : s(e) \notin H_I, v \notin H_I\}$ is a Cuntz-Krieger family for $E\backslash H_I$ with every vertex projection non-zero. If every cycle in this graph has an entry, then the Cuntz-Krieger uniqueness theorem implies that $C^*(E)/I$ is isomorphic to $C^*(E\backslash H_I)$. So we need to identify the subsets H of E^0 which arise as H_I, and find a verifiable condition on E which ensures that every cycle in every $E\backslash H$ has an entry.

The characterisation of the sets H_I uses the preorder $\leq$ on E^0. We say that a subset H of E^0 is *hereditary* if $w \in H$ and $w \leq v$ imply $v \in H$; we say that H is *saturated* if $r^{-1}(v) \neq \emptyset$ and $\{s(e) : r(e) = v\} \subset H$ imply $v \in H$. (We warn that in Chapter 5 we will see that a different definition of saturated is needed when E

is not row-finite.) In every row-finite graph E, E^0 and $\emptyset$ are saturated hereditary sets; we shall refer to the others as the *non-trivial* saturated hereditary sets.

EXAMPLE 4.4. In the following graph E



the sets $\{u\}$ and $\{u, v\}$ are non-trivial hereditary subsets of E^0, but $\{v\}$ is not. The set $\{u, v\}$ is saturated, but $\{u\}$ is not because v satisfies $\{s(e) : r(e) = v\} \subset \{u\}$. If $H = \{u, v\}$, then $E\backslash H$ consists of a single loop at w.

LEMMA 4.5. *Suppose I is a non-zero ideal in the C^*-algebra of the row-finite graph E. Then H_I is saturated and hereditary.*

PROOF. To see that H_I is hereditary, suppose $w \in H_I$ and $w \leq v$. Then there exists $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$, and

$$p_w \in I \Longrightarrow s_\mu = p_{r(\mu)} s_\mu = p_w s_\mu \in I \Longrightarrow p_v = s_\mu^* s_\mu \in I \Longrightarrow v \in H_I.$$

To see that H_I is saturated, suppose that $r^{-1}(v) \neq \emptyset$ and $\{s(e) : r(e) = v\} \subset H_I$. Then for every e with $r(e) = v$, $s_e = s_e p_{s(e)}$ belongs to I, and v is not a source, so

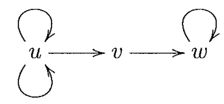
$$p_v = \sum_{r(e)=v} s_e s_e^* \in I,$$

which implies that v belongs to H_I. □

We now introduce our hypothesis on E. We say that E *satisfies Condition* (K) if for every vertex v, either there is no cycle based at v, or there are two distinct paths μ, ν such that $s(\mu) = v = r(\mu)$, $s(\nu) = v = r(\nu)$, $r(\mu_i) \neq v$ for $i < |\mu|$, and $r(\nu_i) \neq v$ for $i < |\nu|$ (we call these distinct *return paths*). If E is a graph which satisfies Condition (K) then every cycle in E has an entry, but the converse is false, as the next examples show.

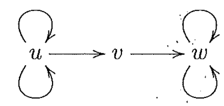
EXAMPLES 4.6. In the graph

(4.1)

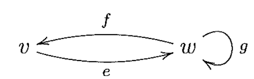


every cycle has an entry, but Condition (K) is not satisfied at the vertex w. When we add another loop at w,

(4.2)



Condition (K) is satisfied. Notice that there is no problem at v because it does not lie on any cycles. It is important that we are not insisting that we can choose the return paths to be cycles: for example, in the graph



$\mu = fe$ and $\nu = fge$ are return paths at v, but there is only one cycle based at v.

If H is a hereditary set, then $E\backslash H := \{E^0\backslash H, s^{-1}(E^0\backslash H), r, s\}$ is a graph. We now check that Condition (K) does what we want:

LEMMA 4.7. *The graph E satisfies Condition* (K) *if and only if for every saturated hereditary subset H of E^0, every cycle in $E\backslash H := \{E^0\backslash H, s^{-1}(E^0\backslash H), r, s\}$ has an entry.*

EXAMPLE 4.8. In the graph (4.1), the set $H = \{u, v\}$ is saturated and hereditary, but $E\backslash H$ consists of a single loop based at w. Thus the single cycle in $E\backslash H$ does not have an entry in $E\backslash H$.

PROOF OF LEMMA 4.7. Suppose E satisfies Condition (K), H is saturated and hereditary, and μ is a cycle in $E\backslash H$ with $v = s(\mu)$. By Condition (K), there is a distinct return path ν in E with $s(\nu) = v$. Choose i such that $\mu_j = \nu_j$ for $j < i$ and $\mu_i \neq \nu_i$. Then ν_i is certainly an entry to μ in E. However, since $v \geq s(\nu_i)$ and H is hereditary, $s(\nu_i)$ cannot be in H. So $\nu_i \in (E\backslash H)^1 = s^{-1}(E^0\backslash H)$, and is an entry to μ in $E\backslash H$.

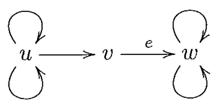
Conversely, suppose that for every saturated hereditary subset H of E^0, every cycle in $E\backslash H$ has an entry in $E\backslash H$. Suppose $v \in E^0$ and there is a cycle μ in E with $s(\mu) = v$. We have to show that v lies on another return path ν. We claim that $H := \{w : v \not\geq w\}$ is hereditary and saturated. If $w \in H$ and $z \geq w$, then $v \geq z$ implies $v \geq w$, so we must have $v \not\geq z$ and $z \in H$. Thus H is hereditary. Now suppose that $z \in E^0$ is not a source and satisfies $\{s(e) : r(e) = z\} \subset H$. If $z \notin H$, then there is a path α such that $s(\alpha) = v$ and $r(\alpha) = z$; but then $s(\alpha_1) \in H$, and $v \geq s(\alpha_1)$ implies $v \in H$, which is a contradiction. So $z \in H$, and H is saturated, as claimed.

The cycle μ lies in $E\backslash H$. So μ has an entry e in $E\backslash H$, say with $r(e) = r(\mu_i)$. Then $s(e)$ is not in H, and hence $v \geq s(e)$; choose a shortest path β with $s(\beta) = v$ and $r(\beta) = s(e)$. Now $\nu = \mu_1 \cdots \mu_{i-1} e\beta$ is the required return path. □

We can now state our main classification theorem. If H is a saturated and hereditary subset of E^0, then $E\backslash H$ is the graph in Lemma 4.7, and E_H denotes the graph $(H, r^{-1}(H), r_E, s_E)$.

THEOREM 4.9. *Suppose E is a row-finite graph which satisfies Condition* (K). *For $H \subset E^0$, let I_H be the ideal generated by $\{p_v : v \in H\}$. Then $H \mapsto I_H$ is a bijection between the saturated hereditary subsets of E^0 and the closed ideals in $C^*(E)$, with inverse given by $I \mapsto H_I = \{v : p_v \in I\}$. The quotient $C^*(E)/I_H$ is isomorphic to $C^*(E\backslash H)$, and $C^*(E_H)$ is isomorphic to the full corner $p_H I_H p_H$ associated to the projection p_H defined in Lemma 2.10.*

EXAMPLE 4.10. Consider the saturated and hereditary set $H = \{u, v\}$ in the graph



of (4.2). When we split E into two graphs E_H and $E\backslash H$, as in Theorem 4.9, the edge e does not belong to either graph: it is not in E_H because $r(e) = w$ is not in H, and it is not in $E\backslash H$ because $s(e) = v$ does not belong to $E^0\backslash H$.

PROOF OF THEOREM 4.9. Let I be an ideal in $C^*(E)$. Then we know from Lemma 4.5 that $H = H_I$ is saturated and hereditary. We claim that $I = I_H$. Since all the generators of I_H are by definition in I, we trivially have $I_H \subset I$. Consider the quotient maps

$$q^I : C^*(E) \to C^*(E)/I, \quad q^{I_H} : C^*(E) \to C^*(E)/I_H, \quad \text{and}$$
$$q^{I/I_H} : C^*(E)/I_H \to C^*(E)/I = (C^*(E)/I_H)/(I/I_H);$$

note that $q^I = q^{I/I_H} \circ q^{I_H}$. Since q^I and q^{I_H} kill exactly the same vertex projections, and hence exactly the same partial isometries s_e, both $\{q^I(s_e), q^I(p_v)\}$ and $\{q^{I_H}(s_e), q^{I_H}(p_v)\}$ are Cuntz-Krieger $(E\backslash H)$-families which generate the respective quotients. Let $\pi : C^*(E\backslash H) \to C^*(E)/I_H$ and $\rho : C^*(E\backslash H) \to C^*(E)/I$ be the corresponding homomorphisms. Then ρ and $q^{I/I_H} \circ \pi$ are homomorphisms which agree on the generators of $C^*(E\backslash H)$, and hence are equal. By Lemma 4.7, we can apply the Cuntz-Krieger uniqueness theorem to $E\backslash H$, and deduce that ρ is injective. Since π is surjective and $\rho = q^{I/I_H} \circ \pi$, this implies that q^{I/I_H} is injective. We deduce that $I = I_H$, as claimed, and that $C^*(E)/I_H$ is isomorphic to $C^*(E\backslash H)$.

Since $I = I_{H_I}$, $H \mapsto I_H$ is surjective. To see that it is injective, we need to show that if H is saturated and hereditary, then $H = \{v : p_v \in I_H\}$. We trivially have $H \subset \{v : p_v \in I_H\}$. To prove the reverse inclusion, consider the canonical $(E\backslash H)$-family $\{t, q\}$ which generates $C^*(E\backslash H)$. We claim that when we define $t_e = 0$ for $s(e) \in H$ and $q_v = 0$ for $v \in H$, $\{t, q\}$ becomes a Cuntz-Krieger E-family. When $s(e) \in H$, we have $t_e^* t_e = 0 = q_{s(e)}$, and the relations at $v \in H$ are also trivially satisfied because hereditarity of H implies that every edge e with $r(e) = v$ has $s(e) \in H$, and hence $t_e = 0$. If v is not a source in $E\backslash H$, the new edges with $r(e) = v$ all have $t_e t_e^* = 0$, so the relation at v is unchanged. If v is a source in $E\backslash H$, all edges e in E with $r(e) = v$ have $s(e) \in H$, which unless v is a source in E implies $v \in H$ by saturation; thus v is a source in E, and there is still no Cuntz-Krieger relation at v. So $\{t, q\}$ is a Cuntz-Krieger E-family, as claimed. Now the universal property of $C^*(E)$ gives a homomorphism $\pi_{t,q}$ of $C^*(E)$ into $C^*(E\backslash H)$. Since $\pi_{t,q}(p_v) = 0$ for $v \in H$, $\ker \pi_{t,q} \supset I_H$, and

$$v \notin H \Longrightarrow q_v \neq 0 \Longrightarrow \pi_{t,q}(p_v) \neq 0 \Longrightarrow p_v \notin \ker \pi_{t,q} \Longrightarrow p_v \notin I_H,$$

so that $H \supset \{v : p_v \in I_H\}$. Thus $H = \{v : p_v \in I_H\}$, and $H \mapsto I_H$ is injective.

For the last statement, suppose H is saturated and hereditary. We claim that

$$I_H = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H\}. \tag{4.3}$$

To see this, notice that if $s(\mu) = s(\nu) \in H$ and α, β are any paths in E, then the formula (1.4) shows that the product $(s_\mu s_\nu^*)(s_\alpha s_\beta^*)$ has the form $s_\sigma s_\tau^*$ for some $\sigma, \tau \in E^*$ with $s(\mu) \leq s(\sigma) = s(\tau)$; since H is hereditary, $s(\sigma)$ then belongs to H, and the product belongs to the right-hand side of (4.3). Thus the right-hand side of (4.3) is an ideal. Since this ideal contains the generators of I_H, and every spanning element $s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^*$ belongs to I_H, it must be I_H, as claimed.

Since compression by the projection p_H is linear and continuous, it follows from (4.3) that

$$p_H I_H p_H = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H, r(\mu) \in H, r(\nu) \in H\}. \tag{4.4}$$

Since H is hereditary, $\{s_e : r(e) \in H\} \cup \{p_v : v \in H\}$ is a Cuntz-Krieger E_H-family in $p_H I_H p_H$, and (4.4) implies that this family generates $p_H I_H p_H$. Every cycle in E_H has an entry in E, and by hereditarity this is also an entry in E_H. Thus the

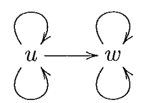
Cuntz-Krieger uniqueness theorem implies that $p_H I_H p_H$ is isomorphic to $C^*(E_H)$. The corner is full because $\{p_v : v \in H\}$ generates I_H. □

REMARK 4.11. In the last two paragraphs, we did not use that H was saturated. In fact, if X is any hereditary subset of E^0, then the smallest saturated set ΣX containing X is also hereditary, and there is an isomorphism of $C^*(E_X)$ onto the corner $p_X I_{\Sigma X} p_X$, which is again full.

REMARK 4.12. When E does not satisfy Condition (K), the saturated hereditary subsets of E^0 parametrise the *gauge-invariant ideals* I for which $\gamma_z(I) \subset I$ for every $z \in \mathbb{T}$. To prove this, follow the argument of Theorem 4.9, but use gauge-invariant ideals everywhere, and use the gauge-invariant uniqueness theorem everywhere we used the Cuntz-Krieger uniqueness theorem. See [**9**, Theorem 4.1] for details. Of importance for us will be the observation that if H is a non-trivial saturated hereditary subset of E^0, then I_H is a non-zero gauge-invariant ideal in $C^*(E)$ with non-zero quotient $C^*(E\backslash H)$.

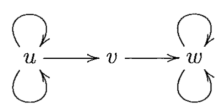
Many important directed graphs do not satisfy Condition (K), such as the graphs used to model non-commutative spheres in [**56**] (see Example 1.25), and it is of considerable interest to learn more about the ideal structure of their C^*-algebras. The ideal structure of graph algebras has now been completely determined by Hong and Szymański [**59**]; since they deal with graphs which are not row-finite, we postpone a discussion of their work to Chapter 5. Their results, however, contain important new information even for row-finite graphs (see [**59**, Corollary 3.5]).

EXAMPLE 4.13. In the following graph E



the only non-trivial hereditary set is $\{u\}$ and it is saturated. So the ideals in $C^*(E)$ are $\{0\} = I_\emptyset$, $I_{\{u\}}$ and $C^*(E) = I_{E^0}$. The quotient $C^*(E)/I_{\{u\}}$ is the C^*-algebra of the graph consisting of two loops based at w, and hence is the Cuntz algebra $\mathcal{O}_2$. The corner $p_u I_{\{u\}} p_u$ is also $\mathcal{O}_2$. The ideal $I_{\{u\}}$ itself is much larger.

If we add an extra vertex to E to obtain the following graph F



then the non-trivial hereditary sets are $\{u\}$ and $\{u, v\}$, and only $\{u, v\}$ is saturated. So the ideals in $C^*(E)$ are $\{0\} = I_\emptyset$, $I_{\{u,v\}}$ and $C^*(E) = I_{E^0}$. The quotient $C^*(E)/I_{\{u,v\}}$ is still $\mathcal{O}_2$, and $p_u I_{\{u,v\}} p_u$ is still $\mathcal{O}_2$, but the ideal $p_{\{u,v\}} I_{\{u,v\}} p_{\{u,v\}}$ discussed in Theorem 4.9 is larger. (Indeed, with a bit of work one can see that it is isomorphic to the C^*-algebra $M_2(\mathcal{O}_2)$ of 2×2 matrices over $\mathcal{O}_2$.) This illustrates why one might prefer to work with ideals generated by more general hereditary sets, as in Remark 4.11.

Theorem 4.9 gives us another criterion for simplicity: $C^*(E)$ is simple if and only if every cycle has an entry and E^0 has no non-trivial saturated hereditary sets.

We can also use the ideals I_H to see that the cofinality criterion of Proposition 4.2 is necessary and sufficient.

THEOREM 4.14. *Suppose E is a row-finite graph. Then $C^*(E)$ is simple if and only if every cycle in E has an entry and E is cofinal.*

PROOF. We proved the "if" direction in Proposition 4.2. So suppose that $C^*(E)$ is simple.

To see cofinality, suppose $\mu \in E^{\leq\infty}$, and consider

$$H_\mu := \{w \in E^0 : w \not\leq v \text{ for every } v \text{ on } \mu\}.$$

Then H_μ is saturated and hereditary. If H_μ were non-trivial, then I_{H_μ} would be a proper ideal in $C^*(E)$ (see Remark 4.12); thus H_μ is either empty or all of E^0. Since $r(\mu)$ does not belong to H_μ, we must have H_μ empty. Thus we can reach every vertex in E from some vertex on μ, and E is cofinal.

Now suppose that μ is a cycle in E which does not have an entry. Then $X := \{s(\mu_i) : 1 \leq i \leq |\mu|\}$ is a nonempty hereditary set, so its saturation ΣX must be all of E^0, and $I_{\Sigma X} = C^*(E)$ by simplicity. Thus by Remark 4.11, $C^*(E_X)$ is isomorphic to the corner $p_X C^*(E) p_X$. However, the graph E_X is a cycle, so we know from Example 2.14 that $C^*(E_X)$ is isomorphic to $C(\mathbb{T}, M_{|\mu|}(\mathbb{C}))$. Let J be a proper ideal in $p_X C^*(E) p_X$, such as the set of functions which vanish at 1. Then

$$C^*(E) J C^*(E) := \overline{\operatorname{span}}\{ajb : a, b \in C^*(E), j \in J\}$$

is a non-zero ideal in $C^*(E)$, and hence is all of $C^*(E)$. Thus

$$p_X C^*(E) J C^*(E) p_X = p_X C^*(E) p_X J p_X C^*(E) p_X = J$$

is all of $p_X C^*(E) p_X$, and we have a contradiction. So every cycle in E has an entry. □

REMARK 4.15. The ideal structure of the Cuntz-Krieger algebras $\mathcal{O}_A$ was analysed by Cuntz in [**15**] under a hypothesis (II) on A, which is equivalent to asking that the associated graph E_A satisfies Condition (K). The ideal theory of $\mathcal{O}_A$ for arbitrary finite A was studied in [**60**]. Theorem 4.9 is [**9**, Theorem 4.4], which slightly generalises previous results in [**83**] and [**67**]. The proof given here is based on that of [**9**], and is substantially different from those of [**15**] and [**60**], which used a complicated approximate identity argument, from that of [**83**], which used the groupoid model for $C^*(E)$ and results of Renault [**115**], and from that of [**67**], which viewed $C^*(E)$ as a Cuntz-Pimsner algebra.

REMARK 4.16. When E is a finite graph with no sinks or sources, every vertex v connects to a cycle μ. If E is cofinal, then the infinite path $\mu\mu\cdots$ connects to every vertex w, and hence so does v. So Theorem 4.14 says that $C^*(E)$ is simple if and only if E is transitive and not a single cycle C_n. This simplicity criterion for Cuntz-Krieger algebras was obtained in [**16**]. Simplicity criteria for various classes of row-finite graphs were obtained in [**96, 83, 67**], and cofinality was introduced in [**83**]. Theorem 4.14 is a mild improvement on [**9**, Proposition 5.1], in that we have amended the definition of cofinality to accommodate graphs with sources.

APPENDIX A

Background material

A.1. Projections and partial isometries

If M is a closed subspace of a Hilbert space $\mathcal{H}$, the *orthogonal projection* of $\mathcal{H}$ on M is a bounded linear operator $P : \mathcal{H} \to \mathcal{H}$ which is characterised by the property that $Ph \in M$ and $h - Ph$ is orthogonal to M for every $h \in \mathcal{H}$. The orthogonal projections are themselves characterised by the relations $P^2 = P = P^*$: every bounded operator $T \in B(\mathcal{H})$ such that $T^2 = T = T^*$ is the orthogonal projection of $\mathcal{H}$ onto the closed subspace $T\mathcal{H}$. We therefore say that an element p of a C^*-algebra A is a *projection* if $p^2 = p = p^*$; whenever $\pi : A \to B(\mathcal{H})$ is a representation of A on $\mathcal{H}$, $\pi(p)$ is then the orthogonal projection of $\mathcal{H}$ on $\pi(p)\mathcal{H}$.

The next two propositions show how geometric properties of closed subspaces of Hilbert space can be encoded using the $*$-algebraic structure of $B(\mathcal{H})$, and hence can be interpreted in an abstract C^*-algebra.

PROPOSITION A.1. *Suppose that P and Q are orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $P\mathcal{H} \subset Q\mathcal{H}$;
(b) $QP = P = PQ$;
(c) $Q - P$ *is a projection;*
(d) $P \leq Q$ *(in the sense that* $(Ph \,|\, h) \leq (Qh \,|\, h)$ *for all* $h \in \mathcal{H}$*).*

PROOF. (a) $\Longrightarrow$ (b). For $h \in \mathcal{H}$, we have $Ph \in P\mathcal{H} \subset Q\mathcal{H}$, so $Q(Ph) = Ph$. Thus $QP = P$. Taking adjoints gives $PQ = P$.

(b) $\Longrightarrow$ (c). We just need to check that

$$(Q-P)^2 = Q^2 - QP - PQ + P^2 = Q - P - P + P = Q - P$$

and $(Q-P)^* = Q^* - P^* = Q - P$.

(c) $\Longrightarrow$ (d). Every projection is a positive operator, and hence if $Q - P$ is a projection, then $Q = P + (Q - P) \geq P$.

(d) $\Longrightarrow$ (a). Suppose $h \in P\mathcal{H}$, so that $h = Ph$. Then $P \leq Q$ implies

$$\|Qh\|^2 = (Qh \,|\, Qh) = (Qh \,|\, h) \geq (Ph \,|\, h) = \|h\|^2. \tag{A.1}$$

Since $\|h\|^2 = \|Qh\|^2 + \|(1-Q)h\|^2$, (A.1) implies that $\|(1-Q)h\| = 0$ and $h = Qh \in Q\mathcal{H}$. □

PROPOSITION A.2. *Suppose that P and Q are orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $P\mathcal{H} \perp Q\mathcal{H}$;
(b) $QP = 0 = PQ$;
(c) $P + Q$ *is a projection.*

PROOF. (a) $\Longrightarrow$ (b). For $h \in \mathcal{H}$, $Qh \in Q\mathcal{H}$ is orthogonal to $P\mathcal{H}$, so $PQh = 0$. Taking adjoints gives $QP = 0$.

(b) $\Longrightarrow$ (c). We just need to check that

$$(P+Q)^2 = P^2 + QP + PQ + Q^2 = P + 0 + 0 + Q = P + Q,$$

and that $P + Q$ is self-adjoint.

(c) $\Longrightarrow$ (b). Suppose $P + Q$ is a projection. Then $(P+Q)^2 = P + Q$ implies that $PQ = -QP$. This implies, first, that $PQPQ = (PQ)^2 = (QP)^2 = QPQP$, and, second, that

$$-PQ = P(-PQ)Q = P(QP)Q = QPQP = Q(-QP)P = -QP = PQ,$$

which in turn implies that $PQ = 0$.

(b) $\Longrightarrow$ (a). We just compute $(Ph \,|\, Qk) = (QPh \,|\, k) = 0$ for every $h, k \in \mathcal{H}$. □

All except the first statements (a) in Propositions A.1 and A.2 make sense in any C^*-algebra A, and these C^*-algebraic statements are preserved by any representation of A as bounded operators on Hilbert space. So the equivalence of (b), (c) and (d) in Proposition A.1 and the equivalence of (b) and (c) in Proposition A.2 are valid in a C^*-algebra. Thus, for example:

COROLLARY A.3. *Suppose that $\{p_i : 1 \leq i \leq n\}$ are projections in a C^*-algebra A. Then $\sum_{i=1}^n p_i$ is a projection if and only if $p_i p_j = 0$ for $i \neq j$, in which case we say that the projections are* mutually orthogonal.

PROOF. If the projections p_i are mutually orthogonal, a straightforward calculation shows that $p := \sum_{i=1}^n p_i$ satisfies $p^2 = p$, and p is clearly self-adjoint. We prove the converse by induction. It is trivially true when $n = 1$. Suppose that the converse is true for $n = k$, and that $\sum_{i=1}^{k+1} p_i$ is a projection. Since each $p_i = p_i^* p_i$ is a positive element of the C^*-algebra A, we have $\sum_{i=1}^{k+1} p_i \geq p_{k+1}$ in A, and hence by Proposition A.1,

$$\sum_{i=1}^{k} p_i = \Big(\sum_{i=1}^{k+1} p_i\Big) - p_{k+1}$$

is a projection in A. Now the inductive hypothesis implies that the projections $\{p_i : 1 \leq i \leq k\}$ are mutually orthogonal, and Proposition A.2 implies that p_{k+1} and $\sum_{i=1}^k p_i$ are mutually orthogonal. Thus for $i \leq k$ we have

$$0 \leq p_{k+1} p_i p_{k+1} \leq p_{k+1}\Big(\sum_{i=1}^{k} p_i\Big) p_{k+1} = 0,$$

which forces $p_{k+1} p_i = 0$. Thus $\{p_i : 1 \leq i \leq k+1\}$ is mutually orthogonal. □

An operator S on Hilbert space is a *partial isometry* if the restriction of S to $(\ker S)^\perp$ is an isometry.

PROPOSITION A.4. *Let S be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) *S is a partial isometry;*
(b) *S^*S is a projection;*
(c) *$SS^*S = S$;*
(d) *SS^* is a projection;*

(e) *$S^*SS^* = S^*$.*

If so, S^*S is the projection on $(\ker S)^\perp$ and SS^* is the projection on the range of S.

PROOF. Taking adjoints in (c) gives (e), and applying (b) $\Longleftrightarrow$ (c) to S^* gives (d) $\Longleftrightarrow$ (e). It therefore suffices to prove that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a).

(a) $\Longrightarrow$ (b). We shall show that S^*S is the orthogonal projection P onto $(\ker S)^\perp$. The polarisation identity

$$4(Th \,|\, k) = \sum_{n=0}^{3} i^n \big(T(h + i^n k) \,|\, h + i^n k\big)$$

shows that it suffices to prove

$$\big((S^*S - P)h \,|\, h\big) = 0 \quad \text{for all } h \in \mathcal{H}. \tag{A.2}$$

If $h \in (\ker S)^\perp$, then

$$(S^*Sh \,|\, h) = \|Sh\|^2 = \|h\|^2 = (h \,|\, h) = (Ph \,|\, h),$$

which gives (A.2) for $h \in (\ker S)^\perp$. But both S^*S and P have range in $(\ker S)^\perp$ and are zero on $\ker S$, so for every $h \in \mathcal{H}$ we have

$$\big((S^*S - P)h \,|\, h\big) = \big(P(S^*S - P)Ph \,|\, h\big) = \big((S^*S - P)Ph \,|\, Ph\big)$$

so we have (A.2) for all $h \in \mathcal{H}$.

(b) $\Longrightarrow$ (c). Suppose S^*S is a projection. Then

$$\|S - SS^*S\|^2 = \|(S - SS^*S)^*(S - SS^*S)\| = \|S^*S - 2(S^*S)^2 + (S^*S)^3\| = 0,$$

which gives (c).

(c) $\Longrightarrow$ (a). We first note that that S^*S must be the projection onto $(\ker S)^\perp$. To see this, recall that the projection P on a closed subspace M is the unique operator such that $Ph \in M$ and $h - Ph \perp M$ for all $h \in \mathcal{H}$. Let $h \in \mathcal{H}$. For $k \in \ker S$, we have $(S^*Sh \,|\, k) = (Sh \,|\, Sk) = 0$, so $S^*Sh \in (\ker S)^\perp$. The relation $(S - SS^*S)h = 0$ implies that $h - S^*Sh \in \ker S$, and hence that $h - S^*Sh$ is orthogonal to $(\ker S)^\perp$. So S^*S is the projection onto $(\ker S)^\perp$. Now for $h \in (\ker S)^\perp$ we have

$$\|Sh\|^2 = (Sh \,|\, Sh) = (S^*Sh \,|\, h) = (h \,|\, h) = \|h\|^2,$$

which says that S is a partial isometry.

That S^*S is the projection on $(\ker S)^\perp$ we established in the course of proving (c) $\Longrightarrow$ (a). We trivially have $SS^*\mathcal{H} \subset S\mathcal{H}$, and (c) implies that $S\mathcal{H} = SS^*S\mathcal{H} \subset SS^*\mathcal{H}$. □

Statements (b), (c), (d) and (e) make sense in an abstract C^*-algebra, and are still equivalent there because we can always represent the C^*-algebra faithfully on Hilbert space. An element s of a C^*-algebra A which satisfies $s = ss^*s$ or one of the equivalent conditions is called a *partial isometry in A*. If s is a partial isometry in A, we call s^*s the *initial projection of s*, and ss^* the *final projection of s*.