Algebraic Topology, Exam problems

(A) Graphs, free groups and their subgroups.

Recall that the free group generated by a set S is $F(S) = \star_{s\in S}\mathbb{Z}$. One can show that $F(S) \cong F(T)$ iff S and T have the same cardinality. In particular we write F_n for the free group with $n \in \mathbb{N}$ generators.

The aim is to prove *Schreier's Theorem* (1929):

THEOREM 0.1 *Let $F = F(S)$ be a free group and $H \subset F$ a subgroup. Then H is a free group, i.e. $H \cong F(T)$ for some set T.*

In particular, if $H \subset F_n$ with $p = [F_n : H] < \infty$ then $H \cong F_m$ with $m = p(n-1)+1$.

Proceed as follows:

1. Define *graphs* and *maximal trees* in them. Prove that every graph contains a maximal tree.
2. Prove that the fundamental group of a connected graph is the free group.
3. Every free group is the fundamental group of a connected graph.
4. Prove that connected graphs are locally path-connected and semilocally simply connected and that a covering space of a graph is a graph.
5. Deduce the first result.
6. Prove the identity $m-1 = p(n-1)$ using the notion of Euler characteristics of a graph.

Remark. There are purely algebraic proofs of the theorem, but the most instructive proof uses *graphs*.

References. **May: A concise course in algebraic topology, Chapter 4.** Hatcher: Algebraic Topology, Section 1.A. Munkres: Topology (But don't use piecewise linear graphs!) Bredon: Topology and geometry, Section III.7 (gives only some of the needed facts).

(B) The long exact homotopy sequence

1. For a space X, a subspace A, a point $x \in A$ and $n \geq 1$, define *relative homotopy sets* $\pi_n(X, A, x)$. Prove that $\pi_n(X, A, x)$ is a group for $n \geq 2$ and abelian for $n \geq 3$.
2. Prove that there exists a long exact sequence

$$\cdots \pi_{n+1}(X, A, x) \to \pi_n(A, x) \to \pi_n(X, x) \to \pi_n(X, A, x) \to \pi_{n-1}(A, x) \cdots$$

   The end

$$\cdots \to \pi_1(X, A, x) \to \pi_0(A) \to \pi_0(X)$$

   of the sequence requires special considerations, since one has no group structures.

Remark. One nice thing about the homotopy exact sequence is that it arises in a purely geometrical way, i.e. without using the fundamental theorem of homological algebra. (Nowadays there is a tendency to base all of algebraic topology on homotopy theory and to avoid homological algebra whenever possible.)

References. **Hatcher: Algebraic topology, Section 4.1.**

(C) Connection between $\pi_1(X, x)$ and $H_1(X)$: Abelianization

If G is a group then $[G, G]$ denotes the normal subgroup of G generated by (i.e. the smallest normal subgroup containing) $[g, h] := ghg^{-1}h^{-1}$, $g, h \in G$. The quotient group $G/[G, G]$ is called the abelianization G_{ab} of G. Prove the following

1. G_{ab} is abelian.
2. Let H be abelian and $\alpha : G \to H$ a homomorphism. Then there exists a unique homomorphism $\beta : G_{ab} \to H$ such that $\alpha = \beta \circ \varphi$, where $\varphi_G : G \to G_{ab}$ is the quotient homomorphism.
3. Let $\alpha : G \to H$ be a homomorphism. There exists a unique homomorphism $\alpha_{ab} : G_{ab} \to H_{ab}$ such that $\alpha_{ab} \circ \varphi_G = \varphi_H \circ \alpha$.
4. Abelianization is a functor from the category of groups to the category of abelian groups.

After these preliminaries, prove the following

THEOREM 0.2 *Let X be a path-connected space and let $x \in X$. Then there exists an isomorphism $\phi : \pi_1(X, x)_{ab} \to H_1(X)$.*

Finally, generalize to spaces that are not path-connected.

References. : **Bredon: Topology and Geometry, Section IV.3**. Hatcher: Algebraic Topology, Section 2.A.

(D) Simplicial Homology

1. Define the purely combinatorial notion of a simplicial complex C.
2. Define the simplicial homology $H_n(C)$ of a simplicial complex C.
3. Define the topological realization $|C|$ of a simplicial complex C.

Remark. Simplicial complexes and the generalization of 'simplicial sets' allow to formulate much of algebraic topology in a purely combinatorical way. Among other things, this allows to apply ideas of algebraic topology in other fields, like algebra and algebraic geometry. You are not expected to prove the equivalence $H_n(C) \cong H_n(|C|)$ of singular and simplicial homology, but I encourage you to look at the proof.

References. **Hatcher: Algebraic topology, Section 2.1.** Bredon: Topology and Geometry, Section IV.21.