3. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$.

(a) Denote by Λ the set of all pairs (n, U) where n is a positive integer, and U is a neighbourhood of 0 in the strong topology of $B(H)$. For (n, U) and (n', U') in Λ, write $(n, U) \leq (n', U')$ if $n \leq n'$ and $U' \subseteq U$. Show that Λ is a poset under the relation $\leq$, and that it is upwards-directed.

(b) Let u denote the unilateral shift on (e_n), and note that (u^{n*}) is strongly convergent to zero. If $\lambda = (n_\lambda, U_\lambda) \in \Lambda$, then $\lim_{n\to\infty}(n_\lambda u^{n*}) = 0$ in the strong topology, so for some n we have $n_\lambda u^{n*} \in U_\lambda$. Set $u_\lambda = n_\lambda u^{n*}$ and $v_\lambda = \frac{1}{n_\lambda} u^n$. Show that $\lim_\lambda u_\lambda = 0$ in the strong topology and $\lim_\lambda v_\lambda = 0$ in the norm topology. (Since $u_\lambda v_\lambda = 1$, this shows that the operation of multiplication

$$B(H) \times B(H) \to B(H), \quad (u, v) \mapsto uv,$$

is not jointly continuous in either the weak or the strong topologies.)

(c) Show that neither the weak nor the strong topologies on $B(H)$ are metrisable, using Exercise 4.2 and the nets (u_λ) and (v_λ) from part (b) of this exercise.

4. Let A be a von Neumann algebra on a Hilbert space H, and suppose that τ is a bounded linear functional on A. We say that τ is *normal* if, whenever an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ in A_{sa} converges strongly to an operator $u \in A_{sa}$, we have $\lim_\lambda \tau(u_\lambda) = \tau(u)$. Show that every σ-weakly continuous functional $\tau \in A^*$ is normal (use Theorem 4.2.10 and show that if $(v_\lambda)_\lambda$ is a bounded net strongly convergent to v, and if $u \in L^1(H)$, then $\lim_\lambda \|v_\lambda u - vu\|_1 = 0$.)

5. The existence and characterisation of extreme points is very important in many contexts (for example, we shall be concerned with this in the next chapter in connection with pure states). See the Appendix for the definition of extreme points.

Let H be a non-zero Hilbert space.

(a) Show that the extreme points of the closed unit ball of H are precisely the unit vectors.

(b) Deduce that the isometries and co-isometries of $B(H)$ are extreme points of the closed unit ball of $B(H)$. (It can be shown that these are all of the extreme points. This follows from [Tak, Theorem I.10.2].)

6. Let A be a C*-algebra.

(a) Show that if A is unital, then its unit is an extreme point of its closed unit ball.

(b) If p is a projection of A, show that it is an extreme point of the closed unit ball of A^+ (use the unital algebra pAp and part (a)). The converse of this result is also true, but more difficult. It follows from [Tak, Lemma I.10.1].

(c) Show that if H is an infinite-dimensional Hilbert space, then the closed unit ball of $B(H)^+$ is not the convex hull of the projections of $B(H)$.

7. Let A be a C*-algebra. Show that if p, q are equivalent projections in A, and r is a projection orthogonal to both (that is, $rp = rq = 0$), then the projections $r + p$ and $r + q$ are equivalent.

If H is a separable Hilbert space and p is a projection not of finite rank, set $\text{rank}(p) = \infty$. If p has finite rank, set $\text{rank}(p) = \dim p(H)$. Show that $p \sim q$ in $B(H)$ if and only if $\text{rank}(p) = \text{rank}(q)$.

Thus, the equivalence class of a projection in a C*-algebra can be thought of as its "generalised rank."

We say a projection p in a C*-algebra A is *finite* if for any projection q such that $q \sim p$ and $q \leq p$ we necessarily have $q = p$. Otherwise, the projection is said to be *infinite*. Show that if p, q are projections such that $q \leq p$ and p is finite, then q is finite.

A projection p in a von Neumann algebra A is *abelian* if the algebra pAp is abelian. Show that abelian projections are finite.

A von Neumann algebra is said to be *finite* or *infinite* according as its unit is a finite or infinite projection. If H is a Hilbert space, show that the von Neumann algebra $B(H)$ is finite or infinite according as H is finite- or infinite-dimensional.

4. Addenda

Let τ be a bounded linear functional on a von Neumann algebra A. The following are equivalent conditions:

(i) τ is normal.
(ii) The restriction of τ to the closed unit ball of A is weakly continuous.
(iii) τ is σ-weakly continuous.

Reference: [Ped, Theorem 3.6.4].

A projection in a von Neumann algebra A is *central* if it commutes with every element of A.

We say that A is *Type* I if every non-zero central projection in A majorises a non-zero abelian projection in A. Thus, abelian von Neumann algebras are trivially Type I. Just as easy, $B(H)$ is Type I for every Hilbert space H.

We say that A is *Type* II if it has no non-zero abelian projections and every non-zero central projection majorises a non-zero finite projection.

We say that A is *Type* III if it contains no non-zero finite projections.

We say that A is *properly infinite* if it has no non-zero finite central projection.

If A is Type II and properly infinite, it is said to be *Type* II_∞, and if it is Type II and finite, it is said to be *Type* II_1.