is a C*-norm and, therefore, it is dominated by the maximal C*-norm $\|.\|_{max}$. Hence, π' is norm-decreasing for $\|.\|_{max}$, so π' can be extended to a $*$-homomorphism from $A \otimes_{max} D$ to $B \otimes_* D$ which we shall also denote by π'.

Let Q be the quotient algebra of $A \otimes_{max} D$ by the closed ideal $\mathrm{im}(j')$, and let $\psi: A \otimes_{max} D \to Q$ be the quotient map. By a construction similar to that carried out in the proof of Theorem 6.5.2, there is a unique $*$-homomorphism $\theta: B \otimes_* D \to Q$ such that $\theta(\pi(a) \otimes d) = a \otimes d + \mathrm{im}(j')$ for all $a \in A$ and $d \in D$ (this uses nuclearity of B). We therefore get a commutative diagram:

$$\begin{array}{ccccc} J \otimes_* D & \xrightarrow{j \otimes_* \mathrm{id}} & A \otimes_* D & \xrightarrow{\pi \otimes_* \mathrm{id}} & B \otimes_* D \\ & \searrow j' & \uparrow \varphi & \nearrow \pi' & \downarrow \theta \\ & & A \otimes_{max} D & \xrightarrow{\psi} & Q. \end{array}$$

Now suppose that $c \in \ker(\varphi)$. Then $0 = \bar{\pi}\varphi(c) = \pi'(c)$, so $0 = \theta\pi'(c) = \psi(c)$. Hence, $c = j'(c_0)$ for some element $c_0 \in J \otimes_* D$, and therefore $\bar{j}(c_0) = \varphi j'(c_0) = \varphi(c) = 0$. Since $\bar{j}$ is injective by Theorem 6.5.1, we have $c_0 = 0$ and therefore $c = j'(c_0) = 0$. Thus, φ is injective and the theorem is proved. □

6.5.1. *Example.* Let **A** denote the Toeplitz algebra (the C*-algebra generated by all Toeplitz operators on the Hardy space H^2 having continuous symbol). This algebra was investigated in Section 3.5, where it was shown that its commutator ideal is $K(H^2)$ (Theorem 3.5.10). The algebras $K(H^2)$ and $\mathbf{A}/K(H^2)$ are nuclear (by Example 6.3.2 and Theorem 6.4.15, respectively), so by Theorem 6.5.3, **A** is nuclear.

6. Exercises

1. Let $(A_n, \varphi_n)_{n=1}^{\infty}$ and $(B_n, \psi_n)_{n=1}^{\infty}$ be direct sequences of C*-algebras with direct limits A and B, respectively. Let $\varphi^n: A_n \to A$ and $\psi^n: B_n \to B$ be the natural maps. Suppose there are $*$-homomorphisms $\pi_n: A_n \to B_n$ such that for each n the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ \downarrow \pi_n & & \downarrow \pi_{n+1} \\ B_n & \xrightarrow{\psi_n} & B_{n+1}. \end{array}$$

Show that there exists a unique $*$-homomorphism $\pi: A \to B$ such that for each n the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi^n} & A \\ \downarrow \pi_n & & \downarrow \pi \\ B_n & \xrightarrow{\psi^n} & B. \end{array}$$

Show that if all the π_n are $*$-isomorphisms, then π is a $*$-isomorphism.

2. Show that every non-zero finite-dimensional C*-algebra admits a faithful tracial state. Give an example of a unital simple C*-algebra not having a tracial state.

3. Let A be a C*-algebra. A *trace* on A is a function $\tau: A^+ \to [0, +\infty]$ such that

$$\begin{aligned} \tau(a+b) &= \tau(a) + \tau(b) \\ \tau(ta) &= t\tau(a) \\ \tau(c^*c) &= \tau(cc^*) \end{aligned}$$

for all $a, b \in A^+$, $c \in A$, and all $t \in \mathbf{R}^+$. We use the convention that $0.(+\infty) = 0$.

The motivating example is the usual trace function on $B(H)$. Another example is got on $C_0(\mathbf{R})$ by setting $\tau(f) = \int f\,dm$ where $f \in C_0(\mathbf{R})^+$ and m is ordinary Lebesgue measure on $\mathbf{R}$.

Traces (and their generalisation, *weights*) play a fundamental role, especially in von Neumann algebra theory ([Ped], [Tak]).

Let

$$A_\tau^2 = \{a \in A \mid \tau(a^*a) < \infty\}.$$

Show that

$$(a+b)^*(a+b) \le 2a^*a + 2b^*b$$

and

$$(ab)^*ab \le \|a\|^2 b^*b,$$

and deduce that A_τ^2 is a self-adjoint ideal of A.

Let A_τ be the linear span of all products ab, where $a, b \in A_\tau^2$. Show that A_τ is a self-adjoint ideal of A.

Show that for arbitrary $a, b \in A$,

$$a^*b = \tfrac{1}{4} \sum_{k=0}^{3} i^k (b + i^k a)^*(b + i^k a),$$