Again suppose X to be an infinite-dimensional Banach space and suppose that $u \in B(X)$. We define the *essential spectrum* of u to be

$$\sigma_e(u) = \{\lambda \in \mathbf{C} \mid u - \lambda \text{ is not Fredholm}\}.$$

Let C denote the quotient algebra $B(X)/K(X)$. This algebra is called the *Calkin* algebra on X. If π is the quotient map from $B(X)$ to C, it is clear from the Atkinson characterisation (Theorem 1.4.16) that $\sigma_e(u) = \sigma_C(\pi u)$. Thus, $\sigma_e(u)$ is a non-empty compact set. Obviously, $\sigma_e(u) \subseteq \sigma(u)$.

1.4.4. *Example.* Suppose that H is a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. The *unilateral shift* on this basis is the operator u in $B(H)$ such that $u(e_n) = e_{n+1}$ for all n. Observe that $\text{nul}(u) = 0$ and $\text{def}(u) = 1$, so u is a Fredholm operator and $\text{ind}(u) = -1$.

If instead we suppose that $(f_n)_{n\in\mathbf{Z}}$ is an orthonormal basis for H, the *bilateral shift* on this basis is the operator v such that $v(f_n) = f_{n+1}$ for all $n \in \mathbf{Z}$. This operator is invertible, so $\text{ind}(v) = 0$. Hence, u and v are not similar (two elements a, b of a unital algebra are *similar* if there is an invertible element c such that $a = c^{-1}bc$).

It follows from Theorem 1.4.16 that if $\pi: B(H) \to B(H)/K(H)$ is the quotient homomorphism, then $\pi(u)$ is invertible. It is natural to ask if one can write $\pi(u) = \pi(w)$ for some invertible operator w in $B(H)$. If this were the case, then $\text{ind}(u) = \text{ind}(w)$, since $u - w \in K(H)$. This is, however, impossible, since $\text{ind}(u) = -1$, and $\text{ind}(w) = 0$. An interesting consequence is that $\pi(u)$ provides an example of an invertible element that cannot be written as an exponential, for if $\pi(u) = e^w$ for some w in the Calkin algebra, then $w = \pi(w')$ for some $w' \in B(H)$, and therefore $\pi(u) = e^{\pi(w')} = \pi(e^{w'})$. But $e^{w'}$ is invertible in $B(H)$, which contradicts what we have just shown. Thus, $\pi(u)$ has no logarithm in the Calkin algebra.

We shall have more to say about shifts in the next chapter.

We shall see further examples and applications concerning compact and Fredholm operators in later chapters. We turn in Chapter 2 to the case where the algebras have involutions and the operators have adjoints. This is the self-adjoint theory, and it is in this setting that some of the deepest results concerning algebras and operators have been proved.

1. Exercises

1. Let $(A_\lambda)_{\lambda\in\Lambda}$ denote a family of Banach algebras. The *direct sum* $A = \oplus_\lambda A_\lambda$ is the set of all $(a_\lambda) \in \prod_\lambda A_\lambda$ such that $\|(a_\lambda)\| = \sup_\lambda \|a_\lambda\|$ is finite. Show that this is a Banach algebra under the pointwise-defined operations

$$(a_\lambda) + (b_\lambda) = (a_\lambda + b_\lambda)$$
$$\mu(a_\lambda) = (\mu a_\lambda)$$
$$(a_\lambda)(b_\lambda) = (a_\lambda b_\lambda),$$

and norm given by $(a_\lambda) \mapsto \|(a_\lambda)\|$. Show that A is unital or abelian if this is the case for all of the algebras A_λ.

The *restricted sum* $B = \bigoplus_\lambda^{c_0} A_\lambda$ is the set of all elements $(a_\lambda) \in A$ such that for each $\varepsilon > 0$ there exists a finite subset F of Λ for which $\|a_\lambda\| < \varepsilon$ if $\lambda \in \Lambda \setminus F$. Show that B is a closed ideal in A.

2. Let A be a Banach algebra and Ω a non-empty set. Denote by $\ell^\infty(\Omega, A)$ the set of all bounded maps f from Ω to A. Show that $\ell^\infty(\Omega, A)$ is a Banach algebra with the pointwise-defined operations and the sup-norm $\|f\| = \sup\{\|f(\omega)\| \mid \omega \in \Omega\}$. If Ω is a compact Hausdorff space, show that the set $C(\Omega, A)$ of all continuous functions from Ω to A is a closed subalgebra of $\ell^\infty(\Omega, A)$.

3. Give an example of a unital non-abelian Banach algebra A in which 0 and A are the only closed ideals.

4. Give an example of a non-modular maximal ideal in an abelian Banach algebra. (If A is the disc algebra, let $A_0 = \{f \in A \mid f(0) = 0\}$. Then A_0 is a closed subalgebra of A and admits an ideal of the type required.)

5. Let A be a unital abelian Banach algebra.

(a) Show that $\sigma(a+b) \subseteq \sigma(a) + \sigma(b)$ and $\sigma(ab) \subseteq \sigma(a)\sigma(b)$ for all $a, b \in A$. Show that this is not true for *all* Banach algebras.

(b) Show that if A contains an *idempotent* e (that is, $e = e^2$) other than 0 and 1, then $\Omega(A)$ is disconnected.

(c) Let $a_1, \ldots, a_n$ generate A as a Banach algebra. Show that $\Omega(A)$ is homeomorphic to a compact subset of $\mathbf{C}^n$. More precisely, set $\sigma(a_1, \ldots, a_n) = \{(\tau(a_1), \ldots, \tau(a_n)) \mid \tau \in \Omega(A)\}$. Show that the canonical map from $\Omega(A)$ to $\sigma(a_1, \ldots, a_n)$ is a homeomorphism.

6. Let A be a unital Banach algebra.

(a) If a is invertible in A, show that $\sigma(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$.

(b) For any element $a \in A$, show that $r(a^n) = (r(a))^n$.

(c) If A is abelian, show that the Gelfand representation is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

7. Let A be a Banach algebra. Show that the spectral radius function $r\colon A \to \mathbf{R}$ is upper semi-continuous. (One can show that r is not in general continuous [Hal, Problem 104].)

8. Show that if B is a maximal abelian subalgebra of a unital Banach algebra A, then B is closed and contains the unit. Show that $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$.

9. Let (Ω, μ) be a measure space. Show that the linear span of the idempotents is dense in $L^\infty(\Omega, \mu)$. Show that the spectrum of the Banach algebra $L^\infty(\Omega, \mu)$ is totally disconnected, by showing that if A is an arbitrary abelian Banach algebra in which the idempotents have dense linear span, its spectrum $\Omega(A)$ is totally disconnected.

10. Let $A = C^1[0,1]$, as in Example 1.2.6. Let $x: [0,1] \to \mathbf{C}$ be the inclusion. Show that x generates A as a Banach algebra. If $t \in [0,1]$, show that τ_t belongs to $\Omega(A)$, where τ_t is defined by $\tau_t(f) = f(t)$, and show that the map $[0,1] \to \Omega(A)$, $t \mapsto \tau_t$, is a homeomorphism. Deduce that $r(f) = \|f\|_\infty$ $(f \in A)$. Show that the Gelfand representation is not surjective for this example.

11. Let A be a unital Banach algebra and set

$$\zeta(a) = \inf_{\|b\|=1} \|ab\| \qquad (a \in A).$$

We say that an element a of A is a *left topological zero divisor* if there is a sequence of unit vectors (a_n) of A such that $\lim_{n\to\infty} aa_n = 0$. Equivalently, $\zeta(a) = 0$.

(a) Show that left topological zero divisors are not invertible.
(b) Show that $|\zeta(a) - \zeta(b)| \leq \|a - b\|$ for all $a, b \in A$. Hence, ζ is a continuous function.
(c) If a is a boundary point of the set $\mathrm{Inv}(A)$ in A, show that there is a sequence of invertible elements (v_n) converging to a such that $\lim_{n\to\infty} \|{v_n}^{-1}\|^{-1} = 0$. Using the continuity of ζ, deduce that $\zeta(a) = 0$. Thus, boundary points of $\mathrm{Inv}(A)$ are left topological zero divisors. In particular, if λ is a boundary point of the spectrum of an element a of A, then $\lambda - a$ is a left topological zero divisor.
(d) Let Ω be a compact Hausdorff space and let $A = C(\Omega)$. Show that in this case the topological zero divisors are precisely the non-invertible elements (if f is non-invertible, then 0 is a boundary point of the spectrum of $\bar{f}f$).
(e) Give an example of a unital Banach algebra and a non-invertible element that is not a left topological zero divisor.

12. A derivation on an algebra A is a linear map $d: A \to A$ such that $d(ab) = adb + d(a)b$. Show that the Leibnitz formula,

$$d^n(ab) = \sum_{r=0}^{n} \binom{n}{r} d^r(a)d^{n-r}(b) \qquad (n = 1, 2, \ldots),$$

holds.

13. Suppose that d is a bounded derivation on a unital Banach algebra A and $\lambda \in \mathbf{C} \setminus \{0\}$ such that $da = \lambda a$. Show that a is nilpotent, that is, that $a^n = 0$ for some positive integer n (use the boundedness of $\sigma(d)$).

14. Suppose that d is a bounded derivation on a unital Banach algebra A, and that $a \in A$ and $d^2 a = 0$. Show that da is quasinilpotent. (Hint: Show that $d^{n+1}(a^n) = 0$ and hence, $d^n(a^n) = n!(da)^n$.) For $a \in A$, the map $b \mapsto [a, b] = ab - ba$ is a bounded derivation on A. Therefore, the Kleinecke–Shirokov theorem holds: If $[a, [a, b]] = 0$, then $[a, b]$ is quasinilpotent.

15. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$, and let u be an operator in $B(H)$ diagonal with respect to (e_n) with diagonal the sequence (λ_n). Show that u is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.

16. Let X be a Banach space. If $p \in B(X)$ is a compact idempotent, show that its rank is finite.

17. Let $u: X \to Y$ be a compact operator between Banach spaces. Show that if the range of u is closed, then it is finite-dimensional. (Hint: Show that the well-defined operator

$$X/\ker(u) \to u(X), \quad x + \ker(u) \mapsto u(x),$$

is an invertible compact operator.)

18. Let X, Y be Banach spaces and suppose that $u \in B(X, Y)$ has compact transpose u^*. Show that u is compact using the fact that u^{**} is compact.

19. Let $u: X \to Y$ and $u': X' \to Y'$ be bounded operators between Banach spaces. Show that the linear map

$$u \oplus u': X \oplus X' \to Y \oplus Y', \quad (x, x') \mapsto (u(x), u'(x')),$$

is bounded with norm $\max\{\|u\|, \|u'\|\}$. Show that if u and u' are Fredholm operators, so is $u \oplus u'$, and $\operatorname{ind}(u \oplus u') = \operatorname{ind}(u) + \operatorname{ind}(u')$.

20. If X is an infinite-dimensional Banach space and $u \in B(X)$, show that

$$\bigcap_{v \in K(X)} \sigma(u + v) = \sigma(u) \setminus \{\lambda \in \mathbf{C} \mid u - \lambda \text{ is Fredholm of index zero}\}.$$