2.5.7. Theorem. *Let u be a normal operator on a Hilbert space H, and suppose that $g\colon \mathbf{C} \to \mathbf{C}$ is a continuous function. Then $(g \circ f)(u) = g(f(u))$ for all $f \in B_\infty(\sigma(u))$.*

Proof. The result is easily seen by first showing it for g a polynomial in z and $\bar{z}$, and then observing that an arbitrary continuous function $g\colon \mathbf{C} \to \mathbf{C}$ is a uniform limit of such polynomials on the compact disc $\Delta = \{\lambda \in \mathbf{C} \mid |\lambda| \leq \|f\|_\infty\}$, using the Stone–Weierstrass theorem applied to $C(\Delta)$. □

We give an application of this to writing a unitary as an exponential.

2.5.8. Theorem. *Let u be a unitary operator in $B(H)$, where H is a Hilbert space. Then there exists a hermitian operator v in $B(H)$ such that $u = e^{iv}$ and $\|v\| \leq 2\pi$.*

Proof. The function

$$f\colon [0, 2\pi) \to \mathbf{T}, \ t \mapsto e^{it},$$

is a continuous bijection with Borel measurable inverse g. Since $\sigma(u) \subseteq \mathbf{T}$, we can set $v = g(u)$. The operator v is self-adjoint because g is real-valued. Moreover, $\|v\| \leq \|g\|_\infty \leq 2\pi$. By Theorem 2.5.7, $(f \circ g)(u) = f(g(u)) = f(v) = e^{iv}$. But $(f \circ g)(\lambda) = \lambda$ for all $\lambda \in \mathbf{T}$, so $(f \circ g)(u) = u$. Therefore, $u = e^{iv}$. □

2. Exercises

1. Let A be a Banach algebra such that for all $a \in A$ the implication

$$Aa = 0 \text{ or } aA = 0 \Rightarrow a = 0$$

holds. Let L, R be linear mappings from A to itself such that for all $a, b \in A$,

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad \text{and} \quad R(a)b = aL(b).$$

Show that L and R are necessarily continuous.

2. Let A be a unital C*-algebra.

(a) If a, b are positive elements of A, show that $\sigma(ab) \subseteq \mathbf{R}^+$.

(b) If a is an invertible element of A, show that $a = u|a|$ for a unique unitary u of A. Give an example of an element of $B(H)$ for some Hilbert space H that cannot be written as a product of a unitary times a positive operator.

(c) Show that if $a \in \mathrm{Inv}(A)$, then $\|a\| = \|a^{-1}\| = 1$ if and only if a is a unitary.

3. Let Ω be a locally compact Hausdorff space, and suppose that the C*-algebra $C_0(\Omega)$ is generated by a sequence of projections $(p_n)_{n=1}^{\infty}$. Show that the hermitian element $h = \sum_{n=1}^{\infty} p_n/3^n$ generates $C_0(\Omega)$.

4. We shall see in the next chapter that all closed ideals in C*-algebras are necessarily self-adjoint. Give an example of an ideal in the C*-algebra $C(\mathbf{D})$ that is not self-adjoint.

5. Let $\varphi: A \to B$ be an isometric linear map between unital C*-algebras A and B such that $\varphi(a^*) = \varphi(a)^*$ $(a \in A)$ and $\varphi(1) = 1$. Show that $\varphi(A^+) \subseteq B^+$.

6. Let A be a unital C*-algebra.
(a) If $r(a) < 1$ and $b = (\sum_{n=0}^{\infty} a^{*n}a^n)^{1/2}$, show that $b \geq 1$ and $\|bab^{-1}\| < 1$.
(b) For all $a \in A$, show that

$$r(a) = \inf_{b \in Inv(A)} \|bab^{-1}\| = \inf_{b \in A_{sa}} \|e^b a e^{-b}\|.$$

7. Let A be a unital C*-algebra.
(a) If $a, b \in A$, show that the map

$$f: \mathbf{C} \to A, \quad \lambda \mapsto e^{i\lambda b} a e^{-i\lambda b},$$

is differentible and that $f'(0) = i(ba - ab)$.
(b) Let X be a closed vector subspace of A which is unitarily invariant in the sense that $uXu^* \subseteq X$ for all unitaries u of A. Show that $ba - ab \in X$ if $a \in X$ and $b \in A$.
(c) Deduce that the closed linear span X of the projections in A has the property that $a \in X$ and $b \in A$ implies that $ba - ab \in X$.

8. Let a be a normal element of a C*-algebra A, and b an element commuting with a. Show that b^* also commutes with a (Fuglede's theorem). (Hint: Define $f(\lambda) = e^{i\lambda a^*} b e^{-i\lambda a^*}$ in $\tilde{A}$ and deduce from Exercise 2.7 that this map is differentiable and $f'(0) = i(a^*b - ba^*)$. Since $e^{i\bar{\lambda} a}$ and b commute, $f(\lambda) = e^{2ic(\lambda)} b e^{-2ic(\lambda)}$, where $c(\lambda) = Re(\lambda a^*)$. Hence, $\|f(\lambda)\| = \|b\|$, so by Liouville's theorem, $f(\lambda)$ is constant.)

In the following exercises H is a Hilbert space:

9. If I is an ideal of $B(H)$, show that it is self-adjoint.