is an isometric unital $*$-homomorphism such that $\varphi_\lambda(u) = v_\lambda$.

Let (H, φ) be the direct sum of the family of representations $(H_\lambda, \varphi_\lambda)_\lambda$ of $\mathbf{A}$. Then $\varphi: \mathbf{A} \to B(H)$ is a unital $*$-homomorphism such that $\varphi(u) = \oplus_\lambda v_\lambda = v$. Moreover, since $\varphi(u) \in B$ and u generates $\mathbf{A}$, therefore $\varphi(\mathbf{A})$ is contained in B.

Now suppose that $vv^* \neq 1$. Then some v_{λ_0} is a unilateral shift. Hence, the representation $(H_{\lambda_0}, \varphi_{\lambda_0})$ is faithful, so (H, φ) is faithful. Therefore, φ is isometric. □

3. Exercises

In Exercises 1 to 7, A denotes an arbitrary C*-algebra.

1. Let a, b be normal elements of a C*-algebra A, and c an element of A such that $ac = cb$. Show that $a^*c = cb^*$, using Fuglede's theorem (Exercise 2.8) and the fact that the element

$$d = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

is normal in $M_2(A)$ and commutes with

$$d' = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

This more general result is called the Putnam–Fuglede theorem.

2. Let τ be a positive linear functional on A.

(a) If I is a closed ideal in A, show that $I \subseteq \ker(\tau)$ if and only if $I \subseteq \ker(\varphi_\tau)$.

(b) We say τ is *faithful* if $\tau(a) = 0 \Rightarrow a = 0$ for all $a \in A^+$. Show that if τ is faithful, then the GNS representation (H_τ, φ_τ) is faithful.

(c) Suppose that α is an automorphism of A such that $\tau(\alpha(a)) = \tau(a)$ for all $a \in A$. Define a unitary on H_τ by setting $u(a + N_\tau) = \alpha(a) + N_\tau$ ($a \in A$). Show that $\varphi_\tau(\alpha(a)) = u\varphi(a)u^*$ ($a \in A$).

3. If $\varphi: A \to B$ is a positive linear map between C*-algebras, show that φ is necessarily bounded.

4. Suppose that A is unital. Let α be an automorphism of A such that $\alpha^2 = \mathrm{id}_A$. Define B to be the set of all matrices

$$c = \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix},$$

where $a, b \in A$. Show that B is a C*-subalgebra of $M_2(A)$. Define a map $\varphi: A \to B$ by setting

$$\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Show that φ is an injective $*$-homomorphism. We can thus identify A as a C*-subalgebra of B. If we set $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then u is a self-adjoint unitary and $B = A + Au$. If C is any unital C*-algebra with a self-adjoint unitary element v, and $\psi: A \to C$ is a $*$-homomorphism such that

$$\psi(\alpha(a)) = v\psi(a)v^* \qquad (a \in A),$$

show that there is a unique $*$-homomorphism $\psi': B \to C$ extending ψ (that is, $\psi' \circ \varphi = \psi$) such that $\psi'(u) = v$.

(This establishes that B is a (very easy) example of a crossed product, namely $B = A \times_\alpha \mathbf{Z}_2$, the crossed product of A by the two-element group $\mathbf{Z}_2$ under the action α. The theory of crossed products is a vast area of the modern theory of C*-algebras. One of its primary uses is to generate new examples of simple C*-algebras. For an account of this theory, see [Ped].)

5. An element a of A^+ is *strictly positive* if the hereditary C*-subalgebra of A generated by a is A itself, that is, if $(aAa)^- = A$.

(a) Show that if A is unital, then $a \in A^+$ is strictly positive if and only if a is invertible.

(b) If H is a Hilbert space, show that a positive compact operator on H is strictly positive in $K(H)$ if and only if it has dense range.

(c) Show that if a is strictly positive in A, then $\tau(a) > 0$ for all non-zero positive linear functionals τ on A.

6. We say that A is *σ-unital* if it admits a sequence $(u_n)_{n=1}^\infty$ which is an approximate unit for A. It follows from Remark 3.1.1 that every separable C*-algebra is σ-unital.

(a) Let a be a strictly positive element of A, and set $u_n = a(a + 1/n)^{-1}$ for each positive integer n. Show that (u_n) is an approximate unit for A. (Hint: Define $g_n: \sigma(a) \to \mathbf{R}$ by $g_n(t) = t^2/(t + 1/n)$. Show that the sequence (g_n) is pointwise-increasing and pointwise-convergent to the inclusion $z: \sigma(a) \to \mathbf{R}$, and use Dini's theorem to deduce that (g_n) converges uniformly to z. Hence, $a = \lim_{n\to\infty} au_n$.)

(b) If $(u_n)_{n=1}^\infty$ is an approximate unit for A, show that $a = \sum_{n=1}^\infty u_n/2^n$ is a strictly positive element of A.

Thus, A is σ-unital if and only if it admits a strictly positive element.