7. Let Ω be a locally compact Hausdorff space. Show that $C_0(\Omega)$ admits an approximate unit $(p_n)_{n=1}^{\infty}$, where all the p_n are projections, if and only if Ω is the union of a sequence of compact open sets. Deduce that if a C*-algebra A admits a strictly positive element a such that $\sigma(a) \setminus \{0\}$ is discrete, then A admits an approximate unit $(p_n)_{n=1}^{\infty}$ consisting of projections. (Show that $C^*(a)$ is $*$-isomorphic to $C_0(\sigma(a) \setminus \{0\})$.)

8. Let $z: \mathbf{T} \to \mathbf{C}$ be the inclusion map. Let $\theta \in [0,1]$. Show that there is a unique automorphism α of $C(\mathbf{T})$ such that $\alpha(z) = e^{i2\pi\theta}z$. Define the faithful positive linear functional $\tau: C(\mathbf{T}) \to \mathbf{C}$ by setting $\tau(f) = \int f\, dm$ where m is normalised arc length on $\mathbf{T}$. Show that $\tau(\alpha(f)) = \tau(f)$ for all $f \in C(\mathbf{T})$. Deduce from Exercise 3.2 that there is a unitary v on the Hilbert space H_τ such that $\varphi_\tau(\alpha(f)) = v\varphi_\tau(f)v^*$ for all $f \in C(\mathbf{T})$. Let u be the unitary $\varphi_\tau(z)$. Show that $vu = e^{i2\pi\theta}uv$. If θ is irrational, the C*-algebra A_θ generated by u and v is called an ***irrational rotation*** algebra, and A_θ can be shown to be simple. See [Rie] for more details concerning A_θ. These algebras form a very important class of examples in C*-algebra theory. They are motivating examples in Connes' development of "non-commutative differential geometry," a subject of great future promise [Con 2].

9. Let m be normalised Haar measure on $\mathbf{T}$. If $\lambda \in \mathbf{C}$, $|\lambda| < 1$, define $\tau_\lambda: H^1 \to \mathbf{C}$ by setting

$$\tau_\lambda(f) = \int \frac{f(w)}{1 - \lambda\bar{w}}\, dmw \qquad (f \in H^1).$$

Show that $\tau_\lambda \in (H^1)^*$. By expanding $(1 - \lambda\bar{w})^{-1}$ in a power series, show that $\tau_\lambda(f) = \sum_{n=0}^{\infty} \hat{f}(n)\lambda^n$. Deduce that the function

$$\tilde{f}: \operatorname{int} \mathbf{D} \to \mathbf{C}, \quad \lambda \mapsto \tau_\lambda(f),$$

is analytic, where $\operatorname{int} \mathbf{D} = \{\lambda \in \mathbf{C} \mid |\lambda| < 1\}$. If $f, g \in H^2$, show that $fg \in H^1$ and $\tau_\lambda(fg) = \tau_\lambda(f)\tau_\lambda(g)$. (Hint: There exist sequences (φ_n) and (ψ_n) in Γ_+ converging to f and g, respectively, in the L^2-norm. Show that the sequence $(\varphi_n\psi_n)$ converges to fg in the L^1-norm, and deduce the result by first showing it for functions in Γ_+.)

10. If $f: \operatorname{int} \mathbf{D} \to \mathbf{C}$ is an analytic function and $0 < r < 1$, define $f_r \in C(\mathbf{T})$ by setting $f_r(\lambda) = f(r\lambda)$. Set $\|f\|_2 = \sup_{0<r<1} \|f_r\|_2$, and let $H^2(\mathbf{D})$ denote the set of all analytic functions $f: \operatorname{int} \mathbf{D} \to \mathbf{C}$ such that $\|f\|_2 < \infty$. If $f \in H^2(\mathbf{D})$, show that $\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |\lambda_n|^2}$, where $f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n$ is the Taylor series expansion of f. Show that $H^2(\mathbf{D})$ is a Hilbert space with inner product $\langle f, g\rangle = \sum_{n=0}^{\infty} \lambda_n \bar{\mu}_n$, where $\lambda_n = f^{(n)}(0)/n!$ and $\mu_n =$