A family $\{A_s\}_{s\in S}$ of subsets of a set X is called a *cover* of X if $\bigcup_{s\in S} A_s = X$. If X is a topological space and all the sets A_s are open (closed), we say that the cover $\{A_s\}_{s\in S}$ is *open (closed)*. A family $\{A_s\}_{s\in S}$ of subsets of a set X is called *point-finite (point-countable)* if for every $x \in X$ the set $\{s \in S: x \in A_s\}$ is finite (countable). Clearly every locally finite cover is point-finite. On the other hand, the open cover of the interval I consisting of I itself and of all intervals $(1/(i+1), 1/i)$, where $i = 1, 2, \ldots$, is point-finite and not locally finite.

1.5.18. THEOREM. *For every point-finite open cover $\{U_s\}_{s\in S}$ of a normal space X there exists an open cover $\{V_s\}_{s\in S}$ of X such that $\overline{V}_s \subset U_s$ for every $s \in S$.*

PROOF. Let $\mathcal{G}$ be the family of all functions G from the set S to the topology $\mathcal{O}$ of the space X subject to the conditions:

$$G(s) = U_s \quad \text{or} \quad \overline{G(s)} \subset U_s, \tag{7}$$

and

$$\bigcup_{s\in S} G(s) = X. \tag{8}$$

Let us order the family $\mathcal{G}$ by defining that $G_1 \le G_2$ whenever $G_2(s) = G_1(s)$ for every $s \in S$ such that $G_1(s) \neq U_s$. We shall show that for each linearly ordered subfamily $\mathcal{G}_0 \subset \mathcal{G}$ the formula $G_0(s) = \bigcap_{G\in\mathcal{G}_0} G(s)$ for $s \in S$ defines a member of $\mathcal{G}$. Condition (7) is clearly satisfied for $G = G_0$; we shall verify condition (8). Take a point $x \in X$; as $\{U_s\}_{s\in S}$ is point-finite, there exists a finite set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that $x \in U_{s_i}$ for $i = 1, 2, \ldots, k$ and $x \notin U_s$ for $s \in S \setminus S_0$. If $G_0(s_i) = U_{s_i}$ for some $s_i \in S_0$, then $x \in G_0(s_i) \subset \bigcup_{s\in S} G_0(s)$. Assume now that for $i = 1, 2, \ldots, k$ there exists a $G_i \in \mathcal{G}_0$ such that $G_i(s_i) \neq U_{s_i}$. Since the family $\mathcal{G}_0$ is linearly ordered, there exists a $j \le k$ such that $G_i \le G_j$ for $i = 1, 2, \ldots, k$. Applying (8) to G_j we find an $i_0 \le k$ such that $x \in G_j(s_{i_0}) = G_0(s_{i_0})$, so that also in this case $x \in \bigcup_{s\in S} G_0(s)$. One easily sees that $G \le G_0$ for every $G \in \mathcal{G}_0$.

From the Kuratowski-Zorn lemma it follows that there exists a maximal element G in $\mathcal{G}$; to complete the proof it suffices to show that $\overline{G(s)} \subset U_s$ for every $s \in S$.

Let us suppose that $\overline{G(s_0)} \cap (X \setminus U_{s_0}) \neq \emptyset$. The set $A = X \setminus \bigcup\{G(s): s \in S \setminus \{s_0\}\} \subset G(s_0)$ is closed. By the normality of X there exists an open set U such that $A \subset U \subset \overline{U} \subset G(s_0)$. Since from (7) it follows that $G(s_0) = U_{s_0}$, the formula

$$G_0(s) = \begin{cases} U & \text{for } s = s_0, \\ G(s) & \text{for } s \neq s_0, \end{cases}$$

defines a function $G_0 \in \mathcal{G}$ such that $G \le G_0$ and $G \neq G_0$. This contradiction to maximality of G shows that $\overline{G(s)} \subset U_s$ for every $s \in S$. ■