From the theorems of Green, Gauss and Stokes to differential forms and ... the theorem of Stokes

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Abstract

This text is meant as a well-motivated introduction to differential forms, using the classical integral theorems mentioned in the title as point of departure.

1 The Classical Theorems of Green, Gauss and Stokes

We assume familiarity with (i) Riemann integration on $\mathbb{R}^n$, (ii) surfaces in $\mathbb{R}^n$, including boundaries and orientations, and (iii) the classical integral theorems. See e.g. [3] and, for a more rigorous treatment of (i) and (ii), [5].

We begin by recalling the following theorems (nicely treated in [3, Chapter 8]):

1. **The Fundamental Theorem of Analysis.** Let $a, b \in \mathbb{R}$, $a < b$ and $f \in C^1([a,b], \mathbb{R})$. Then
$$\int_a^b f'(x)dx = f(b) - f(a).$$

2. **The Theorem of Green.** Let $A \subset \mathbb{R}^2$ be compact such that ∂A is a (piecewise) C^1 curve, and let $f_1, f_2 \in C^1(A, \mathbb{R})$. Then
$$\int_A \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial A} f_1 dx_1 + f_2 dx_2,$$
where ∂A has the boundary orientation induced by the standard orientation $[(1,0),(0,1)]$ of $A \subset \mathbb{R}^2$.

3. **The Theorem of Gauss.** Let $A \subset \mathbb{R}^3$ be compact such that ∂A is a (piecewise) C^1 surface (of dimension 2), and let $f_1, f_2, f_3 \in C^1(A, \mathbb{R})$. Then
$$\int_A \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = \int_{\partial A} (f_1, f_2, f_3)^t \cdot \vec{dA},$$
where $\vec{dA} = \vec{n} dA$, $\vec{n}$ being the outward directed normal vector on ∂A and dA being the surface element.

4. **The Theorem of Stokes.** Let $A \subset \mathbb{R}^3$ be a compact (piecewise) C^1 oriented 2-dimensional surface. Let $f_1, f_2, f_3 \in C^1(U, \mathbb{R})$, where U is some open neighborhood of A. Then
$$\int_A \mathrm{rot}\vec{f} \cdot \vec{dA} = \int_{\partial A} f_1 dx_1 + f_2 dx_2 + f_3 dx_3,$$

where

$$\mathrm{rot}\vec{f} = \left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}, \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}, \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \right)^t.$$

Again, the orientation on the curve ∂A is the one canonically induced by the one on A.

What do these theorems have in common? Can we perhaps prove all of them in one go? Are there higher dimensional versions? We begin with the following observations:

- We start from a compact surface A of dimension k in $\mathbb{R}^n$ and a C^1 function $f : A \to \mathbb{R}^k$. (In order to make sense of the differentiability assumption, in the case $k = n$ we will require that A is the closure of its interior. When $k < n$ one usually assumes that f is defined on some open neighborhood $U \supset A$.
- On the left hand side, one integrates over A and on the right hand side over the boundary ∂A, which is a surface of dimension $k - 1$.
- On the left hand side, the integrand is a linear combination of first derivatives of f, on the right hand side f appears without differentiation.
- We also note that there is a difference between the Theorem of Stokes and the three other results: In the former we have $k = 2 < 3 = n$, whereas in the others we have $k = n$. The latter means that A has the same dimension as $\mathbb{R}^n$, thus A is the closure of some bounded open subset $U \subset \mathbb{R}^n$. We will now try to find a common formulation of the fundamental theorem of analysis and the theorems of Gauss and Green that can be generalized.
- If we replace (f_1, f_2) in Green's theorem by $(-f_2, f_1)$, we obtain the equivalent equation

$$\int_M \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = \int_{\partial M} f_1 dx_2 - f_2 dx_1,$$

  which is known as the 'divergence version' of Green's theorem, cf. [3]. This equals

$$\int_{\partial M} \left(f_1 \frac{\partial x_2}{\partial t} - f_2 \frac{\partial x_1}{\partial t} \right) dt = \int \det \begin{pmatrix} f_1 & \frac{\partial x_1}{\partial t} \\ f_2 & \frac{\partial x_2}{\partial t} \end{pmatrix} dt.$$

  (Since ∂M is a closed curve, we need at least two charts to parametrize it. Thus we should really have a sum of terms, one for each chart (U_i, φ_i), where the U_i are chosen such that the pairwise intersections of their closures have measure zero.)
- In Gauss' theorem, the (normal vector valued) surface element $\vec{dA} = \vec{n} dA$ of ∂M is given by

$$\left(\frac{\partial \vec{x}}{\partial t_1} \times \frac{\partial \vec{x}}{\partial t_2} \right) dt_1 dt_2,$$

  where t_1, t_2 are local parameters. (By definition of the boundary orientation on ∂M, the triple $(\vec{n}, \frac{\partial \vec{x}}{\partial t_1}, \frac{\partial \vec{x}}{\partial t_2})$ has the standard orientation of $\mathbb{R}^3$ iff $\vec{n}$ points to the outside of M, as in the statement of Gauss' theorem.) Thus the right hand side of Gauss' theorem equals

$$\int \vec{f} \cdot \left(\frac{\partial \vec{x}}{\partial t_1} \times \frac{\partial \vec{x}}{\partial t_2} \right) dt_1 dt_2 = \int \det \begin{pmatrix} f_1 & \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ f_2 & \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ f_3 & \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{pmatrix} dt_1 dt_2.$$

  (Again, we really have a sum over charts.)

2 The Generalized Gauss Theorem

In view of the preceding discussion, it is not too difficult to guess the right generalization of the theorems of Gauss and Green:

THEOREM 1 *Let $U \subset \mathbb{R}^n$ bounded, open and connected such that ∂U is a (piecewise) C^1 surface (necessarily of dimension $n-1$). Write $M = \overline{U}$, let $f_1, \ldots, f_n \in C^1(M, \mathbb{R})$ and φ a parametrization of M. Then*

$$\int_M \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) \right) dx = \int_{\partial M} \det \begin{pmatrix} f_1 & \frac{\partial \varphi_1}{\partial t_1} & \cdots & \frac{\partial \varphi_1}{\partial t_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & \frac{\partial \varphi_n}{\partial t_1} & \cdots & \frac{\partial \varphi_n}{\partial t_{n-1}} \end{pmatrix} dt_1 \cdots dt_{n-1}, \qquad (1)$$

where the orientation of ∂M is induced from the standard orientation on M.

REMARK 2 1. Since U is open, $\partial U = \overline{U} - U$.

2. $\int_M f(x)dx$ means the Riemann integral over M, defined as $\int_I (\chi_M f)(x)dx$, where I is an interval containing M. This integral exists since f is continuous, M is bounded and ∂M is a C^1 surface, implying that ∂M is a measure zero subset of $\mathbb{R}^n$.

3. On the r.h.s., $\int_{\partial M}$ should be understood as

$$\sum_i \int dt_1 \cdots dt_{n-1},$$

where the summation is over the (finitely many) non-overlapping charts (U_i, φ_i) of ∂M. The integration variables $t_1, \ldots, t_{n-1}$ are coordinates of the chart $\varphi : I^{n-1} \mapsto U_i$ under consideration. They must be chosen consistently with the boundary orientation, cf. [5, 12.2]. Now, f_i is understood as a function of the t_j via $f_i(\varphi(t))$.

4. If $n = 1$, then ∂M is zero dimensional, thus a (discrete) set. An orientation on a point is a sign $\sigma \in \{+, -\}$, and the oriented boundary of $M = ([a, b], \rightarrow)$ (where $a < b$) is $\partial M = \{(a, -), (b, +)\}$. Since 'integration' of a function f over an oriented set S is defined by $\int_S f = \sum_{(x,\sigma)\in S} \sigma f(x)$, we have $\int_{\partial M} f = f(b) - f(a)$. The generalized Gauss theorem thus reduces to the fundamental theorem of analysis.

4. In order to include cases like M being an interval $I = \prod_{i=1}^n [a_i, b_i]$, we allow ∂M to be only piecewise C^1. This means that ∂M is the union of finitely many components N_i such that each N_i is a C^1 surface and $N_i \cap N_j, j \neq i$ has measure zero as a subset of N_i, i.e. w.r.t. the coordinates $t_1, \ldots, t_{n-1}$ of N_i.

Proof of Theorem 1. We first consider the case where M is an interval $I = \prod_{i=1}^n [a_i, b_i]$. Then for the l.h.s. of (1) we have:

$$\begin{aligned} \int_M \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \right) dx &= \sum_{i=1}^n \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_n \\ &= \sum_{i=1}^n \int_{a_1}^{b_1} \cdots \widehat{\int_{a_i}^{b_i}} \cdots \int_{a_n}^{b_n} F_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) dx_1 \cdots \widehat{dx_i} \cdots dx_n, \qquad (2) \end{aligned}$$

where

$$F_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f_i(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n) - f_i(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n))$$

and $\widehat{\int_{a_i}^{b_i} \cdots dx_i}$ means that integration over x_i is omitted. We have used Fubini's theorem (i.e., the integral can be computed one variable after the other, in arbitrary order) and the fundamental theorem of analysis. With

$$A_i = \{x \in I \mid x_i = a_i\}, \quad B_i = \{x \in I \mid x_i = b_i\}$$

the r.h.s. of (2) can be written as

$$\sum_{i=1}^{n} \left(\widetilde{\int_{B_i}} f_i(x)dx - \widetilde{\int_{A_i}} f_i(x)dx \right), \tag{3}$$

where the $\tilde{}$ means that we use the natural positive volume elements $dx_1 \cdots \widehat{dx_i} \cdots dx_n$ on A_i and B_i. Note that $\partial I = \cup_{i=1}^{n}(A_i \cup B_i)$.

On the other hand, developing the determinant on the right hand side of (1) along the first column, we obtain

$$\int_{\partial M} \sum_{i=1}^{n} (-1)^{i+1} f_i \det D_i,$$

where D_i is the $(n-1) \times (n-1)$-matrix $(\frac{\partial \varphi_r}{\partial t_s})_{1 \le r \le n, 1 \le s \le n-1, r \ne i}$. We coordinatize the faces A_i and B_i of I by $x_1 = t_1, \ldots, x_{i-1} = t_{i-1}, x_{i+1} = t_i, \ldots, x_n = t_{n-1}$, together with $x_i = a_i$ or $x_i = b_i$, respectively. Now it is easy to see that D_i is the unit matrix on A_i and B_i, giving $\det D_i = 1$. For $j \ne i$, $\det D_i$ is zero on A_j and B_j. Thus the r.h.s. of (1) equals

$$\sum_{i=1}^{n} (-1)^{i+1} \left(\int_{A_i} f_i(x) + \int_{B_i} f_i(x) \right), \tag{4}$$

where $A_i \subset \partial M$ and $B_i \subset \partial M$ have the canonical boundary orientation. Now, the standard orientation on $\mathbb{R}^n$ is $[(e_1, \ldots, e_n)]$. Bringing the vector e_i, which is orthogonal to A_i and B_i, to the first position requires $i-1$ exchanges of vectors. Thus

$$[(e_i, e_1, \ldots, \widehat{e_i}, \ldots, e_n)] = (-1)^{i-1}[(e_1, \ldots, e_n)].$$

Now, e_i points in the direction of increasing x_i, thus outside of I on B_i and inside I on A_i. Thus, positive orientation on B_i is defined by $(-1)^{i-1}[(e_1, \ldots, \widehat{e_i}, \ldots, e_n)]$ and on A_i by $(-1)^i[(e_1, \ldots, \widehat{e_i}, \ldots, e_n)]$. Thus

$$\int_{A_i} \cdots = (-1)^i \widetilde{\int_{A_i}} \cdots, \qquad \int_{B_i} \cdots = (-1)^{i-1} \widetilde{\int_{B_i}} \cdots,$$

implying that (4) equals (3) and proving the theorem in the case $M = I$.

Turning to the general case, we chose an interval $I = \prod_{i=1}^{n}[a_i, b_i]$ containing M and define $\int_M \cdots$ as $\int_I \chi_M \cdots$. This integral exists since $\sum_i \frac{\partial f_i}{\partial x_i}$ is continuous and ∂M has measure zero. Let P be a partition of I. (This means that we consider a partition P_i of $[a_i, b_i]$ for each $i = 1, \ldots, n$ and then let I_k run through the products of n intervals, one from each P_i.) Then

$$\int_M \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right) dx = \int_I \chi_M(x) \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right) dx = \sum_j \int_{I_j} \chi_M(x) \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right) dx.$$

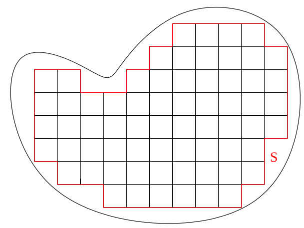
Since ∂M has measure zero, the total volume of those intervals I_j that intersect ∂M tends to zero as $\lambda(P) \to 0$. Together with the fact that the integrand is bounded, this implies that the contribution of these intervals to the sum tends to zero:

$$\int_M \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right) dx = \lim_{\substack{P=\{I_j\} \\ \lambda(P) \to 0}} \sum_{I_j \subset M} \int_{I_j} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right) dx.$$

Since we now integrate over intervals I_j in the interior of M, the integrand is continuous and we can apply the version of this theorem for intervals as proven above:

$$\int_M \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) \right) dx = \lim_{\substack{P=\{I_j\} \\ \lambda(P)\to 0}} \sum_{I_j \subset M} \int_{\partial I_j} \det \begin{pmatrix} f_1 & \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \end{pmatrix}.$$

Now recall that the boundary of each I_j consists of $2n$ faces. Each of these faces belongs to at most one other interval I_k, whose intersection with I_j then is exactly that face. The following picture, in which only the intervals $I_j \subset M$ are drawn, should illustrate this:



Now, a face that belongs to two adjacent intervals I_i and I_j contributes to $\int_{\partial I_i}$ and $\int_{\partial I_j}$ with opposite signs, since it has opposite orientations in ∂I_i and ∂I_j. Thus only the faces that appear only once contribute, and we denote their union by $S(P)$. In fact:

$$S(P) = \partial \left(\bigcup_{I_j \subset M} I_j \right).$$

$S(P)$ is a piecewise C^1 $(n-1)$-dimensional surface, and constitutes an approximation of ∂M, as is clear from the figure. We summarize:

$$\int_M \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) \right) dx = \lim_{\substack{P=\{I_j\} \\ \lambda(P)\to 0}} \int_{S(P)} \det \begin{pmatrix} f_1 & \frac{\partial \varphi_1}{\partial t_1} & \cdots & \frac{\partial \varphi_1}{\partial t_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & \frac{\partial \varphi_n}{\partial t_1} & \cdots & \frac{\partial \varphi_n}{\partial t_{n-1}} \end{pmatrix}. \tag{5}$$

Thus, the proof of the theorem is complete if we can show that the r.h.s. of (5) equals the r.h.s. of (1). This is certainly plausible, since by assumption ∂M is piecewise C^1, leading us to expect that in some sense

$$\text{“} \lim_{\substack{P=\{I_j\} \\ \lambda(P)\to 0}} S(P) = \partial M\text{”}.$$

Unfortunately, turning this into a rigorous proof is somewhat tedious, and we will only give a sketch. We begin by observing that the determinant in (5) can be written as a scalar product $\vec{f} \cdot \vec{n}^S = \sum_{i=1}^N f_i \, n_i^S$, where

$$n_i^S = (-1)^{i-1} \det D_i, \quad \text{where} \quad D_i = \left(\frac{\partial \varphi_r}{\partial t_s} \right)_{1 \le r \le n,\, 1 \le s \le n-1,\, r \ne i}.$$

(We have met the $(n-1) \times (n-1)$-matrices D_i before.) We can thus write the r.h.s. of (5) as

$$\lim_{\substack{P=\{I_j\} \\ \lambda(P)\to 0}} \int_{S(P)} \vec{f}(\varphi(t)) \cdot \vec{n}^S(t) \, dt_1 \ldots dt_{n-1},$$

and similarly with the r.h.s. of (1). Now, the vectors $\partial\vec{x}/\partial t_i,\ i = 1,\dots,n-1$ are a basis of $T_x S(P)$, thus the determinant in (5) is zero if $\vec{f}$ is in $T_x S(P)$, implying that $\vec{n}^S$ is orthogonal to the surface $S(P)$. Furthermore, $\|\vec{n}^S(t)\| dt_1 \cdots dt_{n-1}$ is the surface element. Thus, $\vec{n}^S(t)$ is constant on the faces of which $S(P)$ consists and jumps on the edges of $S(P)$, and therefore it has more and more jump discontinuities as $\lambda(P) \to 0$. But if $x \in \partial M$ is a point where ∂M is C^1, ∂M can be approximated by an $(n-1)$-dimensional plane near x, implying $\lim_{\lambda(P)\to 0} d(x, S(P)) = 0$. Then, as $\lambda(P) \to 0$, the average of $\vec{n}^S(t)$ over a part of $S(P)$ close to x will converge to $\vec{n}^{\partial M}(x)$ (which exists since ∂M is C^1 at x.) Together with the uniform continuity of $\vec{f}$ on compact sets, this implies

$$\lim_{\substack{P=\{I_j\}\\ \lambda(P)\to 0}} \int_{S(P)} \vec{f}(\varphi(t)) \cdot \vec{n}^S(t)\, dt_1 \dots dt_{n-1} = \int_{\partial M} \vec{f}(\varphi(t)) \cdot \vec{n}^{\partial M}(t) dt_1 \dots dt_{n-1},$$

and therefore the theorem. (Note that the maps φ on the left and right hand sides are not the same, since they parametrize (pieces of) $S(P)$ and ∂M, respectively.) ■

Remark 3 1. In the course of the proof we have also seen that the r.h.s. of (1) is just the flux $\int_{\partial M} \vec{f}(t) \cdot d\vec{n}(t)$ of $\vec{f}$ through the surface ∂M. This makes the formal similarity of Theorem 1 with Gauss' theorem even more complete. Therefore we will call Theorem 1 the *Generalized Gauss Theorem*. As we have seen, it contains the classical theorems of Gauss and Green as special cases. However, it does not contain the classical theorem of Stokes, since there we have $M \subset \mathbb{R}^n$ with $k = \dim M = 2 < 3 = n$. We will soon discuss the *Generalized Stokes Theorem*, which contains both Theorem 1 and the classical theorem of Stokes as special cases. We begin by using Theorem 1 to motivate the necessary definitions.

2. Our proof of the general case of the theorem was less than elegant. As it turns out, this was due to our insistence on using elementary methods (and in particular the avoidance of partitions of unity). In fact, the generalized Stokes theorem below admits very elegant proofs.

References

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