Syllabus on Homology Theory

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1 Preliminaries

1.1 Direct products and direct sums of groups, Free abelian groups

1.1 Definition *Let $\{G_\alpha,\ \alpha \in S\}$ be a family of groups. Then the (direct) product $G = \times_{\alpha\in S} G_\alpha$ is defined as the product of the sets $G_\alpha, \alpha \in S$. (Thus an element $g \in G$ is a map, which to every $\alpha \in S$ assigns $g_\alpha \in G_\alpha$.) We endow G with componentwise multiplication ($(gh)_\alpha = g_\alpha h_\alpha$) and unit ($e_\alpha = e_{G_\alpha}$).*

1.2 Definition *Let $\{G_\alpha,\ \alpha \in S\}$ be a family of groups. Then the restricted direct product $\times^{res}_{\alpha\in S} G_\alpha$is the subset of $\times_{\alpha\in S} G_\alpha$ of those $(g_\alpha)_{\alpha\in S}$, which satisfy $g_\alpha = e_{G_\alpha}$ for all but finitely many $\alpha \in S$. It is obvious that this is a subgroup of $\times_{\alpha\in S} G_\alpha$.*

It is clear that $\times_{\alpha\in S} G_\alpha$ and $\times^{res}_{\alpha\in S} G_\alpha$ are abelian if and only if every G_α is abelian. The restricted direct product of a family of *abelian* groups is usually called the direct sum $\oplus_{\alpha\in S} G_\alpha$.

1.3 Definition *For a set S, the free abelian group $F(S)$ generated by S is defined by*

$$F(S) = \oplus_{\alpha\in S}\mathbb{Z}.$$

Thus an element $g \in F(S)$ is a map $S \to \mathbb{Z}$ such that $g_\alpha = 0$ for all but finitely many $\alpha \in S$. Again, the unit (zero) and the group operation are defined componentwise.

To every $\alpha \in S$ we assign the map $\iota_\alpha : S \to \mathbb{Z}$ as follows:

$$\iota_\alpha(\beta) = \begin{cases} 1 & \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\iota_\alpha \in F(S)$. The map $S \to F(S),\ \alpha \mapsto \iota_\alpha$ is injective, thus defines an embedding $\iota_S : S \hookrightarrow F(S)$. Thus every element $g = (g_\alpha) \in F(S)$ can be written as

$$g = \sum_{\alpha\in S} g_\alpha\, \iota_S(\alpha).$$

Here, $\iota_S(\alpha) = \delta_\alpha \in F(S)$, and $g_\alpha\, \iota_S(\alpha)$ is given by the action of $\mathbb{Z}$ on $F(S)$ (as on any abelian group). Finally, the sum over α makes sense since $g_\alpha \neq 0$ for finitely many $\alpha \in S$. We usually suppress ι_S, writing

$$g = \sum_{\alpha\in S} g_\alpha\, \alpha.$$

1.4 Lemma *Every map $f : S \to T$ induces a homomorphism $\hat{f} : F(S) \to F(T)$ of abelian groups, which can be defined as follows:*

$$\hat{f} : \sum_{\alpha \in S} g_\alpha \iota_S(\alpha) \mapsto \sum_{\alpha \in S} g_\alpha \iota_T(f(\alpha)).$$

Usually we will write f instead of $\hat{f}$.

Proof. This is well defined precisely because the $\iota_S(\alpha)$ are a set of free generators of $F(S)$. The above notation is very convenient and will be used below, but the following may be helpful: An element $g \in F(S)$ is, by definition, a map $S \to \mathbb{Z}$, $\alpha \mapsto g_\alpha$ where $g_\alpha = 0$ for all but finitely many $\alpha \in S$. Now $\hat{f}(g) \in F(T)$ is the map $T \to \mathbb{Z}$ given by

$$\beta \mapsto \sum_{\substack{\alpha \in S \\ f(\alpha)=\beta}} g_\alpha.$$

This defines an element of $F(T)$ since (1) the g_α are elements of $\mathbb{Z}$, thus can be added, (2) the sum over α is finite since only finitely many g_α are non-zero, and (3) the resulting map $T \to \mathbb{Z}$ is non-zero only for finitely many $\beta \in T$. ■

1.2 Standard Simplices

1.5 Definition *For $n \geq 0$ we define the standard n-simplex*

$$\Delta_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \ \ \forall i, \ \sum_i t_i = 1\}.$$

Is is easy to see that Δ_0 is a one-point set, $\Delta_1 \cong I$ etc.

1.6 Definition *We define inclusion maps $\delta_i^n : \Delta_{n-1} \to \Delta_n$ for $n \geq 1$ and $0 \leq i \leq n$ by*

$$\delta_i^n : (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Thus δ_i^n inserts a zero at the i-th position, counting from zero. (If $i = 0$ omit the sequence $t_0, \ldots, t_{i-1}$ on the r.h.s., if $i = n$ omit $t_i, \ldots, t_{n-1}$.)

For $n \geq 2$ and $i \geq j$ we compute

$$(t_0, \ldots, t_{n-2}) \xmapsto{\delta_i^{n-1}} (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-2}) \xmapsto{\delta_j^n} (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}, 0, t_i, \ldots t_{n-2})$$

and

$$(t_0, \ldots, t_{n-2}) \xmapsto{\delta_j^{n-1}} (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-2}) \xmapsto{\delta_{i+1}^n} (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}, 0, t_i, \ldots t_{n-2}).$$

(If $i = j$ then the sequence $t_j, \ldots, t_{i-1}$ disappears.) We have proven

1.7 Lemma *For $n \geq 2$ and $i \geq j$ we have the identity*

$$\delta_j^n \circ \delta_i^{n-1} = \delta_{i+1}^n \circ \delta_j^{n-1}$$

of maps from Δ_{n-2} to Δ_n.

2 Singular Homology

2.1 Definition of $H_n(X)$

2.1 Definition *Let X be a space. Then $S_n(X), n \geq 0$ is the set of continuous maps $\Delta_n \to X$ and $C_n(X) = F(S_n(X))$ is the free abelian group generated by $S_n(X)$.*

2.2 Remark The elements $\sigma : \Delta_n \to X$ of $S_n(X)$ are called 'singular n-simplices in X'. The motivation is that σ must be continuous, but it is not required to be a homeomorphism between Δ_n and $\sigma(\Delta_n)$. The image $\sigma(\Delta_n)$ can be self-intersecting or even a one-point set. $\square$

2.3 Definition *For $n \geq 1$ we define a homomorphism $d_n : C_n(X) \to C_{n-1}(X)$ by*

$$d_n : \sum_{\sigma \in S_n(X)} a_\sigma\, \sigma \mapsto \sum_{\sigma \in S_n(X)} a_\sigma \sum_{i=0}^{n} (-1)^i\, \sigma \circ \delta_i^n.$$

Note that $\sigma \circ \delta_i^n \in S_{n-1}(X)$.

2.4 Remark Equivalently, the element $a \in C_n(X)$ (which is a map $S_n(X) \to \mathbb{Z}$) is mapped to

$$d_n(a) : S_{n-1} \to \mathbb{Z}, \quad \sigma \mapsto \sum_{i=0}^{n} (-1)^i \sum_{\substack{\sigma' \in S_n(X) \\ \delta_i^n(\sigma) = \sigma'}} a_{\sigma'}.$$

$\square$

Thus we have a semi-infinite chain of abelian groups and homomorphisms:

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \;\ldots\; C_1(X) \xrightarrow{d_1} C_0(X). \tag{2.1}$$

2.5 Lemma *For every $n \geq 2$ we have $d_{n-1} \circ d_n : C_n(X) \to C_{n-2}$ is the zero homomorphism.*

Proof. Let $\eta \in C_n(X)$. Then

$$\begin{aligned}
d_{n-1} \circ d_n(\eta) &= d_{n-1} \sum_{j=0}^{n} (-1)^j\, \eta \circ \delta_j^n \\
&= \sum_{i=0}^{n-1} (-1)^i \sum_{j=0}^{n} (-1)^j\, \eta \circ \delta_j^n \circ \delta_i^{n-1} \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} + \sum_{0 \leq j \leq i < n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} + \sum_{0 \leq j \leq i < n} (-1)^{i+j}\, \eta \circ \delta_{i+1}^n \circ \delta_j^{n-1} \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} + \sum_{0 \leq j < i \leq n} (-1)^{i+j+1}\, \eta \circ \delta_i^n \circ \delta_j^{n-1} \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} - \sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \eta \circ \delta_j^n \circ \delta_i^{n-1} \\
&= 0.
\end{aligned}$$

Explanation: To obtain the third equality we have split the summation over i, j into two sums: one where $i < j$ and one where $j \leq i$. (One has to pay attention to the upper summation bounds!) The next manipulations concern only the $j \leq i$-term: The fourth equality follows by application of Lemma 1.7. The fifth equality is obtained by replacing $i+1$ by i and correcting the summation bounds. The sixth follows by interchanging i and j. Now the second term is the negative of the first, thus we are done. ∎

We have thus shown that the composition of any two adjacent maps d_n, d_{n-1} in the diagram (2.1) is the zero map. The general situation is the following

2.6 Definition *Let $A_n, n \geq 0$ be abelian groups and $d_n : A_n \to A_{n-1}, n \geq 1$ homomorphisms:*

$$\cdots \longrightarrow A_{n+1}(X) \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \quad \cdots \quad A_1 \xrightarrow{d_1} A_0$$

If $d_{n-1} \circ d_n = 0$ for all $n \geq 2$ then (A_n, d_n) is called a non-negative chain complex, or simply a complex.

With

$$\begin{aligned} Z_n &= \ker d_n &\subset A_n, \\ B_n &= \operatorname{im} d_{n+1} &\subset A_n \end{aligned}$$

we have an inclusion $B_n \subset Z_n \subset A_n$ of abelian groups for all $n \geq 1$.

2.7 Definition *The homology groups $H_n(A)$ of the complex A are defined by*

$$\begin{aligned} H_n(A) &= Z_n/B_n, \quad n \geq 1, \\ H_0(A) &= A_n/B_n. \end{aligned}$$

(If we declare $d_0 : A_n \to 0$ to be the zero homomorphism then we also have $H_0(A) = Z_0/B_0$.)

Having shown that $(C_n(X), d_n)$ is a complex we can define:

2.8 Definition *Let X be a space. Then the homology groups $H_n(X)$ are defined as the homology groups of the complex $(C_n(X), d_n)$.*

2.2 Some First Computations

2.9 Lemma *Let $X = \{x\}$ be a one-point space. Then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0, n \geq 1$.*

Proof. For every $n \geq 0$ there exists precisely one singular n-simplex: the constant map σ_n from Δ_n to $x \in X$. Thus all $C_n(X) \cong \mathbb{Z}$. Clearly, $\sigma_n \circ \delta_i^n = \sigma_{n-1}$ for all i. The map $d_n : C_n(X) \to C_{n-1}(X)$ maps the generator σ_n of $C_n(X)$ to

$$\sum_{i=0}^{n} (-i)^i \sigma_n \circ \delta_i^n = \sum_{i=0}^{n} (-i)^i \sigma_{n-1} \ \in C_{n-1}(X).$$

We thus have

$$d_n = \begin{cases} 0 & n \text{ odd} \\ \mathrm{id}_{\mathbb{Z}} & n \text{ even} \end{cases}$$

Now, the complex

$$\cdots C_3 = \mathbb{Z} \xrightarrow{0} C_2 = \mathbb{Z} \xrightarrow{\mathrm{id}} C_1 = \mathbb{Z} \xrightarrow{0} C_0 = \mathbb{Z}$$

has

$$\begin{aligned} n \text{ odd}: &\quad Z_n = B_n = \mathbb{Z}, \\ n \text{ even}, n > 0: &\quad Z_n = B_n = 0 \end{aligned}$$

and thus $H_n = 0 \ \forall n \geq 1$. Since $A_0 = \mathbb{Z}, B_0 = 0$ we have $H_0 = \mathbb{Z}$. ∎

2.10 Remark This lemma amounts to the statement that singular homology theory satisfies the 'dimension axiom'. The latter requires $H_n(\{x\}) = 0\ \forall n \geq 1$. □

2.11 Proposition *If X is non-empty and path-connected then $H_0(X) \cong \mathbb{Z}$.*

Proof. Since $\Delta_0 = \{1\}$, we may and will identify the set $S_0(X)$ of singular 0-simplices with X. We define a map $\tilde{\varepsilon} : C_0(X) \to \mathbb{Z}$ by

$$\tilde{\varepsilon} : \sum_{x \in X} a_x\, x \ \mapsto \ \sum_{x \in X} a_x.$$

Since X is non-empty, $\tilde{\varepsilon}$ is surjective. Let $a \in C_1(X)$, thus $a = \sum_{\sigma \in S_1(X)} a_\sigma \sigma$. Then

$$d_1\, a = \sum_{\sigma \in S_1(X)} a_\sigma(\sigma(0,1) - \sigma(1,0)) \ \in C_0(X).$$

Obviously, $\tilde{\varepsilon} \circ d_1 = 0$. Thus there is a unique map $\varepsilon : H_0(X) \to \mathbb{Z}$ such that

$$\begin{tikzcd} C_0(X) \arrow[r] \arrow[dr, "\tilde{\varepsilon}"'] & H_0(X) \arrow[d, "\varepsilon"] & = C_0(X)/\operatorname{im} d_1 \\ & \mathbb{Z} & \end{tikzcd}$$

commutes. Again, ε is surjective, and in order to establish the isomorphism $H_0(X) \cong \mathbb{Z}$ it remains to prove that ε is injective. This is the case iff $\ker \tilde{\varepsilon} \subset C_0(X)$ coincides with $\operatorname{im} d_1$. We already know $\operatorname{im} d_1 \subset \ker \tilde{\varepsilon}$, thus it remains to prove $\ker \tilde{\varepsilon} \subset \operatorname{im} d_1$.

To this purpose fix some point $x_0 \in X$. Since X is path connected we may choose, for every $x \in X$, a path $l_x : I \to X$ from x_0 to x. (Thus $l_x(0) = x_0$ and $l_x(1) = x$.) Let λ_x be the associated singular 1-simplex: $\lambda_x(t_0, t_1) = l_x(t_1)$.

Let now $\sigma = \sum_{x \in X} a_x\, x \ \in C_0(X)$ and assume $\sigma \in \ker \tilde{\varepsilon}$. The latter just means $\sum_{x \in X} a_x = 0$. Put $\sigma' = \sum_{x \in X} a_x\, \lambda_x \ \in C_1(X)$. We compute

$$\begin{aligned} d_1\, \sigma' &= \sum_{x \in X} a_x\, (\lambda_x(0,1) - \lambda_x(1,0)) \\ &= \sum_{x \in X} a_x\, (l_x(1) - l_x(0)) \\ &= \sum_{x \in X} a_x\, (x - x_0) \\ &= \sum_{x \in X} a_x x \ - \ (\sum_{x \in X} a_x)\, x_0 \\ &= \sum_{x \in X} a_x x \\ &= \sigma. \end{aligned}$$

(In the third-last equality we have used $\sum_{x \in X} a_x = 0$.) Thus $\ker \tilde{\varepsilon} \subset \operatorname{im} d_1$ as desired. ■

The next result will follow from the functoriality properties of H_n, proved in the next section.

2.12 Proposition *$H_0(X)$ is the free abelian group generated by $\pi_0(X)$, the set of path-components.*

In the remainder of this subsection we state without proof two further relations between homology and homotopy groups. (The first of them will be one of the homework problems!)

2.13 PROPOSITION *Let X be path-connected and $x \in X$. Then there exists an isomorphism $\phi : \pi_1(X,x)_{ab} \to H_1(X)$.*

Here, $\pi_1(X,x)_{ab}$ is the *abelianization* of $\pi_1(X,x)$. For any group G, the abelianization G_{ab} is defined as the quotient group $G/[G,G]$, where $[G,G] \subset G$ is the smallest normal subgroup containing $[g,h] := ghg^{-1}h^{-1}$ for all $g,h \in G$. One easily proves the following facts:

1. G_{ab} is abelian.
2. Let $\alpha : G \to H$ be a homomorphism, where H is abelian. Then there exists a unique homomorphism $\alpha' : G_{ab} \to H$ such that $\alpha = \alpha' \circ \varphi_G$, where $\varphi_G : G \to G_{ab} = G/[G,G]$ is the quotient map.
3. If $\alpha : G \to H$ is a homomorphism, there exists a unique homomorphism $\alpha_{ab} : G_{ab} \to H_{ab}$ such that $\alpha_{ab} \circ \varphi_G = \varphi_H \circ \alpha$.
4. $F(G) = G_{ab}$, $F(\alpha) = \alpha_{ab}$ defines a functor F from the category of groups to the category of abelian groups.

Note: It can happen that $H_1(X) = 0$ even though $\pi_1(X)$ is non-trivial. This is the case precisely when $G = \pi_1(X)$ satisfies $[G,G] = G$. Such groups are called perfect. For example, every non-abelian simple group like $A_n, n \geq 5$ is perfect.

2.14 THEOREM *Let X be simply connected (i.e. path-connected and $\pi_1(X) = 0$). Then*

$$\min\{n \in N \mid H_n(X) \neq 0\} = \min\{n \in N \mid \pi_n(X,x) \neq 0\}.$$

(Here the choice of $x \in X$ does not matter since X is path-connected.) If this expression is finite, we denote it by m and there exists an isomorphism $\pi_m(X,x) \cong H_m(X)$.

Thus for a simply connected space X the first non-vanishing homology and homotopy groups appear for the same $m \in \mathbb{N}$ and they coincide. However, there is no simple relation between the homology and homotopy groups for $m' > m$.

2.3 Functoriality, Additivity and Homotopy Invariance

2.15 DEFINITION *Let (C_n, d_n) and (C'_n, d'_n) be chain complexes. A family of homomorphisms $f_n : C_n \to C'_n$, $n \geq 0$ is called a chain map iff the diagram*

$$\begin{tikzcd} C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[d, "f_n"] \\ C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \end{tikzcd}$$

commutes for all $n \geq 0$.

2.16 LEMMA *A chain map $(f_n) : (C_n, d_n) \to (C'_n, d'_n)$ induces homomorphisms $H_n(f) : H_n \to H'_n$, for all $n \geq 0$, such that $H_n(f)([a]) = [f_n(a)]$ where $a \in Z_n$ is a representer of the class $[a] \in H_n$.*

Usually we will write f_* rather than $H_n(f)$.

Proof. Let $(f_n) : (C_n, d_n) \to (C'_n, d'_n)$ be a chain map. Let $a \in Z_n$, thus $d_n(a) = 0$. Then $d'_n(f_n(a)) = f_{n-1}(d_n(a)) = f_{n-1}(0) = 0$, thus $f_n(a) \in Z'_n$. Let $a \in B_n$, thus there exists $a' \in C_{n+1}$ such that $a = d_{n+1}(a')$. Thus $f_n(a) = f_n(d_{n+1}(a')) = d_{n+1}(f_{n+1}(a'))$, thus $f_n(a) \in B'_{n+1}$.

If $a \in Z_n$ then $f_n(a) \in Z'_n$. Let $[a]$ be the image of a in $H_n = Z_n/B_n$. We define $H_n(f)([a]) = [f_n(a)]$. This is well defined since changing a by an element $a' \in B_n(A)$ changes $f_n(a)$ by an element of B'_n, which does not affect $[f_n(a)] \in H'_n$. ■

We now apply this to singular homology:

2.17 Proposition *A map $f : X \to Y$ gives rise to a chain map $(f_n) : (C_n(X), d_n) \to (C_n(Y), d_n)$ and thus to induced homomorphisms $H_n(f) : H_n(X) \to H_n(Y)$. We have $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and $H_n(id_X) = id_{H_n(X)}$, thus $H_n, n \geq 0$ is a functor* Top $\to$ Ab.

Usually we will write f_* rather than $H_n(f)$.

Proof. If $\sigma \in S_n(X)$ is a singular n-simplex in X, thus $\sigma : \Delta_n \to X$, and $f : X \to Y$ then $f \circ \sigma : \Delta_n \to Y$ is an element of $S_n(Y)$. Thus f gives rise to maps $f_* : S_n(X) \to S_n(Y)$ for all $n \geq 0$. By Lemma 1.4 this induces homomorphisms $f_* : C_n(X) \to C_n(Y)$ for all $n \geq 0$. Let now $a = \sum_{\sigma \in S_n(X)} a_\sigma \sigma \in C_n(X)$. Then

$$\begin{aligned} f_*(d_n(\sigma)) &= f_*\left(\sum_{\sigma \in S_n(X)} a_\sigma \sum_{i=0}^{n} (-1)^i \sigma \circ \delta_i^n\right) \\ &= \sum_{\sigma \in S_n(X)} a_\sigma \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \delta_i^n, \end{aligned}$$

whereas

$$\begin{aligned} d_n(f_*(\sigma)) &= d_n\left(\sum_{\sigma \in S_n(X)} a_\sigma \ f \circ \sigma\right) \\ &= \sum_{\sigma \in S_n(X)} a_\sigma \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \delta_i^n. \end{aligned}$$

Thus the family of homomorphisms $C_n(X) \to C_n(Y)$ is a chain map, and we can apply Lemma 2.16.

The proof of functoriality is obvious and therefore omitted. ■

For spaces that are not path-connected we can now state:

2.18 Proposition *Let X be a space and X_α, $\alpha \in \pi_0(X)$ its path-components. Then the inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X$ induce isomorphisms*

$$H_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha) \quad \forall n \geq 0.$$

Proof. Let $\sigma \in S_n(X)$ be a singular n-simplex in X. Since $\sigma : \Delta_n \to X$ is continuous, the image $\sigma(\Delta_n)$ is contained in one path-component X_α of X. Thus

$$S_n(X) \cong \coprod_{\alpha \in \pi_0(X)} S_n(X_\alpha), \qquad C_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} C_n(X_\alpha),$$

w.r.t. the obvious maps $C_n(X_\alpha) \to C_n(X)$. Now, if $\sigma \in S_n(X)$ maps into X_α then the same is true for all $\sigma \circ \delta_i^n \in S_{n-1}(X), 0 \leq i \leq n$. Thus d_n maps $C_n(X_\alpha)$ into $C_{n-1}(X_\alpha)$. We conclude that

$$H_n(X) = Z_n(X)/B_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} Z_n(X_\alpha)/B_n(X_\alpha) = \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha).$$

$\blacksquare$

The following property of singular homology is called the 'additivity axiom':

2.19 Corollary *Let $X = \coprod_{i \in S} X_i$ be a disjoint union of spaces. Then the inclusions $\iota_i : X_i \hookrightarrow X$ induce isomorphisms*

$$H_n(X) \cong \bigoplus_{i \in S} H_n(X_i) \quad \forall n \geq 0.$$

Proof. This is an immediate consequence of the preceding result, since every path-component X_α of X is contained in one of the subspaces X_i. $\blacksquare$

Since all functors in algebraic topology are supposed to be homotopy invariant, the following is very important:

2.20 Theorem *Let $f, g : X \to Y$ be homotopic. Then the induced maps on all homology groups coincide: $f_* = g_* : H_n(X) \to H_n(Y)$ $\forall n \geq 0$.*

Recall that the proof of homotopy invariance of the functors $\pi_1 : Top_* \to Grp$ and $\pi_n : Top_* \to Ab$, $n \geq 2$ was immediate since π_n was defined in terms of homotopy classes. For homology, the proof is considerable more involved, since the definition of the homology functors H_n does *not* appeal to homotopy classes. We don't have the time to prove this important result. You can find it, e.g., in [1, Sect. IV.16] or [2, Sect. 2.1].

We have seen that the one-point space has the homology groups $H_0(\{x\}) = \mathbb{Z}$ and $H_n(\{x\}) = 0$, $n \geq 1$. Then the preceding theorem implies the following, which we will soon prove directly.

2.21 Proposition *If X is contractible then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$, $n \geq 1$.*

As an algebraic preliminary we need the following (which is also used in the proof of Theorem 2.20).

2.22 Definition *Let $(C_n, d_n), (C'_n, d'_n)$ be chain complexes and $(f_n : C_n \to C'_n)$, $(g_n : C_n \to C'_n)$ chain maps. f and g are chain homotopic, denoted $f \simeq g$, if there exist maps $D_n : C_n \to C'_{n+1}$, $n \geq 0$*

$$\begin{tikzcd}
\cdots & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "{f_{n+1}, g_{n+1}}"'] & C_n \arrow[r, "d_n"] \arrow[d, "{f_n, g_n}"] \arrow[dl, "D_n"'] & C_{n-1} \arrow[d, "{f_{n-1}, g_{n-1}}"] \arrow[dl, "D_{n-1}"'] & \cdots \\
\cdots & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} & \cdots
\end{tikzcd}$$

such that

$$f_n - g_n = d'_{n+1} \circ D_n + D_n \circ d_n \;:\; C_n \to C'_n \qquad \forall n \geq 0.$$

(Here $d_0 = 0$.) The family (D_n) is called a chain homotopy.

2.23 Lemma *Let (D_n) be a chain homotopy between chain maps $(f_n), (g_n)$ from (C_n, d_n) to $(C'_n \to d'_n)$. Then $H_n(f) = H_n(g)$ for all $n \geq 0$, or just $f_* = g_*$.*

Proof. Let $a \in Z_n$ be a representer of the class $[a] \in H_n = Z_n/B_n$. Then

$$f_n(a) - g_n(a) = d'_{n+1} \circ D_n(a) + D_n \circ d_n(a) = d'_{n+1} \circ D_n(a).$$

Here the first equality holds by existence of the chain homotopy D and the second since $a \in Z_n = \ker d_n$. We conclude $f_*(a) - g_*(a) \in B'_n$, and therefore $(f_* - g_*)([a]) = 0$ in $H'_n = Z'_n/B'_n$. $\blacksquare$

2.24 Remark The part of algebra dealing with complexes and their homology, chain maps, chain homotopies etc. is called *homological algebra*. It has many applications, not only in algebraic topology, but also in number theory, algebraic geometry and algebra itself! □

Proof of Proposition 2.21. Contractibility is equivalent to the existence of a map $F : X \times I \to X$ such that $F(x,0) = x, F(x,1) = x_0$ for all $x \in X$, where x_0 is some fixed point of X. In particular, X is path-connected, thus $H_0(X) = \mathbb{Z}$ by Proposition 2.11.

For $\sigma \in S_n(X)$ we define $\tilde{\sigma} \in S_{n+1}(X)$ as follows:

$$\tilde{\sigma}(t_0, \ldots, t_{n+1}) = F(\sigma(t_1/\lambda, \ldots, t_n/\lambda), t_0), \quad \lambda = 1 - t_0.$$

As long as $t_0 \neq 1$, we have $(t_1/\lambda, \ldots, t_n/\lambda) \in \Delta_n$ and $\tilde{\sigma}$ is clearly a continuous map from Δ_{n+1} to X. As $t = (t_0, \ldots, t_{n+1})$ tends to $(1, 0, \ldots, 0)$, $F(\sigma(t_1/\lambda, \ldots, t_n/\lambda), t_0)$ converges to x_0, thus $\tilde{\sigma} : \Delta_{n+1} \to X$ is continuous everywhere and thus is an element of $S_{n+1}(X)$. With Definition 1.6 it is trivial to compute $\tilde{\sigma} \circ \delta_i^{n+1} \in S_n(X)$:

$$\begin{aligned} \tilde{\sigma} \circ \delta_0^{n+1} &= \sigma, \\ \tilde{\sigma} \circ \delta_i^{n+1} &= \widetilde{\sigma \circ \delta_{i-1}^n}, \quad 1 \leq i \leq n. \end{aligned}$$

Now, $\sigma \mapsto \tilde{\sigma}$ defines a map $S_n(X) \to S_{n+1}(X)$ which, by Lemma 1.4, extends to a homomorphism $D_n : C_n(X) \to C_{n+1}(X)$. Let $\sigma \in S_n(X)$, considered as a basis element of $C_n(X)$. Using the preceding computatation we find, for $n \geq 1$,

$$\begin{aligned} d_{n+1} \circ D_n(\sigma) &= \sum_{i=0}^{n+1} (-1)^i \ \tilde{\sigma} \circ \delta_i^{n+1} = \sigma + \sum_{i=1}^{n+1} (-1)^i \ \widetilde{\sigma \circ \delta_{i-1}^n} = \sigma - \sum_{i=0}^{n} (-1)^i \ \widetilde{\sigma \circ \delta_i^n}, \\ D_{n-1} \circ d_n(\sigma) &= D_{n-1} \left(\sum_{i=0}^{n} (-1)^i \ \sigma \circ \delta_i^n \right) = \sum_{i=0}^{n} (-1)^i \ \widetilde{\sigma \circ \delta_i^n}, \end{aligned}$$

and thus

$$(d_{n+1} \circ D_n + D_{n-1} \circ d_n)(\sigma) = \sigma \quad \forall n \geq 1.$$

Since the $\sigma \in S_n(X)$ generate $C_n(X)$, we see by comparison with Definition 2.22 that $(D_n, n \geq 1)$ is a chain homotopy between the chain maps $f_n = \mathrm{id}$ and $g_n \equiv 0$ from $(C_n(X), d_n)$ to itself, restricted to $n \geq 1$. Now Lemma 2.23 implies that the identity map and the zero map on $H_n(X)$ coincide, thus $H_n(X) = 0$ for all $n \geq 1$. ■

3 Exact Sequences, short and long

3.1 Some Homological Algebra

3.1 Definition *Let (C_n, d_n) be a complex, possibly two-sided, i.e. $n \in \mathbb{Z}$:*

$$\cdots \quad C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \quad \cdots$$

Recall that $d_n \circ d_{n+1} = 0$ for all n, thus $\operatorname{im} d_{n+1} \subset \ker d_n$. The complex is called acyclic iff all these inclusions are equalities, thus $\operatorname{im} d_{n+1} = \ker d_n$ for all n.

Clearly, a complex C is acyclic iff all homology groups $H_n(C) = \ker d_n / \operatorname{im} d_{n+1}$ vanish, thus the latter are a measure of the complex' deviation from being exact. An acyclic complex is usually called an *exact sequence*.

An exact sequence does not need to be 'long', i.e. infinite. For example, exactness of the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is equivalent to $\ker g = \operatorname{im} f$. Consider the special cases

$$0 \longrightarrow A \xrightarrow{f} B,$$

$$C \xrightarrow{g} D \longrightarrow 0.$$

(The maps $0 \to A$ and $D \to 0$ are not labelled since there is only one possibility: the zero map.) Exactness of these diagrams amounts to $\ker f = \operatorname{im} 0 = 0$ and $\operatorname{im} g = \ker 0 = B$, respectively. In other words, f is injective and g is surjective. Putting this together, we consider the *short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \tag{3.1}$$

Again, $g : B \to C$ is surjective, implying $C \cong B/\ker g$. But $\ker g = \operatorname{im} f$, thus $C \cong B/f(A)$. Since f is injective, we have $f(A) \cong A$, thus the exact sequence just means that C is the quotient of B by a subgroup $f(A)$ isomorphic to A. (The language of exact sequences is often also applied to non-abelian groups, algebras or other structures.) It is clear that an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

can always be extended to one of the form (3.1) by putting $C = B/f(A)$ and taking $g : B \to C$ to be the quotient map.

3.2 Definition *A short exact sequence $0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$ of complexes consists of two chain maps of complexes $(f_n) : (C_n, d_n) \to (C'_n, d'_n)$, $(g_n) : (C'_n, d'_n) \to (C''_n, d''_n)$ (in the sense of Definition 2.15) such that*

$$0 \longrightarrow C_n \xrightarrow{f_n} C'_n \xrightarrow{g_n} C''_n \longrightarrow 0 \tag{3.2}$$

is exact for all n.

Thus a short exact sequence of complexes is a commutative diagram where the horizontals are complexes and the verticals are short exact sequences, cf. Figure 1.

3.3 Lemma *Let $0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$ be a short exact sequence of complexes. Then*

$$H_n \xrightarrow{f_*} H'_n \xrightarrow{g_*} H''_n$$

is exact (i.e. $\ker g_* = \operatorname{im} f_* \subset H'_n$) for all n.

Proof. As a consequence of $g_n \circ f_n = 0$ we have $g_* \circ f_* = (g \circ f)_* = 0$ for all n. Thus it remains to show that $\ker g_* \subset \operatorname{im} f_*$. Let thus $x \in H'_n = Z'_n/B'_n$ such that $f_*(x) = 0$. If $a \in Z'_n$ is a representer of x, then $g_*(x) = 0$ amounts to $g_n(a) \in B''_n = \operatorname{im} d''_{n+1}$. Thus there exists $b \in C''_{n+1}$ such that $d''_{n+1} = g_n(a)$. Since g_{n+1} is surjective, there exists $c \in C'_{n+1}$ such that $g_{n+1}(c) = b$. Now, $g_n(a) = d''_{n+1} \circ g_{n+1}(c) = g_n \circ d'_{n+1}(c)$, or equivalently $g_n(a - d'_{n+1}(c)) = 0$. Since $\ker g_n = \operatorname{im} f_n$, this means that there exists $d \in C_n$ such that $f_n(d) = a - d'_{n+1}(c)$. Thus $a = f_n(d) + d'_{n+1}(c)$. But in $H'_n = Z'_n/B'_n = Z'_n/\operatorname{im} d'_{n+1}$ we have $x = [a] = [f_n(d) + d'_{n+1}(c)] = [f_n(d)]$. Thus $x = f_*([d]) \in \operatorname{im} f_*$, as desired. ■

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] & \cdots \\
\cdots & C'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "g_{n+1}"'] & C'_n \arrow[r, "d'_n"] \arrow[d, "g_n"'] & C'_{n-1} \arrow[d, "g_{n-1}"] & \cdots \\
\cdots & C''_{n+1} \arrow[r, "d''_{n+1}"] \arrow[d] & C''_n \arrow[r, "d''_n"] \arrow[d] & C''_{n-1} \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 1: Short exact sequence of complexes

3.4 Remark Let $0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$ be a short exact sequence of complexes. Here the unlabeled maps can be considered as chain maps from the zero complex (where all $C_n = 0$) to f and from g to the zero complex, respectively. One might expect that the induced maps on the homology groups give rise to short exact sequences $0 \to H_n \xrightarrow{f_*} H'_n \xrightarrow{g_*} H''_n \to 0$ on the homology groups. We have just proven exactness at H'_n, i.e. $\ker g_* = \operatorname{im} f_* \subset H'_n$. In general, however, $f_* : H_n \to H'_n$ need not be injective: If $x \in H_n$ and $a \in Z_n$ is a representer, then $f_*(x) = [f_n(a)]$. Even if $x \neq 0$, equivalent to $a \notin B_n$, we cannot exclude $f_n(a) \in B'_n$, i.e. $f_*(x) = 0$. Similarly, if $x \in H''_n$ and $a \in Z_n$ is a representer, then surjectivity of g_n implies the existence of $b \in C'_n$ such that $g_n(b) = a$, but there is no guarantee that such b can be found in Z'_n, as is needed to conclude surjectivity of g_*. □

It thus remains to understand $\ker f_* \subset H_n$ and $\operatorname{im} g_* \subset H''_n$. This is achieved by the following

3.5 Theorem *Let $0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$ be a short exact sequence of complexes. Then there exist 'connecting' homomorphisms $\partial_n : H''_n \to H_{n-1}$ for all $n \geq 1$ such that the sequence*

$$\cdots \xrightarrow{g_*} H''_{n+1} \xrightarrow{\partial_{n+1}} H_n \xrightarrow{f_*} H'_n \xrightarrow{g_*} H''_n \xrightarrow{\partial_n} H_{n-1} \xrightarrow{f_*} \cdots$$

is exact.

Proof. To define $\partial_n : H''_n \to H_{n-1}$, let $x \in H''_n = Z''_n/B''_n$ and $a \in Z''_n$ a representer. Since g_n is surjective, there is $b \in C'_n$ such that $g_n(b) = a$. Since $a \in Z''_n = \ker d''_n$, we have $0 = d''_n(a) = d''_n \circ g_n(b) = g_{n-1} \circ d'_n(b)$. By exactness, $d'_n(b) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$. Thus there exists $c \in C_{n-1}$ such that $f_{n-1}(c) = d'_n(b)$. We have $f_{n-2} \circ d_{n-1}(c) = d'_{n-1} \circ f_{n-1}(c) = d'_{n-1} \circ d'_n(b) = 0$. Since f_{n-2} is injective this implies $d_{n-1}(c) = 0$, thus $c \in Z_{n-1}$, and we can define $\partial(x) = [c] \in H_{n-1} = Z_{n-1}/B_{n-1}$.

We now have to check that this is well defined, i.e. independent of the a, b chosen in the construction, and that ∂_n is homomorphism. We note first that, once b is chosen, the rest of the construction is unique, since f_{n-1} is injective. If we replace a by $a' \in Z''_n$ then $a - a' \in B''_n = \operatorname{im} d''_{n+1}$. By surjectivity of

g_{n+1} there thus exists $e \in C'_{+1}n$ such that $a - a' = d''_{n+1} \circ g_{n+1}(e) = g_n \circ d'_{n+1}(e)$. Thus the replacement $a \to a'$ amounts to replacing b by $b - d'_{n+1}(e)$. But $d'_n \circ d'_{n+1}(e) = 0$, thus c is unaffected. On the other hand, b is determined by $g_n(b) = a$ only up to addition of something in $\ker g_n = \operatorname{im} f_n$. But adding $f_n(k)$ (with $k \in C_n$) to b affects $c = f_{n-1}^{-1} \circ d$ by addition of $d_n(k)$. But this is in $\operatorname{im} d_n = B_{n-1}$ and therefore $[c'] = [c]$ in H_{n-1}. Next, $\partial_n(0) = 0$ is obvious, since we may choose $a = b = 0$ in the above construction. Additivity $\partial_n(x + \tilde{x}) = \partial_n(x) + \partial_n(\tilde{x})$ is shown in the same way by adding the b's and c's used in the definition of $\partial_n(x)$ and $\partial_n(\tilde{x})$, respectively.

By Lemma 3.3 we already know that the sequence is exact at H'_n, i.e. $\ker g_* = \operatorname{im} f_*$. Verifying $f_* \circ \partial_n$ is easy: We defined $\partial_n(x) = [c]$, where $c = f_{n-1}^{-1}(d'_n(b))$. Thus $f_* \circ \partial_{n-1}([c]) = [d'_n(b)]$, but this is zero in H'_{n-1} since $d'_n(b) \in B'_{n-1}$. Also $\partial_n \circ g_n = 0$ is easy: Let $y \in H'_n$ and $u \in Z'_n$ a representer of y. Then $g_n(u)$ is a representer of $g_*(y)$, thus we may take $a = g_n(u)$ and $b = u$ in the construction of $\partial_n(x)$. But $u \in Z'_n$ implies $d'_n(u) = 0$ and thus $c = f_{n-1}^{-1} \circ d'_n(u) = 0$.

We still have to prove exactness at H''_n and H_{n-1}, i.e. $\ker \partial_n \subset \operatorname{im} g_* \subset H''_n$ and $\ker f_* \subset \operatorname{im} \partial_n \subset H_{n-1}$. To begin with the latter, let $y \in H_{n-1}$ such that $f_*(y) = 0$, and let $r \in Z_{n-1}$ be a representer of y. $f_*(y) = 0$ is equivalent to $f_{n-1}(r) \in B'_{n-1}$, thus there exists $s \in C'_n$ such that $d'_n(s) = f_{n-1}(r)$. Then $d''_n \circ g_n(s) = g_{n-1} \circ d'_n(s) = g_{n-1} \circ f_{n-1}(r) = 0$ in view of $g_n \circ f_n = 0$. Thus $g_n(s) \in Z''_n$. If we now consider $x = [g_n(s)] \in H''_n$ and apply our construction of ∂_n to it, we can choose $a = g_n(s)$ and $b = s$. This implies $\partial_n(x) = y$, thus $\ker f_* \subset \operatorname{im} \partial_n$.

To prove $\ker \partial_n \subset \operatorname{im} g_* \subset H''_n$, let $x \in H''_n$ be such that $\partial_n(x) = 0$, and let a, b, c as earlier. $\partial_n(x) = 0$ means that the $c \in Z_{n-1}$ representing $\partial_n(x)$ is in B_{n-1}. But this means that there is $p \in C_n$ such that $c = d_n(p)$. Then $f_{n-1} \circ d_n(p) = d'_n \circ f_n(p) = d'_n(b)$, which implies $d'_n(b - f_n(p)) = 0$. Thus $b - f_n(p) \in Z'_n$. The computation $[g_n(b - f_n(p))] = [g_n(b) - g_n \circ f_n(b)] = [g_n(b)] = [a] = x$, where we used $g_n \circ f_n = 0$, shows that $x = g_*([b - f_n(p)])$. Since $b - f_n(p) \in Z'_n$ this means $x \in g_*(H'_n)$ as desired. ■

3.6 Remark The above is a typical case of proof by 'diagram chase'. The theorem can be derived in a fancier way using other standard facts from homological algebra, but ultimately all these proofs boil down to the above computations. □

3.7 Proposition *Consider the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "f"] \arrow[d, "a"] & C' \arrow[r, "g"] \arrow[d, "b"] & C'' \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \overline{C} \arrow[r, "h"] & \overline{C}' \arrow[r, "k"] & \overline{C}'' \arrow[r] & 0
\end{tikzcd}$$

where $C, C', C'', \overline{C}, \overline{C}', \overline{C}''$ are chain complexes, f, g, h, k, a, b, c are chain maps, and the horizontals are exact sequences of complexes. Then

$$\begin{tikzcd}
\cdots & H_n \arrow[r, "f_*"] \arrow[d, "a_*"] & H'_n \arrow[r, "g_*"] \arrow[d, "b_*"] & H''_n \arrow[r, "\partial_n"] \arrow[d, "c_*"] & H_{n-1} \arrow[d, "a_*"] & \cdots \\
\cdots & \overline{H}_n \arrow[r, "h_*"] & \overline{H}'_n \arrow[r, "k_*"] & \overline{H}''_n \arrow[r, "\overline{\partial}_n"] & \overline{H}_{n-1} & \cdots
\end{tikzcd}$$

commutes for all $n \geq 1$.

Proof. The two left squares commute because of the identity $h \circ a = b \circ f$ of chain maps. Commutativity of the last square is left as an exercise. ■

3.2 Mayer-Vietoris sequence

There are many applications of Theorem 3.5, not only in algebraic topology. As the first of them we consider the theorem of Mayer and Vietoris, which allows to reduce the computation of $H_n(X)$ to that of $H_n(U), H_n(V)$ and $H_n(U \cap V)$ when $X = U \cup V$. (This is somewhat analogous to the van Kampen theorem for the fundamental group.)

Let $U, V \subset X$ be such that $X = U \cup V$. Considering $U, V, U \cap V$ with the subspace topologies, we have four complexes: $C_n(U \cap V), C_n(U), C_n(V), C_n(X)$. By Proposition 2.17, the canonical inclusion maps $i_1 : U \cap V \hookrightarrow U$, $i_2 : U \cap V \hookrightarrow V$ and $j_1 : U \hookrightarrow X$, $j_2 : V \hookrightarrow X$ give rise to chain maps, and we consider

$$0 \longrightarrow C_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} C_n(U) \oplus C_n(V) \xrightarrow{j_{1*} - j_{2*}} C_n(X).$$

(Here $(i_{1*} \oplus i_{2*})(a) \equiv (i_{1*}(a), i_{2*}(a))$ and $(j_{1*} - j_{2*})(a, b) = j_{1*}(a) - j_{2*}(a)$.) This sequence is exact: (1) Injectivity of $i_{1*} \oplus i_{2*}$ follows from injectivity of i_{1*} (or i_{2*}). (2) We have $(j_{1*} - j_{2*}) \circ (i_{1*} \oplus i_{2*})(a) = j_{1*} \circ i_{1*}(a) - j_{2*} \circ i_{2*}(a) = 0$ since $j_1 \circ i_1 = j_2 \circ i_2$. (3) If $(a, b) \in \ker j_{1*} - j_{2*}$ then $j_{1*}(a) = j_{2*}(b)$, but this is possible only if a, b are linear combinations of singular simplices in $U \cap V$. Then there exists $c \in C_n(U \cap V)$ such that $a = i_{1*}(c),\ b = i_{2*}(c)$, thus $\ker(j_{1*} - j_{2*}) = \mathrm{im}(i_{1*} \oplus i_{2*})$.

Unfortunately, the last morphism of the sequence is not surjective. It is easy to see that the image $(j_{1*} - j_{2*})(C_n(U) \oplus C_n(V)) \subset C_n(X)$ consists precisely of the subgroup generated by the singular simplices $\sigma \in C_n(X)$ whose image $\sigma(\Delta_n)$ is completely contained in U or in V. This leads us to the following

3.8 Definition *Let X be a space and $\mathcal{U} = (U_\alpha \subset X, \alpha \in S)$ such that $\cup_\alpha U_\alpha = X$. Let $S_n^U(X) \subset S_n(X)$ be the set of those singular n-simplices in X which are contained in one of the sets U_α, more precisely:*

$$S_n^{\mathcal{U}}(X) = \{\sigma \in S_n(X) \mid \exists \alpha \in S \text{ such that } \sigma(\Delta_n) \subset U_\alpha\}.$$

Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$ be the subgroup generated by $S_n^{\mathcal{U}}(X)$.

With this definition one has the following

3.9 Lemma *We have $d_n(C_n^{\mathcal{U}}(X)) \subset C_{n-1}^{\mathcal{U}}(X)$. Thus $(C_n^{\mathcal{U}}, d_n)$ is a complex and the inclusions $\iota_n : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ form a chain map. The homology of $(C_n^{\mathcal{U}}, d_n)$ is denoted by $H_n^{\mathcal{U}}(X)$.*

Proof. If $\sigma \in S_n^{\mathcal{U}}(X)$, thus $\sigma(\Delta_n) \subset U_\alpha$ then $\sigma \circ \delta_i^n(\Delta_{n-1}) \subset U_\alpha$, thus $\sigma \circ \delta_i^n \in S_{n-1}^{\mathcal{U}}(X)$. This implies $d_n(C_n^{\mathcal{U}}(X)) \subset C_{n-1}^{\mathcal{U}}(X)$, and it is obvious that (ι_n) is a chain map. ■

3.10 Lemma *Let X, U, V be as in the preceding paragraph. Then the following is an exact sequence*

$$0 \longrightarrow C_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} C_n(U) \oplus C_n(V) \xrightarrow{j_{1*} - j_{2*}} C_n^{\mathcal{U}}(X) \longrightarrow 0$$

for every $n \geq 0$. Furthermore, these exact sequences combine to an exact sequence of chain complexes $0 \to C(U \cap V) \to C(U) \oplus C(V) \to C(X) \to 0$.

Proof. We only need to show surjectivity of $j_{1*} - j_{2*}$. But this is obvious since, by the very definition, $C_n^{\mathcal{U}}(X)$ is generated by the singular simplices with image in U or V. Clearly all of them are contained in the union of $j_{1*}(C_n(U))$ and $j_{2*}(C_n(V))$. ■

Now we can apply Theorem 3.5 and obtain:

3.11 Corollary *Let $U, V \subset X$ such that $X = U \cup V$. Then there is a long exact sequence*

$$\cdots H_{n+1}(X) \xrightarrow{\partial_{n+1}} H_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} H_n(U) \oplus H_n(V) \xrightarrow{j_{1*} - j_{2*}} H_n^{\mathcal{U}}(X) \xrightarrow{\partial_n} H_{n-1}(U \cap V) \ldots,$$

where the maps ∂_n are as in Theorem 3.5.

This result is not yet very useful since we don't know how the groups $H_n^{\mathcal{U}}(X)$ are related to $H_n(X)$. The answer is given by the following technical result, which has other applications in singular homology theory. Its proof is quite long (4-5 pages) and therefore omitted. See [1, 2]. Recall that the interior A^0 of a subset $A \subset X$ is the union of all subsets of A that are open.

3.12 Theorem *Let X be a space and $\mathcal{U} = (U_\alpha \subset X,\ \alpha \in S)$ such that $\cup_\alpha U_\alpha^0 = X$. Then the homomorphisms $\iota_* : H_n^{\mathcal{U}}(X) \to H_n(X)$ of homology groups induced by the chain map $\iota : (C_n^{\mathcal{U}}(X), d_n) \hookrightarrow (C_n(X), d_n)$ are isomorphisms.*

We now apply this to coverings of X by just two subsets U, V, subject to the condition that $U^0 \cup V^0 = X$. Thus $\mathcal{U} = \{U, V\}$. Let i_1, i_2, j_1, j_2 be as before.

3.13 Theorem (Mayer-Vietoris) *Let X be a space and $U, V \subset X$ subspaces such that $U^0 \cup V^0 = X$. Then there exists a long exact sequence*

$$\cdots\ H_{n+1}(X) \xrightarrow{\partial_{n+1}} H_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} H_n(U) \oplus H_n(V) \xrightarrow{j_{1*} - j_{2*}} H_n(X) \xrightarrow{\partial_n} H_{n-1}(U \cap V) \ldots,$$

where $i_1 : U \cap V \hookrightarrow U,\ i_2 : U \cap V \hookrightarrow V$ and $j_1 : U \hookrightarrow X,\ j_2 : V \hookrightarrow X$ are the obvious inclusion maps.

Proof. Use Theorem 3.12 to replace $H_n^{\mathcal{U}}(X)$ by $H_n(X)$ in Corollary 3.11. ■

3.14 Remark In most applications of Theorem 3.13 one actually does not need to know how the connecting morphisms ∂_n are defined. This will become clear in the computation of $H_n(S^m)$ in Section 4. □

If $f : X \to X$ satisfies $f(U) \subset U,\ f(V) \subset V$, then Proposition 3.7 immediately implies:

3.15 Corollary *Let $f : X \to X$ be such that $f(U) \subset U,\ f(V) \subset V$. Then the following diagram commutes:*

$$\begin{tikzcd}
\cdots H_n(U \cap V) \arrow[r, "i_{1*} \oplus i_{2*}"] \arrow[d, "f_*"] & H_n(U) \oplus H_n(V) \arrow[r, "j_{1*} - j_{2*}"] \arrow[d, "f_* \oplus f_*"] & H_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_*"] & H_{n-1}(U \cap V) \ldots \arrow[d, "f_*"] \\
\cdots H_n(U \cap V) \arrow[r, "i_{1*} \oplus i_{2*}"] & H_n(U) \oplus H_n(V) \arrow[r, "j_{1*} - j_{2*}"] & H_n(X) \arrow[r, "\partial_n"] & H_{n-1}(U \cap V) \ldots
\end{tikzcd}$$

4 Computation of $H_n(S^m)$

The aim of this section is to prove the following

4.1 Theorem

1. *For $m \geq 1$ we have*

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0, m \\ 0 & \text{otherwise} \end{cases}$$

2. *For $m \geq 1$ and $f : S^m \to S^m$ define $\deg f \in \mathbb{Z}$ by $f_*(a) = (\deg f)a$ for all $a \in H_m(S^m) \cong \mathbb{Z}$. Then the map $f_m : S^m \to S^m,\ (x_0, \ldots, x_m) \mapsto (-x_0, \ldots, x_m)$ has degree -1.*

We are going to prove this result using the Mayer-Vietoris theorem. In fact, the theorem can be proven using only the standard axioms satisfied by singular homology, among which the Mayer-Vietoris theorem is not counted. Since Mayer-Vietoris can be derived from the other axioms, our proof is just as good, but it is more intuitive.

1. Since $S^m, m \geq 1$ is connected, Proposition 2.11 implies $H_0(S^m) \cong \mathbb{Z}$.

2. We consider $S^1 = \{(x_0, x_1) \in \mathbb{R}^2 \mid x_0^2 + x_1^2 = 1\}$. Let $0 < \varepsilon < 1$ and consider the subsets

$$\begin{aligned} U &= \{(x_0, x_1) \in S^1 \mid x_1 \geq -\varepsilon\}, \\ V &= \{(x_0, x_1) \in S^1 \mid x_1 \leq +\varepsilon\}. \end{aligned}$$

Since X, U, V are path-connected we have $H_0(X) \cong H_0(U) \cong H_0(V) \cong \mathbb{Z}$. Furthermore, U and V are contractible, thus $H_1(U) = H_1(V) = 0$. Clearly, $U^0 \cup V^0 = X$ and $U \cap V = A \cup B$, with A, B non-empty and path-connected. Thus $H_0(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$. Considering the end of the Mayer-Vietoris sequence, we therefore find:

$$\begin{tikzcd}
H_1(U) \oplus H_1(V) \arrow[r] \arrow[d, equal] & H_1(X) \arrow[r, "\partial_1"] \arrow[d, equal] & H_0(U \cap V) \arrow[r, "i_{1*} \oplus i_{2*}"] \arrow[d, equal] & H_0(U) \oplus H_0(V) \arrow[r, "j_{1*} - j_{2*}"] \arrow[d, equal] & H_0(X) \arrow[d, equal] \\
0 \oplus 0 \arrow[r] & H_1(X) \arrow[r, "\partial_1"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "i_{1*} \oplus i_{2*}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_{1*} - j_{2*}"] & \mathbb{Z}
\end{tikzcd}$$

Since the lower left horizontal arrow is zero, exactness implies $\ker \partial_1 = \operatorname{im} 0 = 0$, thus ∂_1 is injective, and therefore $H_1(X) \cong \operatorname{im} \partial_1 \cong \ker(i_{1*} \oplus i_{2*})$. In view of the construction of the Mayer-Vietoris sequence we have

$$i_{1*} \oplus i_{2*} : \ \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}, \quad (a, b) \mapsto (a + b, a + b),$$

$$j_{1*} - j_{2*} : \ \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}, \quad (a, b) \mapsto a - b.$$

Thus $\ker(i_{1*} \oplus i_{2*}) = \{(a, -a) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \in \mathbb{Z}\} \ \cong \ \mathbb{Z}$, and we conclude

$$H_1(S^1) \cong \mathbb{Z}.$$

(Of course, this also follows from $\pi_1(X) \cong \mathbb{Z}$ and Proposition 2.13, but we didn't prove the latter.)

3. Next, we compute $\deg f$ for $f_1 : S^1 \to S^1$, $(x_0, x_1) \mapsto (-x_0, x_1)$. We have $f_1(U) \subset U, f_1(V) \subset V$. In general, if X is path connected and $f : X \to X$ is arbitrary then $f_* : H_0(X) \to H_0(X)$ is the identity, since f maps the only path component of X into itself. Thus the induced endomorphisms of $H_0(X), H_0(U), H_0(V)$ are all given by $f_{1*} = \mathrm{id}$. Now Corollary 3.15 gives us

$$\begin{tikzcd}
0 \oplus 0 \arrow[r] & H_1(X) \arrow[r, "\partial_1"] \arrow[d, "f_{1*}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "i_{1*} \oplus i_{2*}"] \arrow[d, "f_{1*}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_{1*} - j_{2*}"] \arrow[d, "\mathrm{id} \oplus \mathrm{id}"] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
0 \oplus 0 \arrow[r] & H_1(X) \arrow[r, "\partial_1"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "i_{1*} \oplus i_{2*}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_{1*} - j_{2*}"] & \mathbb{Z}
\end{tikzcd}$$

Now we notice that $f_1 : S^1 \to S^1$ exchanges the two components A, B of $U \cap V$. Thus

$$f_{1*} : H_1(U \cap V) \to H_1(U \cap V), \quad (a, b) \mapsto (b, a).$$

On $\ker(i_{1*} \oplus i_{2*}) = \{(a, -a), a \in \mathbb{Z}\}$, which is isomorphic to $H_1(X)$, this is the multiplication by -1, thus $\deg f_1 = -1$, as desired.

4. Since U, V are contractible, by Proposition 2.21 we have $H_n(U) = H_n(V) = 0$ for all $n \geq 1$. Furthermore, $U \cap V = A \cup B$ with A, B contractible. Thus by Corollary 2.19 we also have $H_n(U \cap V) = 0$ for all $n \geq 1$. Consider now the following part of the Mayer-Vietoris sequence:

$$\cdots H_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} H_n(U) \oplus H_n(V) \xrightarrow{j_{1*} - j_{2*}} H_n(S^1) \xrightarrow{\partial_n} H_{n-1}(U \cap V) \ldots$$

In view of the mentioned facts, for $n \geq 2$ this becomes

$$\cdots \quad 0 \xrightarrow{i_{1*} \oplus i_{2*}} 0 \oplus 0 \xrightarrow{j_{1*} - j_{2*}} H_n(S^1) \xrightarrow{\partial_n} 0 \quad \cdots$$

and exactness implies $H_n(S^1) = 0$ for $n \geq 2$. (If $0 \to 0 \to A \to 0$ is exact then $A = 0$.)

5. Consider $X = S^m, m \geq 2$ and the subsets

$$\begin{aligned} U &= \{(x_0, \ldots, x_m) \in S^1 \mid x_m \geq -\varepsilon\}, \\ V &= \{(x_0, \ldots, x_m) \in S^1 \mid x_m \leq +\varepsilon\} \end{aligned}$$

for $0 < \varepsilon < 1$. We claim (proof as exercise!) that U and V are contractible and that $U \cap V$ is homotopy equivalent to S^{m-1}. ($U \cap V$ deformation retracts to the equator of S^m which is $\{(x_0, \ldots, x_m) \in S^m \mid x_m = 0\}$.) Thus $H_0(U) \cong H_0(V) \cong H_0(U \cap V) \cong \mathbb{Z}$ and $H_n(U) = H_n(V) = 0$ for $n \geq 1$. Now the end of the Mayer-Vietoris sequence looks like

$$\begin{tikzcd}
H_1(U) \oplus H_1(V) \arrow[r] \arrow[d, equal] & H_1(S^m) \arrow[r] \arrow[d, equal] & H_0(U \cap V) \arrow[r] \arrow[d, equal] & H_0(U) \oplus H_0(V) \arrow[r] \arrow[d, equal] & H_0(S^m) \arrow[d, equal] \\
0 \oplus 0 \arrow[r] & H_1(S^m) \arrow[r, "\partial_1"] & \mathbb{Z} \arrow[r, "i_{1*} \oplus i_{2*}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_{1*} - j_{2*}"] & \mathbb{Z}
\end{tikzcd}$$

Now $i_{1*} \oplus i_{2*} : a \mapsto (a, a)$ is injective, thus $\operatorname{im} \partial_1 = \ker(i_{1*} \oplus i_{2*}) = 0$. On the other hand, $\ker \partial_1 = \operatorname{im} 0 = 0$. Thus ∂_1 is injective, but since $\operatorname{im} \partial_1 = 0$ this means $H_1(S^m) = 0$ for all $m \geq 2$.

6. We summarize what we have proven so far:

$$\begin{aligned} H_0(S^m) &= \mathbb{Z},\ m \geq 1, \\ H_1(S^1) &= \mathbb{Z}, \\ H_n(S^1) &= 0,\ n \geq 2, \\ H_1(S^m) &= 0,\ m \geq 2. \end{aligned} \tag{4.1}$$

It thus remains to compute $H_n(S^m)$ for $n, m \geq 2$. Assume $n, m \geq 2$ and consider

$$H_n(U) \oplus H_n(V) \longrightarrow H_n(S^m) \longrightarrow H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V)$$

In view of what we said about U and V, this reduces to

$$0 \longrightarrow H_n(S^m) \longrightarrow H_{n-1}(U \cap V) \longrightarrow 0$$

But exactness of $0 \to A \xrightarrow{f} B \to 0$ is equivalent to $f : A \to B$ being an isomorphism. With $U \cap V \simeq S^{m-1}$ (homotopy equivalence) we conclude

$$H_n(S^m) \cong H_{n-1}(S^{m-1}) \quad \forall n, m \geq 2.$$

Using this to reduce n and m until we arrive one of the situations in (4.1), we obtain the proof of the first half of the theorem.

7. To prove the second half of the theorem, apply naturality of the Mayer-Vietoris sequence, cf. Corollary 3.15:

$$\begin{tikzcd}
0 \arrow[r] & H_n(S^n) \arrow[r] \arrow[d, "f_{m*}"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "(f_{m-1})_*"] & 0 \\
0 \arrow[r] & H_n(S^n) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Namely, this implies that the map $f_m : S^m \to S^m : (x_0, \ldots, x_m) \mapsto (-x_0, x_1, \ldots, x_m)$ has the same degree as f_{m-1}. Together with what we proved in 3., this completes the proof.

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