Acyclic and frugal colourings of graphs

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Given graph $G = (V, E)$, a *colouring* of $G$ is a mapping

$$f : V \rightarrow \{1, \ldots, x\}$$

for some integer $x$. Graph colouring can model interference in a network and we want colourings which map to *few* colours. Typically, graph theorists are concerned with colourings which are *proper*, but we won’t presume this here.
Introduction

We consider graphs with bounded maximum degree $\Delta(G) = d$ and the asymptotic behaviour as $d \to \infty$.

Lovász Local Lemma

Let $\mathcal{E}$ be a set of (typically bad) events such that for each $A \in \mathcal{E}$

1. $\Pr(A) \leq p < 1$, and
2. $A$ is mutually independent of all but at most $\delta$ other events.

If $ep(\delta + 1) < 1$, then with positive probability none of the events in $\mathcal{E}$ occur.
Frugal colouring

Given $G = (V, E)$ and $t \geq 1$, a colouring of $G$ is $t$-frugal if no colour appears more than $t$ times in any neighbourhood.

Notation:

- $t$-frugal chromatic number $\phi^t$
- Proper $t$-frugal chromatic number $\chi^t$
Frugal colouring

Given $G = (V, E)$ and $t \geq 1$, a colouring of $G$ is $t$-frugal if no colour appears more than $t$ times in any neighbourhood.

Notation:

- $t$-frugal chromatic number $\rightarrow \varphi^t$
- Proper $t$-frugal chromatic number $\rightarrow \chi^t_{\varphi}$

Note: $\varphi^1$ also known as injective chromatic number (HKSS ‘02) and $\chi^1_{\varphi}(G)$ is the same as $\chi(G^2)$.

$$\varphi^t(d) := \sup \{ \varphi^t(G) \mid \Delta(G) = d \}.$$

We may allow $t = t(d)$ to vary/grow as a function of $d$. 

Acyclic and frugal colouring

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Frugal colouring

Example: $t = 1$, point-line incidence graph of Fano plane.

$$\implies \varphi^1(3) = 7.$$
Frugal colouring lower bounds

Observation: $\varphi^t(d) \geq d/t$ since $\varphi^t(G) \geq \Delta(G)/t$. 
Frugal colouring lower bounds

Observation: $\varphi^t(d) \geq d/t$ since $\varphi^t(G) \geq \Delta(G)/t$.

Proposition (Alon, cf. HMR ‘97)

For any $t \geq 1$ and any prime power $n$,

$$\varphi^t(n^t + \cdots + 1) \geq (n^{t+1} + \cdots + 1)/t.$$ 

Corollary

Suppose that $t = t(d) \geq 2$ and $t = o(\ln d / \ln \ln d)$. Then,

$$\varphi^t(d) = \Omega(d^{1+1/t}/t).$$
Proper frugal colouring

Theorem (MoRe ‘09)
For sufficiently large $d$, $\chi_{\varphi}^{50\ln d / \ln \ln d}(d) \leq d + 1$.

Theorem (HMR ‘97)
For any $t \geq 1$ and sufficiently large $d$,

$$\chi_{\varphi}^t(d) \leq \max \left\{ (t + 1)d, \left\lceil e^3 d^{1+1/t} / t \right\rceil \right\}.$$
Proper frugal colouring

**Theorem (MoRe ‘09)**

For sufficiently large $d$, $\chi_{\phi}^{50 \ln d / \ln \ln d} (d) \leq d + 1$.

**Theorem (HMR ‘97)**

For any $t \geq 1$ and sufficiently large $d$,

$$\chi_{\phi}^t (d) \leq \max \left\{ (t + 1) d, \left\lceil e^3 d^{1+1/t} / t \right\rceil \right\}.$$

$t \geq 50 \ln d / \ln \ln d \implies \chi_{\phi}^t (d) = \Theta(d)$,

t = o(\ln d / \ln \ln d) \implies \chi_{\phi}^t (d) = \Theta(d^{1+1/t} / t),$

t = o(\ln d / \ln \ln d) \implies \phi^t (d) = \Theta(d^{1+1/t} / t)$.
Frugal colouring, $t$ large

Theorem
Suppose $t = \omega(\ln d)$. Then, $\forall \varepsilon > 0$,

$$\varphi^t(d) \leq \lceil (1 + \varepsilon)d/t \rceil$$

for sufficiently large $d$. 
Frugal colouring, $t$ large

**Theorem**

Suppose $t = \omega(\ln d)$. Then, $\forall \varepsilon > 0$, 

$$
\varphi^t(d) \leq \lceil (1 + \varepsilon)d/t \rceil
$$

for sufficiently large $d$.

**Proof.**

Assume $G = (V, E)$ is $d$-regular. Let $f : V \rightarrow \{1, \ldots, x\}$ be a random colouring where $x = \lceil (1 + \varepsilon)d/t \rceil$. Let $A_{v,i}$ be event that $v$ has $> t$ neighbours coloured $i$. Each event independent of all but at most $d^2 x \ll d^3$ other events and $\Pr(A_{v,i}) = \exp(-\Omega(t))$ using Chernoff. By LLL, none of the events hold and $f$ is $t$-frugal with positive probability.
Frugal colouring summary

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\phi^t(d)$</th>
<th>$\chi^t_\phi(d)$</th>
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<tbody>
<tr>
<td></td>
<td>lower</td>
<td>upper</td>
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<tr>
<td>$O\left(\frac{\ln d}{\ln \ln d}\right)$</td>
<td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td>
<td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td>
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<tr>
<td>$O(\ln d)$</td>
<td>$\left\lceil \frac{d}{t} \right\rceil$</td>
<td>$\left\lceil \frac{d}{t^{1-\varepsilon}} \right\rceil$</td>
</tr>
<tr>
<td>$\omega\left(\frac{\ln d}{\ln \ln d}\right)$</td>
<td>$(1 + \varepsilon)\frac{d}{t}$</td>
<td></td>
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</tbody>
</table>
Acyclic colouring

Given $G = (V, E)$, a colouring of $G$ is *acyclic* if the bipartite subgraph induced by the edges between any two colours is acyclic. (Any graph has an acyclic 1-colouring!)

Notation:

- acyclic $t$-frugal chromatic number $\rightarrow \varphi^t_a$
- acyclic proper $t$-frugal chromatic number $\rightarrow \chi^t_{\varphi,a}$
- (acyclic proper chromatic number $\rightarrow \chi_a$)

Note: $\chi^2_{\varphi,a}$ also known as *linear chromatic number* (Yuster ‘98) and $\chi^1_{\varphi,a}(G)$ is the same as $\chi(G^2)$.

$$\varphi^t_a(d) := \sup \{ \varphi^t_a(G) \mid \Delta(G) = d \}.$$
Acyclic (proper) colouring

Notes:

- Acyclic *proper* colouring has been studied extensively . . .
- Any planar graph is acyclically properly 5-colourable (Borodin ‘79).
- $\chi_a(d) \leq \lceil 50d^{4/3} \rceil$ (AMR ‘91).
Acyclic proper frugal colouring

Theorem (AMR ‘91)
\[ \chi_a(d) \leq \lceil \frac{50d^4}{3} \rceil. \quad \chi_a(d) = \Omega\left(\frac{d^4}{3}/(\ln d)^{1/3}\right). \]

▷ \[ \chi_{\phi, a}^1(d) = d^2 + 1. \]
▷ \[ \chi_{\phi, a}^2(d) \leq \lceil 50d^{3/2} \rceil \text{ (Yuster ‘98)}. \]
Acyclic proper frugal colouring

Theorem
\[ \chi_{\phi,a}^3(d) \leq \lceil 50d^{4/3} \rceil. \]
Acyclic proper frugal colouring

Theorem
$$\chi_{\phi,a}^3(d) \leq \lceil 50d^4/3 \rceil.$$  

Proof outline.
We extend the theorem of AMR ‘91 by adding a fifth event to ensure that the random colouring $f$ is 3-frugal:

For vertices $v, v_1, v_2, v_3, v_4$ with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1, \ldots, v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$.  

□
Acyclic proper frugal colouring

Theorem
\( \chi_{\varphi,a}(d) \leq \lceil 50d^{4/3} \rceil. \)

Proof outline.
We extend the theorem of AMR ‘91 by adding a fifth event to ensure that the random colouring \( f \) is 3-frugal:

For vertices \( v, v_1, v_2, v_3, v_4 \) with \( \{v_1, v_2, v_3, v_4\} \subseteq N(v) \), let \( E_{\{v_1,\ldots,v_4\}} \) be the event that \( f(v_1) = f(v_2) = f(v_3) = f(v_4) \).

\[ \implies \chi_{\varphi,a}^1(d) = \Theta(d^2), \quad \chi_{\varphi,a}^2(d) = \Theta(d^{3/2}), \quad \chi_{\varphi,a}^3(d) = \Theta(d^{4/3}) \]
and, for any \( t \geq 4 \),
\[ \chi_{\varphi,a}^t(d) = \Omega(d^{4/3}/(\ln d)^{1/3}), \]
\[ \chi_{\varphi,a}^t(d) = O(d^{4/3}). \]
Acyclic frugal colouring lower bounds

Clearly, $\varphi_1^a(d) = \Theta(d^2)$, $\varphi_2^a(d) = \Theta(d^{3/2})$, $\varphi_3^a(d) = \Theta(d^{4/3})$. For larger $t$, we have the upper bound $\varphi_t^a(d) = O(d^{4/3})$, but can no longer borrow the lower bound on $\chi_a(d)$.

We use bounds on the *acyclic $t$-improper chromatic number* $\chi_t^a$. 
Acyclic frugal colouring lower bounds

Clearly, $\varphi_1^a(d) = \Theta(d^2)$, $\varphi_2^a(d) = \Theta(d^{3/2})$, $\varphi_3^a(d) = \Theta(d^{4/3})$. For larger $t$, we have the upper bound $\varphi_t^a(d) = O(d^{4/3})$, but can no longer borrow the lower bound on $\chi_a(d)$.

We use bounds on the *acyclic $t$-improper chromatic number* $\chi_t^a$.

Given $G = (V, E)$ and $t \geq 0$, a colouring of $G$ is *$t$-improper* if no vertex has more than $t$ neighbours of its same colour. Any $t$-frugal colouring is $t$-improper.
Acyclic improper colouring

Theorem (AEKMP ‘09+)
For any $t = t(d) \leq d - 10\sqrt{d \ln d}$ and sufficiently large $d$,

$$\chi_a^t(d) \geq \frac{(d - t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$ 

Theorem (AKM ‘09+)
$\chi_a^{d-1}(d) \geq d/8$. 

The same lower bounds hold for $\varphi_a^t(d)$. 

Acyclic and frugal colouring
Acyclic frugal colouring, $t$ large

**Theorem**

*For any $t = t(d) \geq 1$ and sufficiently large $d$,*

$$\varphi_t^a(d) \leq d \cdot \max\{3(d - t), 31 \ln d\} + 2.$$

$$\implies \varphi_a^{d^{-1}}(d) = O(d \ln d).$$
Acyclic frugal colouring, \( t \) large

Given \( d \)-regular \( G = (V, E) \) and \( k \in \{1, \ldots, d\} \), let

\[ \psi(G, k) = \inf \left\{ k' \mid \exists S : \forall v \in V, \ k \leq |N(v) \cap S| \leq k' \right\}. \]

**Lemma**

*For any \( 1 \leq k \leq d \) and sufficiently large \( d \), if \( G \) is \( d \)-regular, then

\[ \psi(G, k) \leq \max\{3k, 31 \ln d\}. \]
Acyclic frugal colouring, $t$ large

**Theorem**

For any $t = t(d) \geq 1$ and sufficiently large $d$,

$$\varphi_a^t(d) \leq d \cdot \max\{3(d - t), 31 \ln d\} + 2.$$ 

**Proof.**

Assume $G = (V, E)$ is $d$-regular with $d$ large enough for Lemma. Let $S$ be such that $d - t \leq |N(v) \cap S| \leq x$ for all $v \in V$ where $x = \max\{3(d - t), 31 \ln d\}$. Colour $G \setminus S$ with colour 1 and colour $G^2[S]$ greedily using at most $dx + 1$ other colours. What results is an acyclic and $t$-frugal colouring of $G$.  

$\square$
## Acyclic frugal colouring summary

<table>
<thead>
<tr>
<th>$d - t$</th>
<th>$\varphi_a^t(d')$</th>
<th>\text{lower}</th>
<th>\text{upper}</th>
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</thead>
<tbody>
<tr>
<td>$d - 1$</td>
<td>$\Omega(d^2)$</td>
<td>$O(d^2)$</td>
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<tr>
<td>$d - 2$</td>
<td>$\Omega(d^{3/2})$</td>
<td>$O(d^{3/2})$</td>
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<tr>
<td>$d - 3$</td>
<td>$\Omega(d^{4/3})$</td>
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<tr>
<td>$\omega(d^{3/4}(\ln d)^{1/4})$</td>
<td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td>
<td>$O(d^{4/3})$</td>
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<tr>
<td>$O(d^{1/2})$</td>
<td>$\Omega(d)$</td>
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<td>$O((d - t)d)$</td>
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<td>$O(d^{1/3})$</td>
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<td>$O(d \ln d)$</td>
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<td>$0$</td>
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