Acyclic and frugal colourings of graphs

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Introduction

Given graph $G = (V, E)$, a *colouring* of G is a mapping

$$f : V \to \{1, \ldots, x\}$$

for some integer x. Graph colouring can model interference in a network and we want colourings which map to *few* colours.

Typically, graph theorists are concerned with colourings which are *proper*, but we won't presume this here.

Introduction

We consider graphs with bounded maximum degree $\Delta(G) = d$ and the asymptotic behaviour as $d \to \infty$.

Lovász Local Lemma

Let $\mathscr{E}$ be a set of (typically bad) events such that for each $A \in \mathscr{E}$

1. $\Pr(A) \leq p < 1$, *and*
2. *A is mutually independent of all but at most δ other events.*

If $ep(\delta + 1) < 1$, then with positive probability none of the events in $\mathscr{E}$ occur.

Frugal colouring

Given $G = (V, E)$ and $t \geq 1$, a colouring of G is *t-frugal* if no colour appears more than t times in any neighbourhood.

Notation:

$$t\text{-frugal chromatic number} \longrightarrow \varphi^t$$
$$\text{proper } t\text{-frugal chromatic number} \longrightarrow \chi_\varphi^t$$

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Notation:

$$\begin{aligned} t\text{-frugal chromatic number} &\longrightarrow \varphi^t \\ \text{proper } t\text{-frugal chromatic number} &\longrightarrow \chi_\varphi^t \end{aligned}$$

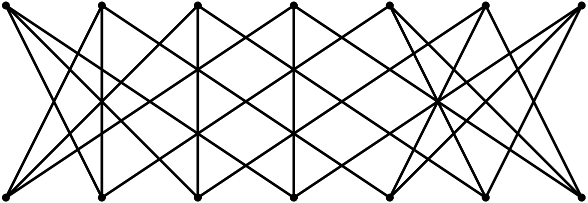
Note: φ^1 also known as *injective chromatic number* (HKSS '02) and $\chi_\varphi^1(G)$ is the same as $\chi(G^2)$.

$$\varphi^t(d) := \sup\left\{\varphi^t(G) \mid \Delta(G) = d\right\}.$$

We may allow $t = t(d)$ to vary/grow as a function of d.

Frugal colouring

Example: $t = 1$, point-line incidence graph of Fano plane.



$\implies \varphi^1(3) = 7$.

Frugal colouring lower bounds

Observation: $\varphi^t(d) \geq d/t$ since $\varphi^t(G) \geq \Delta(G)/t$.

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Proposition (Alon, cf. HMR ‘97)

For any $t \geq 1$ and any prime power n,

$$\varphi^t(n^t + \cdots + 1) \geq (n^{t+1} + \cdots + 1)/t.$$

Corollary

Suppose that $t = t(d) \geq 2$ and $t = o(\ln d / \ln \ln d)$. Then, $\varphi^t(d) = \Omega(d^{1+1/t}/t)$.

Proper frugal colouring

Theorem (MoRe '09)

For sufficiently large d, $\chi_\varphi^{50\ln d/\ln\ln d}(d) \le d+1$.

Theorem (HMR '97)

For any $t \ge 1$ and sufficiently large d,

$$\chi_\varphi^t(d) \le \max\left\{(t+1)d, \left\lceil e^3 d^{1+1/t}/t \right\rceil\right\}.$$

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$$t \ge 50\ln d/\ln\ln d \implies \chi_\varphi^t(d) = \Theta(d),$$
$$t = o(\ln d/\ln\ln d) \implies \chi_\varphi^t(d) = \Theta(d^{1+1/t}/t),$$
$$t = o(\ln d/\ln\ln d) \implies \varphi^t(d) = \Theta(d^{1+1/t}/t).$$

Frugal colouring, t large

Theorem

Suppose $t = \omega(\ln d)$. Then, $\forall \varepsilon > 0$,

$$\varphi^t(d) \le \lceil (1+\varepsilon)d/t \rceil$$

for sufficiently large d.

Frugal colouring, t large

Theorem

Suppose $t = \omega(\ln d)$. Then, $\forall \varepsilon > 0$,

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for sufficiently large d.

Proof.

Assume $G = (V,E)$ is d-regular. Let $f : V \to \{1,\ldots,x\}$ be a random colouring where $x = \lceil (1+\varepsilon)d/t \rceil$. Let $A_{v,i}$ be event that v has $> t$ neighbours coloured i. Each event independent of all but at most $d^2 x \ll d^3$ other events and $\Pr(A_{v,i}) = \exp(-\Omega(t))$ using Chernoff. By LLL, none of the events hold and f is t-frugal with positive probability. $\square$

Frugal colouring summary

<table>
<tr><th rowspan="2" colspan="2">t</th><th colspan="2">$\varphi^t(d)$</th><th colspan="2">$\chi^t_\varphi(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td colspan="2">$O\left(\frac{\ln d}{\ln\ln d}\right)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td></tr>
<tr><td rowspan="2">$\omega\left(\frac{\ln d}{\ln\ln d}\right)$</td><td>$O(\ln d)$</td><td rowspan="2">$\left\lceil \frac{d}{t} \right\rceil$</td><td>$\left\lceil \frac{d}{t^{1-\varepsilon}} \right\rceil$</td><td rowspan="2">$d+1$</td><td rowspan="2">$d+1$</td></tr>
<tr><td></td><td>$\left\lceil (1+\varepsilon)\frac{d}{t} \right\rceil$</td></tr>
</table>

Acyclic colouring

Given $G = (V, E)$, a colouring of G is *acyclic* if the bipartite subgraph induced by the edges between any two colours is acyclic. (Any graph has an acyclic 1-colouring!)

Notation:

$$\begin{aligned} \text{acyclic } t\text{-frugal chromatic number} &\longrightarrow \varphi_a^t \\ \text{acyclic proper } t\text{-frugal chromatic number} &\longrightarrow \chi_{\varphi,a}^t \\ \big(\text{acyclic proper chromatic number} &\longrightarrow \chi_a\big) \end{aligned}$$

Note: $\chi_{\varphi,a}^2$ also known as *linear chromatic number* (Yuster '98) and $\chi_{\varphi,a}^1(G)$ is the same as $\chi(G^2)$.

$$\varphi_a^t(d) := \sup\{\varphi_a^t(G) \mid \Delta(G) = d\}.$$

Acyclic (proper) colouring

Notes:

- Acyclic *proper* colouring has been studied extensively . . .
- Any planar graph is acyclically properly 5-colourable (Borodin '79).
- $\chi_a(d) \leq \lceil 50d^{4/3} \rceil$ (AMR '91).

Acyclic proper frugal colouring

Theorem (AMR '91)

$\chi_a(d) \leq \lceil 50 d^{4/3} \rceil$. $\chi_a(d) = \Omega(d^{4/3}/(\ln d)^{1/3})$.

- $\chi^1_{\varphi,a}(d) = d^2 + 1$.
- $\chi^2_{\varphi,a}(d) \leq \lceil 50 d^{3/2} \rceil$ (Yuster '98).

Acyclic proper frugal colouring

Theorem

$\chi^3_{\varphi,a}(d) \le \lceil 50d^{4/3} \rceil.$

Acyclic proper frugal colouring

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$\chi^3_{\varphi,a}(d) \le \lceil 50 d^{4/3} \rceil$.

Proof outline.

We extend the theorem of AMR '91 by adding a fifth event to ensure that the random colouring f is 3-frugal:

- V For vertices v, v_1, v_2, v_3, v_4 with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1,\dots,v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$.

$\square$

Acyclic proper frugal colouring

Theorem

$\chi^3_{\varphi,a}(d) \leq \lceil 50 d^{4/3} \rceil$.

Proof outline.

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□

$$\implies \chi^1_{\varphi,a}(d) = \Theta(d^2),\ \chi^2_{\varphi,a}(d) = \Theta(d^{3/2}),\ \chi^3_{\varphi,a}(d) = \Theta(d^{4/3})$$

and, for any $t \geq 4$, $\chi^t_{\varphi,a}(d) = \Omega(d^{4/3}/(\ln d)^{1/3})$,

$$\chi^t_{\varphi,a}(d) = O(d^{4/3})\ .$$

Acyclic frugal colouring lower bounds

Clearly, $\varphi_a^1(d) = \Theta(d^2)$, $\varphi_a^2(d) = \Theta(d^{3/2})$, $\varphi_a^3(d) = \Theta(d^{4/3})$. For larger t, we have the upper bound $\varphi_a^t(d) = O(d^{4/3})$, but can no longer borrow the lower bound on $\chi_a(d)$.

We use bounds on the *acyclic t-improper chromatic number* χ_a^t.

Acyclic frugal colouring lower bounds

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We use bounds on the *acyclic t-improper chromatic number* χ_a^t.

Given $G = (V, E)$ and $t \geq 0$, a colouring of G is *t-improper* if no vertex has more than t neighbours of its same colour.
Any t-frugal colouring is t-improper.

Acyclic improper colouring

Theorem (AEKMP '09+)

For any $t = t(d) \leq d - 10\sqrt{d \ln d}$ *and sufficiently large* d*,*

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

Theorem (AKM '09+)

$\chi_a^{d-1}(d) \geq d/8$.

The same lower bounds hold for $\varphi_a^t(d)$.

Acyclic frugal colouring, t large

Theorem

For any $t = t(d) \geq 1$ and sufficiently large d,

$$\varphi_a^t(d) \leq d \cdot \max\{3(d-t), 31 \ln d\} + 2.$$

$$\implies \varphi_a^{d-1}(d) = O(d \ln d).$$

Acyclic frugal colouring, t large

Given d-regular $G = (V,E)$ and $k \in \{1,\ldots,d\}$, let

$$\psi(G,k) = \inf\{k' \mid \exists S : \forall v \in V,\ k \leq |N(v) \cap S| \leq k'\}.$$

Lemma

For any $1 \leq k \leq d$ and sufficiently large d, if G is d-regular, then

$$\psi(G,k) \leq \max\{3k, 31 \ln d\}.$$

Acyclic frugal colouring, t large

Theorem

For any $t = t(d) \geq 1$ and sufficiently large d,

$$\varphi_a^t(d) \leq d \cdot \max\{3(d-t), 31 \ln d\} + 2.$$

Proof.

Assume $G = (V, E)$ is d-regular with d large enough for Lemma. Let S be such that $d - t \leq |N(v) \cap S| \leq x$ for all $v \in V$ where $x = \max\{3(d-t), 31 \ln d\}$. Colour $G \setminus S$ with colour 1 and colour $G^2[S]$ greedily using at most $dx + 1$ other colours. What results is an acyclic and t-frugal colouring of G. $\square$

Acyclic frugal colouring summary

<table>
<tr><th rowspan="2">$d - t$</th><th colspan="2">$\varphi_a^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th></tr>
<tr><td>$d-1$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td></tr>
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<tr><td>$\omega(d^{3/4}(\ln d)^{1/4})$</td><td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td></tr>
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<tr><td>$O(d^{1/3})$</td><td>$O((d-t)d)$</td></tr>
<tr><td>$O(\ln d)$</td><td>$O(d \ln d)$</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
</table>