Largest sparse subgraphs of random graphs

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The stability of random graphs

Notation:

- $G_{n,p}$ — Erdős-Rényi random graph on $n$ vertices, $0 < p < 1$.
- A property holds asymptotically almost surely (a.a.s.) if it holds with probability tending to one as $n \to \infty$.
- Denote $b = \frac{1}{1-p}$. (Note $\log b \to p$ if $p \to \infty$.)
- $\chi(G)$ denotes chromatic number of $G$.
- $\alpha(G)$ denotes the stability of $G$. 
The stability of random graphs

Theorem (Bollobás and Erdős 1976, Matula 1976)

If

$$\alpha_{0,p}(n) = 2\log_b n - 2\log_b \log_b(np) + 2\log_b(\frac{e}{2}) + 1,$$

then

$$\lfloor \alpha_{0,p}(n) - \delta \rfloor \leq \alpha(G_{n,p}) \leq \lceil \alpha_{0,p}(n) + \delta \rceil \text{ a.a.s.}$$
The chromatic number of random graphs

Theorem (Bollobás 1988, Matula and Kučera 1990)

\[ \chi(G_{n,p}) = (1 + o(1)) \frac{n}{2\log_b n} \text{ a.a.s.} \]

Extensions to more general parameters

A graph property $\mathcal{P}$ is *hereditary* if it is closed under taking induced subgraphs. The $\mathcal{P}$-stability $\alpha_\mathcal{P}(G)$ of $G$ is the order of a largest vertex subset of $G$ that induces a subgraph which satisfies $\mathcal{P}$.

The $t$-stability $\alpha^t(G)$ of $G$ is the order of a largest vertex subset of $G$ that induces a subgraph of maximum degree at most $t$. The $t$-sparsity $\hat{\alpha}^t(G)$ of $G$ is the order of a largest vertex subset of $G$ that induces a subgraph of average degree at most $t$. Note $\alpha^0(G) = \hat{\alpha}^0(G) = \alpha(G)$. 
Extensions to more general parameters

Theorem (Bollobás and Thomason 2000)

For fixed $0 < p < 1$ and non-trivial hereditary $\mathcal{P}$, there exists $c_p, \mathcal{P}$ such that a.a.s.

$$(c_p, \mathcal{P} - \delta) \log n \leq \alpha_{\mathcal{P}}(G_{n,p}) \leq (c_p, \mathcal{P} + \delta) \log n.$$
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Indeed, a.a.s.

$$\chi_{\mathcal{P}}(G_{n, p}) = \frac{n}{(c_p, \mathcal{P} + o(1)) \log n}.$$
Extensions to more general parameters

Theorem (K and McDiarmid 2010)

For fixed $0 < p < 1$, there exists $\kappa_p(\tau)$, continuous, strictly increasing for $\tau \in [0, \infty)$, with $\kappa_p(0) = \frac{2}{\log b}$ and $\kappa_p(\tau) \sim \frac{\tau}{p}$ as $\tau \to \infty$ such that a.a.s.

$$(\kappa_p\left(\frac{t}{\log n}\right) - \delta) \log n \leq \alpha_t(G_{n,p}) \leq \hat{\alpha}_t(G_{n,p}) \leq (\kappa_p\left(\frac{t}{\log n}\right) + \delta) \log n$$

if $t(n) = o(n)$. 
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An analogous statement for $\chi_t(G_{n,p})$ and $\hat{\chi}_t(G_{n,p})$. 
The \( t \)-stability of random graphs

**Theorem (Fountoulakis, K and McDiarmid 2010)**

For fixed \( 0 < p < 1 \), \( \delta > 0 \) and \( t \geq 0 \), if

\[
\alpha_{t,p}(n) = 2 \log_b n + (t - 2) \log_b \log_b np + \log_b \left( \frac{t^t}{t!^2} \right) + t \log_b \left( \frac{2bp}{e} \right) + 2 \log_b \left( \frac{e}{2} \right) + 1,
\]

then

\[
\left\lfloor \alpha_{t,p}(n) - \delta \right\rfloor \leq \alpha_t(G_{n,p}) \leq \left\lceil \alpha_{t,p}(n) + \delta \right\rceil \text{ a.a.s.}
\]
The \( t \)-sparsity of random graphs

**Theorem**

For fixed \( 0 < p < 1 \) and \( t \geq 0 \), if \( \delta = \delta(n) = \frac{(\log \log n)^2}{\log n} \) and

\[
\hat{\alpha}_{t,p}(n) = 2 \log_b n + (t - 2) \log_b \log_b np - t \log_b t \\
+ t \log_b (2bpe) + 2 \log_b \left( \frac{e}{2} \right) + 1,
\]

then

\[
\lfloor \hat{\alpha}_{t,p}(n) - \delta \rfloor \leq \hat{\alpha}_t(G_{n,p}) \leq \lceil \hat{\alpha}_{t,p}(n) + \delta \rceil \text{ a.a.s.}
\]
The difference

\[ \hat{\alpha}_{t,p}(n) - \alpha_{t,p}(n) = 2 \log_b \frac{t!}{(t/e)^t} \sim \log_b(2\pi t) \text{ as } t \to \infty. \]
Concluding remarks

- Rather than analytic techniques, large deviations estimates for both first and second moment are applied to obtain tight bounds.
- These techniques extend modestly to the case where $p \to 0$ as $n \to \infty$, though new ideas may be necessary for very sparse random graphs.
- Some precise bounds for the analogous chromatic numbers have been obtained.