Extension from precoloured sets of edges

Ross J. Kang*

Radboud University Nijmegen

Graphs and Matroids

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Constrained colouring

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proper colouring of elements of a (multi)graph

SUBJECT TO

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It could be, for example, that the adversary

- sets a (large enough) list of possible colours for each element OR
- properly precolours some (sparse enough) set of elements.

Constrained colouring

Thomassen's Theorem (1994)

THEOREM. *Let G be a near-triangulation; i.e., G is a planar graph which has no loops or multiple edges and which consists of a cycle $C: v_1 v_2 \cdots v_p v_1$, and vertices and edges inside C such that each bounded face is bounded by a triangle. Assume that v_1 and v_2 are colored 1 and 2, respectively, and that $L(v)$ is a list of at least three colors if $v \in C - \{v_1, v_2\}$ and at least five colors if $v \in G - C$. Then the coloring of v_1 and v_2 can be extended to a list coloring of G.*

Albertson's Theorem (1998)

THEOREM 4. *Suppose $\chi(G) = r$ and $W \subseteq V(G)$ such that the distance between any two vertices in W is at least 4. Any $(r+1)$-coloring of W can be extended to an $(r+1)$-coloring of G.*

The List Colouring Conjecture

$\chi'(G)$ is the chromatic index, the fewest colours in proper edge-colouring of G.

$\mathrm{ch}'(G)$ is the list chromatic index, the minimum k such that however adversary assigns k-lists to edges, there is always proper edge-colouring of G from lists.

The List Colouring Conjecture (late 1970's)

$ch'(G) = \chi'(G)$ for any multigraph G.

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- Galvin (1995) proved it for bipartite multigraphs.
- Kahn (1996/2000) proved an asymptotic, approximate version.
- Other settled cases: odd complete graphs, planar graphs of large maximum degree, graphs of large girth, . . .

Precolouring extension for edge-colourings

What conditions guarantee a proper edge-colouring of a (multi)graph subject to an adversarial proper edge-precolouring?

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(Note: Marcotte and Seymour (1990).)

Conditions (today) can be on

- the set $\mathcal{K}$ (which we call the palette) of globally available colours — usually we use $\mathcal{K} = [K] = \{1, \dots, K\}$ for some K — and
- the precoloured set M of edges — usually we demand M to be a matching, in which case a proper precolouring of M is an arbitrary colouring from $\mathcal{K}$.

So we can have $\mathcal{K}$ large or M sparse or combinations.

Precolouring extension for edge-colourings

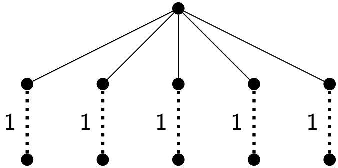
Conjecture (EHKS 2014+)

For G a multigraph with maximum degree $\Delta(G)$, maximum multiplicity $\mu(G)$, using palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

An "adversarial" strengthening of Vizing–Gupta Theorem (1960's).

Tree examples

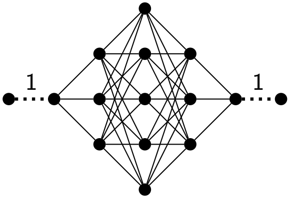
The conjecture is sharp with respect to $\mathcal{K}$, even for trees:



$\chi' = \Delta = 5$, no extension with $\mathcal{K} = [5]$, matching is *induced*.

Long bipartite examples

One might wonder if sparser M permits smaller $\mathcal{K}$, but, alas, no:



$\chi' = \Delta = 6$, no extension with $\mathcal{K} = [6]$.

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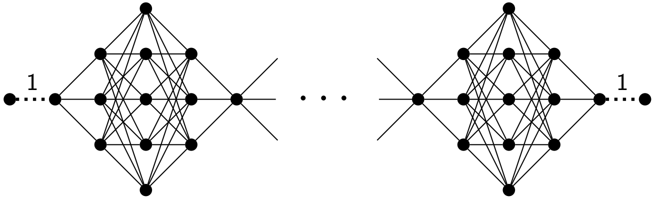

$\chi' = \Delta = 6$, no extension with $\mathcal{K} = [6]$, the two edges have arbitrary distance.

Relationship with earlier work

Theorem (Berge and Fournier 1991)

For G a multigraph with maximum degree $\Delta(G)$, maximum multiplicity $\mu(G)$, using palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any monochromatically precoloured matching can be extended to a proper edge-colouring of all of G.

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Proposition (Albertson 1998)

For G a multigraph with maximum degree $\Delta(G)$, maximum multiplicity $\mu(G)$, using palette $\mathcal{K} = [\Delta(G) + \mu(G) + 1]$, any precoloured distance-3 matching can be extended to a proper edge-colouring of all of G.

Relationship with list edge-colouring

Proposition

For G a multigraph with list chromatic index $ch'(G)$, using palette $\mathcal{K} = [ch'(G)+2]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

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Each edge of G incident to ≤ 2 edges of M, so after forbidding colours of M from incident uncoloured edges, each still has a list of $\geq \mathrm{ch}'(G)$ colours. $\square$

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Corollary (Kahn 1996/2000)

For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds. For any multigraph G with $\chi'(G) \geq C_\varepsilon$, using palette $\mathcal{K} = [(1+\varepsilon)\chi'(G)]$, any precoloured matching can be extended to a proper edge-colouring of all of G.

More supporting evidence for our conjecture

Conjecture (EHKS 2014+)

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- *conjecture is true for bipartite multigraphs (sharp);*
- *conjecture is true for subcubic multigraphs (NB: subcubic OPEN for LCC);*
- *conjecture is true for planar graphs of large maximum degree (sharp);*
- *for planar graphs G of large maximum degree, it suffices that $\mathcal{K} = [\Delta(G)]$ and M be distance-3 (a sharp strengthening of a result of Vizing 1965);*
- *precolouring analogue of Shannon's Theorem (1949), except for extra $+\frac{1}{2}$.*

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Next: even more evidence? . . . or counterexamples?

Shameless announcement

Utrecht Combinatorics Day

Friday, 7 November 2014

Note: Pretty, historic city; close to Schiphol; convenient access to Belgian beer.

Organisers: Tobias Müller (UU) and me.

Thank you!