Guarantees of sparse or dense subgraphs

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*The talk covers joint works with Eoin Long, Janos Pach, Viresh Patel and Guus Regts.
A clique has all possible edges and a stable set has none.

The \textit{clique number} \( \omega \) is the size of a largest clique.

The \textit{stability number} \( \alpha \) is the size of a largest stable set.
\( \alpha_c \) and \( \omega_c \)

Consider sets “close” to cliques or stable sets, tuned by a parameter\(^\dagger \) \( c \in [0, 1] \).

\(^{\dagger}\) Note (or foreshadowing): for now consider \( c \) as fixed.
Consider sets “close” to cliques or stable sets, tuned by a parameter $c \in [0, 1]$.

A vertex subset with $\ell$ vertices
- of minimum degree $\geq c(\ell - 1)$ is called a $c$-clique;
- of maximum degree $\leq (1 - c)(\ell - 1)$ is called a $c$-stable set.

$\alpha_c$ is size of a largest $c$-stable set.

$\omega_c$ is size of a largest $c$-clique.

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is called a \( c \)-clique;

is called a \( c \)-stable set.

\( \omega_c \) is size of a largest \( c \)-clique.

\( \alpha_c \) is size of a largest \( c \)-stable set.

How does the behaviour change as we tune \( c \) between 0 and 1?

†Note (or foreshadowing): for now consider \( c \) as fixed.
Ramsey numbers‡

Ramsey (1930) proved the existence of

\[ R(k) = \min \{ n : |V(G)| = n \Rightarrow \max \{ \alpha(G), \omega(G) \} = k \}. \]

‡Picture borrowed from the cover of Soifer (2009).
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Theorem (Erdős 1947, Erdős and Szekeres 1935)

\[ \sqrt{2^{k + o(k)}} \leq R(k) \leq 4^{k - o(k)} \] as \( k \to \infty. \)

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Theorem (Spencer 1977, Conlon 2009)

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Quasi-Ramsey numbers

Ramsey (1930) still implies, for any $c \in [0, 1]$, the existence of

$$R^*_c(k) = \min \{n : |V(G)| = n \Rightarrow \max\{\alpha_c(G), \omega_c(G)\} = k\}$$ and

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Note that $R^*_c(k) \geq R_c(k)$ always, and both parameters are monotone in $c$.

Moreover, $R_0(k) = R^*_0(k) = k$ and $R_1(k) = R^*_1(k) = R(k) = \exp(\Theta(k))$. 

As we tune $c$ between 0 and 1, how does $R^*_c(k)$ or $R_c(k)$ change? From when is it superlinear in $k$? ... superpolynomial? ... exponential?
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Proposition (Erdős and Pach 1983)
Fix \( c \in [0, 1] \).

- If \( c < 1/2 \), then \( R^*_c(k) = \Theta(k) \).
- If \( c > 1/2 \), then \( R_c(k) = \exp(\Theta(k)) \).
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An intuition for this transition comes from \( \max \{ \alpha_c(G_{n,1/2}), \omega_c(G_{n,1/2}) \} \).
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What happens at \( c = 1/2 \)?
Magnification

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What happens at $c = 1/2$? 

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- of minimum degree $\geq c(\ell - 1)$ is called a $c$-clique;
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Consider $c = 1/2 + \varepsilon$ where $\varepsilon = \varepsilon(\ell)$ is a real function tending to 0 as $\ell \to \infty$. 
The “variable” quasi-Ramsey numbers

\[ R_c(k) = \min \{ n : |V(G)| = n \Rightarrow \max \{ \alpha_c(G), \omega_c(G) \} \geq k \}. \]
Variable sharp threshold

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Theorem (Erdős and Pach 1983)

\[ R_{1/2}(k) = O(k \log k) \] and \[ R_{1/2}(k) = \Omega(k \log k / \log \log k). \]
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Theorem (Kang, Pach, Patel and Regts 2015)

*For some nonnegative real function \( \nu = \nu(\ell) \), let\( c = 1/2 + \nu \sqrt{\log \ell/\ell}. \)\)
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For some nonnegative real function $$\nu = \nu(\ell)$$, let $$c = 1/2 + \nu \sqrt{\log \ell / \ell}$$.

- If $$\nu = o(1)$$ as $$\ell \to \infty$$, then $$R_c(k) = k^{1+o(1)}$$ as $$k \to \infty$$.
- If $$\nu = \Theta(1)$$ as $$\ell \to \infty$$, then $$R_c(k) = k^{\Theta(1)}$$ as $$k \to \infty$$.
- If $$\nu = \omega(1)$$ as $$\ell \to \infty$$, then $$R_c(k) = k^{\omega(1)}$$ as $$k \to \infty$$. 

Again an intuition for this transition comes from $$\max\{\alpha_c(G, 1/2), \omega_c(G, 1/2)\}.$$
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- If \( \nu = \Theta(1) \) as \( \ell \to \infty \), then \( R_c(k) = k^{\Theta(1)} \) as \( k \to \infty \).
- If \( \nu = \omega(1) \) as \( \ell \to \infty \), then \( R_c(k) = k^{\omega(1)} \) as \( k \to \infty \).

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Theorem (Erdős and Pach 1983)

\( R^*_{1/2}(k) = O(k^2). \) (\( R^*_{1/2}(k) = \Omega(k \log k / \log \log k) \) by previous.)
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Problem (Erdős and Pach 1983)

“We suspect that the order of magnitude of $R^*_{1/2}(k)$ is in fact close to $k \log k$.”
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Theorem (Kang, Long, Patel and Regts 2016+)
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Proof outline

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of $k$ vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.
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Call a subset of $\ell$ vertices excessive if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.
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1. A “variable” quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
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Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

1. A “variable” quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
2. Reduction from a $\Omega(\sqrt{\ell})$ excessive set of $Dk$ vertices, $D > 1$ fixed, to an excessive set of exactly $k$ vertices.
3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.
Proof ingredient 1: graph discrepancy

Lemma (1)

For fixed $\nu \geq 0$, let $c = \frac{1}{2} + \frac{\nu}{\sqrt{\ell} - 1}$. Then $R_c(k) = O(k \log k)$.

$\iff$

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of $\ell \geq k$ vertices inducing minimum degree $\ell/2 + \Omega(\sqrt{\ell})$ in the graph or complement.
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Note: This improves on the Erdős and Pach bound $R_{1/2}(k) = O(k \log k)$. 
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Theorem (Erdős and Spencer 1972)

\[ e(S) - \frac{1}{2} \left( \frac{|S|}{2} \right) \]
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Theorem (Erdős and Spencer 1972)
For $n$ large any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega \left( n^{3/2} \right).$$
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Theorem (Erdős and Spencer 1974)
For $n$ large and $\frac{1}{2} \log_2 n < t \leq n$ any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V, |S| \leq t} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega \left( t^{3/2} \sqrt{\log(n/t)} \right).$$
Proof ingredient 2: set system discrepancy

Lemma (2)

Suppose $X$ is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu \sqrt{\ell}$. Some $X' \subseteq X$ of size $k$ induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$. 
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Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)
For $A_1, \ldots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all $i$ $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.

"Six standard deviations suffice."
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Proof of Lemma (2).
Writing $X = [\ell]$, let $A_i \subseteq X$ be neighbourhood $N(i)$ of $i \in [\ell - 1]$, and $A_\ell = X$. 

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Apply Theorem to $A_1, \ldots, A_\ell$ with $p = (k + 1 + 6\sqrt{\ell})/\ell$ (done if $p > 1$)
to produce $Y \subseteq [\ell]$ such that $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell}$. 
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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell / 2 + \nu \sqrt{\ell}) - 6\sqrt{\ell} \geq k / 2 + \nu k / \sqrt{\ell} + 1 - 3\sqrt{\ell}$. 
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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}$.
Proof ingredient 2: set system discrepancy

**Lemma (2)**

Suppose $X$ is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu \sqrt{\ell}$. Some $X' \subseteq X$ of size $k$ induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.

**Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)**

For $A_1, \ldots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all $i$

$$||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}.$$  

"Six standard deviations suffice."

**Proof of Lemma (2).**

Writing $X = [\ell]$, let $A_i \subseteq X$ be neighbourhood $N(i)$ of $i \in [\ell - 1]$, and $A_\ell = X$. Apply Theorem to $A_1, \ldots, A_\ell$ with $p = (k + 1 + 6\sqrt{\ell})/\ell$ (done if $p > 1$) to produce $Y \subseteq [\ell]$ such that $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell}$.

$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu \sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}$.

Take $X' \subseteq [\ell - 1]$ arbitrary with $|X'| = k$. By the above, for all $i \in X'$

$$|N(i) \cap X'| \geq k/2 + \nu k/\sqrt{\ell} - 15\sqrt{\ell} = k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}.$$  

$\square$
Lemma (3)

Suppose $X$ is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$. Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either $X_1$ induces minimum degree $\geq \delta_1$ or $X_2$ induces minimum degree $\geq \delta_2$.
Lemma (3)
Suppose $X$ is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$. Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either $X_1$ induces minimum degree $\geq \delta_1$ or $X_2$ induces minimum degree $\geq \delta_2$.

Proof.
Start with $X_1, X_2$ an arbitrary partition of $X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$. 
Proof ingredient 3: greedy swaps

Lemma (3)
Suppose $X$ is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\delta \geq \delta = \delta_1 + \delta_2$. Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either $X_1$ induces minimum degree $\geq \delta_1$ or $X_2$ induces minimum degree $\geq \delta_2$

Proof.
Start with $X_1, X_2$ an arbitrary partition of $X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$. If $a \in X_1$ has $\deg_{X_1}(a) \leq \delta_1 - 1$ and $b \in X_2$ has $\deg_{X_2}(b) \leq \delta_2 - 1$, swap them.
Lemma (3)

Suppose $X$ is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$. Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either $X_1$ induces minimum degree $\geq \delta_1$ or $X_2$ induces minimum degree $\geq \delta_2$.

Proof.

Start with $X_1, X_2$ an arbitrary partition of $X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$. If $a \in X_1$ has $\deg_{X_1}(a) \leq \delta_1 - 1$ and $b \in X_2$ has $\deg_{X_2}(b) \leq \delta_2 - 1$, swap them. The number of edges in $X_1$ increases by at least

$$\deg_{X_1}(b) - \deg_{X_1}(a) - 1 \geq \delta - \deg_{X_2}(b) - \deg_{X_1}(a) - 1 \geq \delta - \delta_2 - \delta_1 + 1 = 1$$

(where the $-1$ accounts for the possibility of the edge $ab$).
Lemma (3)

Suppose $X$ is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$. Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either $X_1$ induces minimum degree $\geq \delta_1$ or $X_2$ induces minimum degree $\geq \delta_2$.

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$$\deg_{X_1}(b) - \deg_{X_1}(a) - 1 \geq \delta - \deg_{X_2}(b) - \deg_{X_1}(a) - 1 \geq \delta - \delta_2 - \delta_1 + 1 = 1$$

(where the $-1$ accounts for the possibility of the edge $ab$).

At some point we cannot find two vertices to swap, but then we are done. \qed
Proof ingredients

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of $k$ vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

Call a subset of $\ell$ vertices excessive if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

1. A “variable” quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
2. Reduction from a $\Omega(\sqrt{\ell})$ excessive set of $Dk$ vertices, $D > 1$ fixed, to an excessive set of exactly $k$ vertices.
3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

First apply 1.
Proof ingredients

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size $Dk$, with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2.
Proof ingredients

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of $k$ vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size $Dk$, with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2. Otherwise apply 3 repeatedly to partition an excessive set of size $\equiv 0 \pmod{k}$ into roughly equal parts of size $\equiv 0 \pmod{k}$, one excessive.
Summary and open questions

For the quasi-Ramsey numbers

\[ R^*_c(k) = \min \{ n : |V(G)| = n \Rightarrow \max \{\alpha_c(G), \omega_c(G)\} = k \} \] and

\[ R_c(k) = \min \{ n : |V(G)| = n \Rightarrow \max \{\alpha_c(G), \omega_c(G)\} \geq k \}, \]

- identified a sharp transition for \( R_c(k) \) at \( c = 1/2 + \Theta(\sqrt{\log \ell / \ell}) \), and
- solved a problem of Erdős and Pach by showing \( R^*_{1/2}(k) = O(k \log k) \).
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- The remaining \( \log \log k \) factor for \( R^*_1/2(k) \)?
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- The remaining log log \( k \) factor for \( R^*_{1/2}(k) \)?
- How strict could the inequality \( R_c(k) \leq R^*_c(k) \) be?
For the quasi-Ramsey numbers

\[ R_c^*(k) = \min \{ n : |V(G)| = n \Rightarrow \max \{ \alpha_c(G), \omega_c(G) \} = k \} \quad \text{and} \]
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Open questions:

- The remaining \( \log \log k \) factor for \( R_{1/2}^*(k) \)?
- How strict could the inequality \( R_c(k) \leq R_c^*(k) \) be?
- For fixed \( c \in (1/2, 1) \), is \( \limsup_{k \to \infty} k^{-1} \log(R(k)/R_c(k)) > 0 \)?

Hypergraphs? (See next page.)
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\[ R_c^*(k) = \min \{ n : |V(G)| = n \Rightarrow \max\{\alpha_c(G), \omega_c(G)\} = k \} \] and
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- Hypergraphs? (See next page.)
Thank you!

    Discrepancy and large dense monochromatic subsets.
    Submitted, 14 pp. arXiv:1610.06359

    On a Ramsey-type problem of Erdős and Pach.

    A precise threshold for quasi-Ramsey numbers.

[ErPa83] Paul Erdős and János Pach.
    On a quasi-Ramsey problem.
Heterogeneously weighted random graph

\[ R_{1/2}(k) = \Omega(k \log k / \log \log k). \]

i.e. there is some graph on \( Ck \log k / \log \log k \) vertices such that any set of \( \ell \geq k \) vertices is excessive in neither the graph nor complement.
Heterogeneously weighted random graph


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i.e. there is some graph on \( Ck \log k / \log \log k \) vertices such that any set of \( \ell \geq k \) vertices is excessive in neither the graph nor complement.

Let \( z = \frac{\zeta \log k}{\log \log k} \) for some suitably chosen fixed \( \zeta > 0 \).

Let \( V = V_1 \cup \cdots \cup V_z \) where \( |V_1| = \cdots = |V_z| = \left( 1 - \frac{1}{2z} \right) k \).

Generate \( E \) randomly for any \( v_i \in V_i \) and \( v_j \in V_j \) by

\[
P(v_i; v_j \in E) = \begin{cases} 
\frac{1}{2} - (2z)^{-4(i+j)-1} & \text{if } i \neq j; \\
\frac{1}{2} + (2z)^{-8i} & \text{if } i = j.
\end{cases}
\]

There is a chance the graph \( G = (V, E) \) has the desired properties.
Proof ingredient 2: set system discrepancy

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For \( A_1, \ldots, A_n \subseteq [n] \) and \( p \in [0, 1] \), there exists \( Y \subseteq [n] \) such that for all \( i \)
\[
||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}.
\]

For \( \mathcal{H} = \{A_1, \ldots, A_n\} \subseteq 2^{[n]} \), define the discrepancy of \( \mathcal{H} \) as

\[
\text{disc}(\mathcal{H}) := \min_{\chi \in \{-1, 1\}^n} \max_{S \in \mathcal{H}} \sum_{i \in S} \chi(i).
\]

Spencer showed that \( \text{disc}(\mathcal{H}) \leq 6\sqrt{n} \) for any such \( \mathcal{H} \).
Proof ingredient 2: set system discrepancy

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For $A_1, \ldots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all $i$ \[ \|A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}. \]

For $\mathcal{H} = \{A_1, \ldots, A_n\} \subseteq 2^{[n]}$, define the discrepancy of $\mathcal{H}$ as
\[ \text{disc}(\mathcal{H}) := \min_{\chi \in \{-1, 1\}^V} \max_{S \in \mathcal{H}} \sum_{i \in S} \chi(i). \]

Spencer showed that $\text{disc}(\mathcal{H}) \leq 6\sqrt{n}$ for any such $\mathcal{H}$.

If $A$ is the incidence matrix of $\mathcal{H}$, i.e. $A$ is the $n \times n$ matrix given by
\[ A_{ij} = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise}. \end{cases} \]

then the linear discrepancy is
\[ \text{lindisc}(\mathcal{H}) := \max_{c \in [0,1]^V} \min_{x \in \{0,1\}^V} \|A(x - c)\|_\infty. \]

Via Lovász, Spencer and Vesztergombi, $\text{lindisc}(\mathcal{H}) \leq 6\sqrt{n}$ for any such $\mathcal{H}$. 
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Apply this result with $c$ the all $p$ vector.
Proof ingredient 1: graph discrepancy

Lemma (1)
For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$. 

Outline proof of Lemma (1).
Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$
\[ D_\nu(X) := |D(X)| - \nu |X|^{3/2} \]
Let $V_0 = V$. Form $V_{i+1}$ in step $i+1$ by letting $X_i \subseteq V_i$ maximise $D_\nu(X_i)$ and $V_{i+1} = V_i \setminus X_i$. Stop after step $t+1$ if $|V_{t+1}| < 1/2 |V|$.
Let $\{i_1, \ldots, i_m\} \subseteq [t]$ be those $i$ with $D(X_{i_j}) > 0$. Wlog $\sum_{j \in [m]} |X_{i_j}| \geq 1/4 |V|$.

Claim 1
For any $j \in [m]$, $H_{i_j}$ has minimum degree $\geq 1/2 (|X_{i_j}| - 1) + \nu \sqrt{|X_{i_j}| - 1}$.

Claim 2
For any $\ell \in [m-3]$, $D(X_{i_{\ell+3}}) \leq 5/6 D(X_{i_\ell})$.

Then $\left(\frac{5}{6}\right) (m-1)/3 D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m-1 \leq 3 \log \frac{6}{5} / \frac{5}{6} D(X_{i_1}) \leq 6 \log \frac{6}{5} / 5 k$.
Pigeonhole guarantees some $|X_{i_j}| \geq |V| \log (6/5)^{25 \log k} = C \log (6/5)^{25 k} \geq k$ if $C \geq 25 \log (6/5)$.
Proof ingredient 1: graph discrepancy

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Claim 1 implies that $X_i$ is the desired subset.
Proof ingredient 1: graph discrepancy

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Claim 1
For any $j \in [m]$, $H_{i_j}$ has minimum degree $\geq \frac{1}{2} (|X_{i_j}| - 1) + \nu \sqrt{|X_{i_j}| - 1}$. 

Claim 2
For any $\ell \in [m - 3]$, $D(X_{i_\ell + 3}) \leq \frac{5}{6} D(X_{i_\ell})$. Then $(\frac{5}{6})^{(m - 1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log \frac{6}{5} \cdot \frac{5}{5} k \leq 6 \log \frac{6}{5} k$. 

Pigeonhole guarantees some $|X_{i_j}| \geq |V| \log \left(\frac{6}{5}\right)^{25 \log k} \geq k$ if $C \geq 25 \log \left(\frac{6}{5}\right)$. Claim 1 implies that $X_{i_j}$ is the desired subset.
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For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

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Let $\{i_1, \ldots, i_m\} \subseteq [t]$ be those $i$ with $D(X_i) > 0$. Wlog $\sum_{j \in [m]} |X_{i_j}| \geq \frac{1}{4}|V|$.

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Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k.$
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Lemma (1)
For fixed \( \nu \geq 0 \), let \( c = \frac{1}{2} + \frac{\nu}{\sqrt{\ell - 1}} \). Then \( R_c(k) = O(k \log k) \).

Outline proof of Lemma (1).
Let \( G = (V, E) \) be a graph on \( Ck \log k \) vertices and define for any \( X \subseteq V \)
\[
D_\nu(X) := |D(X)| - \nu |X|^{3/2}.
\]

Let \( V_0 = V \). Form \( V_{i+1} \) in step \( i + 1 \) by letting \( X_i \subseteq V_i \) maximise \( D_\nu(X_i) \) and \( V_{i+1} = V_i \setminus X_i \). Stop after step \( t + 1 \) if \( |V_{t+1}| < \frac{1}{2} |V| \).
Let \( \{i_1, \ldots, i_m\} \subseteq [t] \) be those \( i \) with \( D(X_i) > 0 \). Wlog \( \sum_{j \in [m]} |X_{i_j}| \geq \frac{1}{4} |V| \).

Claim 1 For any \( j \in [m] \), \( H_{i_j} \) has minimum degree \( \geq \frac{1}{2} (|X_{i_j}| - 1) + \nu \sqrt{|X_{i_j}| - 1} \).

Claim 2 For any \( \ell \in [m - 3] \), \( D(X_{i_{\ell + 3}}) \leq \frac{5}{6} D(X_{i_\ell}) \).

Then \( \left( \frac{5}{6} \right)^{(m-1)/3} D(X_1) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_1) \leq 6 \log_{6/5} k \).

Pigeonhole guarantees some \( |X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k \) if \( C \geq \frac{25}{\log(6/5)} \).
Proof ingredient 1: graph discrepancy

Lemma (1)
For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

Outline proof of Lemma (1).
Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$

$$D_\nu(X) := |D(X)| - \nu |X|^{3/2}.$$

Let $V_0 = V$. Form $V_{i+1}$ in step $i + 1$ by letting $X_i \subseteq V_i$ maximise $D_\nu(X_i)$ and $V_{i+1} = V_i \setminus X_i$. Stop after step $t + 1$ if $|V_{t+1}| < \frac{1}{2}|V|$.

Let $\{i_1, \ldots, i_m\} \subseteq [t]$ be those $i$ with $D(X_i) > 0$. Wlog $\sum_{j \in [m]} |X_{i_j}| \geq \frac{1}{4}|V|$.

Claim 1 For any $j \in [m]$, $H_{i_j}$ has minimum degree $\geq \frac{1}{2}(|X_{i_j}| - 1) + \nu \sqrt{|X_{i_j}| - 1}$.

Claim 2 For any $\ell \in [m - 3]$, $D(X_{i_{\ell+3}}) \leq \frac{5}{6} D(X_{i_\ell})$.

Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k$.

Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

Claim 1 implies that $X_{i_j}$ is the desired subset. \qed
Claim 1 For any $j \in [m]$, $H_{ij}$ has minimum degree $\geq \frac{1}{2}(|X_{ij}| - 1) + \nu \sqrt{|X_{ij}| - 1}$.

Proof of Claim 1.
If not there exists $x \in X_{ij}$ with $\deg_{H_{ij}}(x) < \frac{1}{2}(|X_{ij}| - 1) + \nu \sqrt{|X_{ij}| - 1}$.
Let $X'_{ij} = X_{ij} \setminus \{x\}$. Since $D(X_{ij}) > 0$,

$$D_\nu(X'_{ij}) = e(X'_{ij}) - \frac{1}{2}\left(\frac{|X_{ij}| - 1}{2}\right) - \nu(|X_{ij}| - 1)^{3/2}$$

$$> e(X_{ij}) - \frac{1}{2}\left(\frac{|X_{ij}|}{2}\right) - \nu \sqrt{|X_{ij}| - 1} - \nu(|X_{ij}| - 1)^{3/2}$$

$$> e(X_{ij}) - \frac{1}{2}\left(\frac{|X_{ij}|}{2}\right) - \nu |X_{ij}|^{3/2} = D_\nu(X_{ij})$$

(since $n^{3/2} > \sqrt{n - 1 + (n - 1)^{3/2}}$), contradicting maximality of $D_\nu(X_{ij})$.  \qed