Improper choosability and Property B

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List colouring

A restricted form of graph colouring. An adversary is allowed to choose the palette for each vertex. Each palette is of a guaranteed size.

Introduced by Vizing (1976) and Erdős-Rubin-Taylor (1980) as a method of attacking unrestricted graph colouring problems.

A challenging area of research attacked from a variety of angles:

- algebraic (Alon-Tarsi, 1992),
- topological (Voigt, 1993; Thomassen, 1994), and
- probabilistic (Kahn, 1996; Molloy-Reed, 1998).

Improper colouring

A degree-based relaxation of "proper". Each vertex is permitted to have t neighbours with the same colour. A natural generalisation of colouring first studied in the 1980s*.

A general line of enquiry: how many fewer colours required?
e.g.

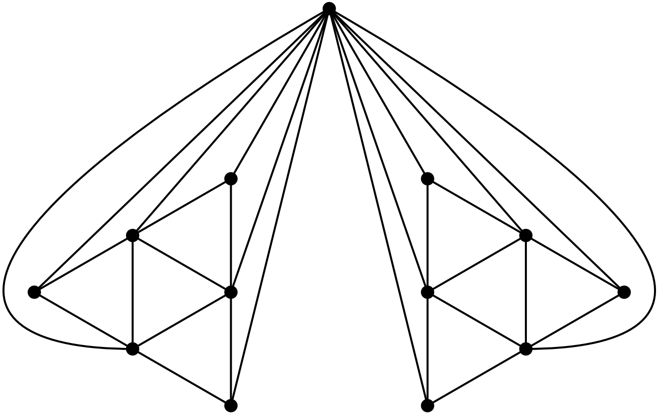
An observation: $$\frac{\chi}{t+1} \leq \chi^t \leq \chi.$$

Lovász (1966): $$\chi^t \leq \frac{\Delta+1}{t+1}.$$

(*So I lied: it was first considered by teenaged Lovász.)

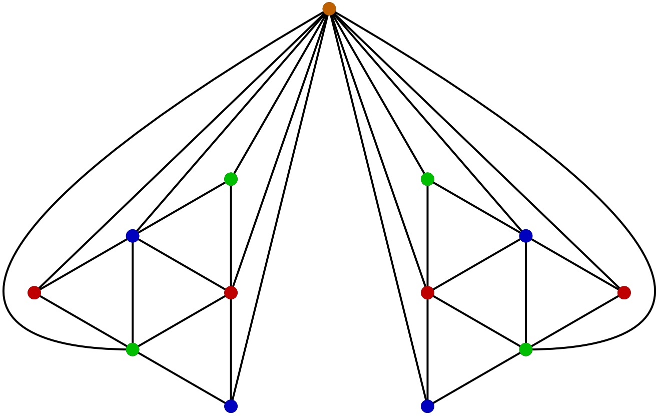
Improper colouring

t



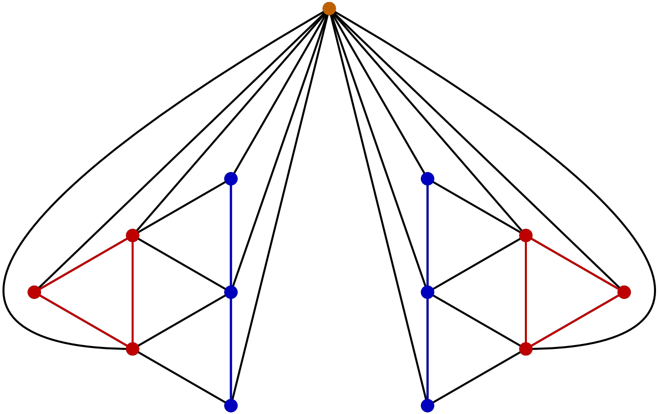
Improper colouring

$t = 0$:



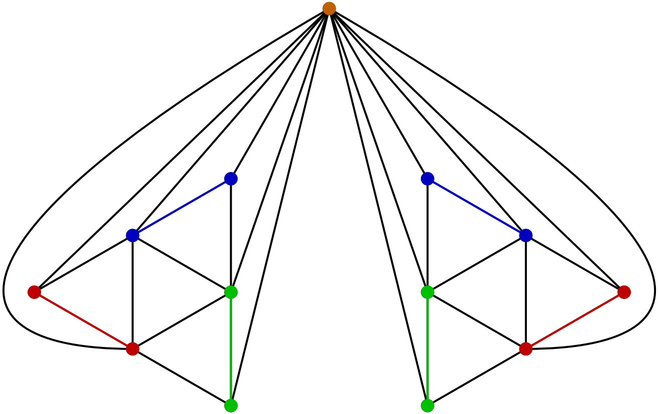
Improper colouring

$t = 2$:



Improper colouring

$t = 1$:



Improper colouring of planar graphs

Theorem (Cowen-Cowen-Woodall, 1986)

- $\chi^2(\mathcal{P}) \leq 3$.
- $\chi^1(\mathcal{P}) \leq 4$ *(sans 4CT).*
- $\chi^t(\mathcal{P}) \geq 3$ *for any* t.
- $\chi^1(\mathcal{P}) \geq 4$.

Improper choosability

The list variant of improper colouring. Let $G = (V, E)$ be a graph.

- $[\ell] = \{1, \ldots, \ell\}$ – *spectrum of colours*. For $k \leq \ell$, $L : V \to \binom{[\ell]}{k}$ is a (k, ℓ)*-list-assignment* of G. And $c : V \to [\ell]$ is an *L-colouring* if $c(v) \in L(v), \forall v \in V$.
- G is *t-improperly* (k, ℓ)*-choosable* if for any (k, ℓ)-list-assignment L there is a t-improper L-colouring. The *t-improper choosability* $\mathrm{ch}^t(G)$ is the least k such that G is t-improperly (k, ℓ)-choosable $\forall \ell \geq k$.

Note t-improperly k-colourable $\equiv$ t-improperly (k, k)-choosable; thus, $\mathrm{ch}^t(G) \geq \chi^t(G)$. Note also $\mathrm{ch}^t(G) \leq \mathrm{ch}(G)$.

Improper choosability of planar graphs

Theorem (Voigt, 1993; Thomassen, 1994)

$\mathrm{ch}^0(\mathcal{P}) = 5$.

Improper choosability first studied, independently, by Eaton-Hull (1999) and Škrekovski (1999); both strengthening Cowen-Cowen-Woodall.

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$\mathrm{ch}^2(\mathcal{P}) \leq 3$.

The case $t = 1$ was open for some time . . .

Theorem (Cushing-Kierstead, 2010)
$\mathrm{ch}^1(\mathcal{P}) \leq 4$.

A general line of enquiry

1. General bounds on ch^t? Does it hold that

$$\mathrm{ch}^t \geq f(\mathrm{ch}),$$

for some increasing function f? Furthermore, does it hold that

$$\mathrm{ch}^t \geq \frac{\mathrm{ch}}{t+1}?$$

2. What about probabilistic methods?

Property B, a.k.a. hypergraph colourability

A family $\mathcal{F}$ has *Property B* if there exists a set B which meets every set in $\mathcal{F}$ but contains no set in $\mathcal{F}$.

For $k, \ell \geq 2$, let $M(k)$ (resp. $M(k, \ell)$) be the size of a smallest family of k-sets (resp. smallest subfamily of $\binom{[\ell]}{k}$) **without** Property B.

(Note $M(k, 2k-1) = \binom{2k-1}{k}$.)

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Asymptotically, best upper bound due to Erdős (1964) and best lower bound by Radhakrishnan-Srinivasan (2000):

$$\Omega\left(\sqrt{\frac{k}{\ln k}}2^k\right) \leq M(k) \leq O\left(k^2 2^k\right).$$

Complete t-improperly multipartite graphs

Let $\mathcal{K}_t(n * j)$ be the class of graphs that admit a vertex partition $(V_1, \ldots, V_j)$ such that, for all i, i' with $i \neq i'$, $|V_i| = n$, $V_i \times V_{i'} \subseteq E$ and V_i induces a subgraph of maximum degree at most t.

Theorem (1)

For fixed $t \geq 0, j \geq 2$, for any $K \in \mathcal{K}_t(n * j)$, as $n \to \infty$,

$$\mathrm{ch}^t(K) = (1 + o(1)) \frac{\ln n}{\ln\left(\frac{j}{j-1}\right)}.$$

Improper choosability and bounded spectrum

Král'-Sgall (2005) were the first to explicitly treat bounded spectrum:

1. Given k, does there exist ℓ_k such that G is properly k-choosable if it is properly (k, ℓ_k)-choosable?
2. For $3 \leq k \leq \ell$ does there exist a number $K_{k,\ell}$ such that each properly (k, ℓ)-choosable graph G is properly $K_{k,\ell}$-choosable?

Improper choosability and bounded spectrum

We extend their answer to the second question to t-improper colouring:

Theorem (2)

Fix $t \geq 0$ and $3 \leq k \leq \ell$, and let $M = M(k,\ell)$ and

$$D = 12M^2 \cdot \ln M \cdot \ln k \cdot \left(1 + \sqrt{1 + \frac{(tk+1)}{3\ln M}}\right)^2.$$

Then $\delta(G) \geq D$ implies G is not t-improperly (k,ℓ)-choosable.

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Then $\delta(G) \geq D$ implies G is not t-improperly (k, ℓ)-choosable.

Corollary

If G is t-improperly $(k, 2k-1)$-choosable, then $\mathrm{ch}(G) = O(2^{4k} kt \ln k)$.

Improper choosability and minimum degree

Recall $\mathrm{ch}(G) \le \delta^*(G) + 1$ where $\delta^*(G) = \max_{H \subseteq G} \delta(H)$.

Another corollary to the last result extends a theorem first established in the case $t = 0$ by Alon (1993):

Corollary

$\mathrm{ch}^t \ge (1/2 + o(1)) \log_2 \delta^*$.

By Theorem (1), $j = 2$, this is correct up to a factor of 2. Furthermore,

Corollary

$\mathrm{ch}^t = \Omega(\ln(\mathrm{ch}))$.

Outline proof of Theorem (2)

Two stages of randomness:

1. Simultaneously, choose small subset $A \subseteq V$ and assign lists to A from subfamily $\mathcal{F}$ of $\binom{[\ell]}{k}$.
    - w.p.p. large number of members of $V \setminus A$ are **good**.
2. Choose a list assignment of the good vertices, again chosen from $\mathcal{F}$.
    - with *negligible* probability, a list colouring of A can be extended to a list colouring of good vertices (using properties of $\mathcal{F}$ and "good").

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- $\mathcal{F}$ is family of size M without Property B.
- “Good”: in A, it sees $tk + 1$ copies of each $F \in \mathcal{F}$.

Open questions

1. Given t, k, does there exist $\ell_{t,k}$ such that G is t-improperly k-choosable if it is t-improperly $(k, \ell_{t,k})$-choosable?
2. $$\mathrm{ch}^t \geq \frac{\mathrm{ch}}{t+1}?$$