Bounded palette list colouring

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List colouring

A classic notion in graph theory from 1970’s.
Aim is to properly colour vertices from individual lists.
Lists chosen by adversary subject to uniform minimum list size.
The optimal minimum so we can always colour is the *choosability*,
or *list chromatic number* or *choice number*, denoted \( \text{ch}(\cdot) \).
List colouring examples

The standard example, $\text{ch}(K_{3,3}) > 2 (= \chi(K_{3,3}))$:

\[
\begin{align*}
\{1, 2\} & \quad \{1, 3\} & \quad \{2, 3\} \\
\{1, 2\} & \quad \{1, 3\} & \quad \{2, 3\}
\end{align*}
\]

More generally, $\text{ch}(K_{n,n}) \sim \log_2 n$ as $n \to \infty$. 
List colouring examples

Another standard example, \( \text{ch}(K_{2,4}) > 2(= \chi(K_{2,4})) \):

\[
\begin{align*}
\{1, 2\} & \quad \{3, 4\} \\
\{1, 3\} & \quad \{1, 4\} & \{2, 3\} & \{2, 4\}
\end{align*}
\]

More generally, \( \text{ch}(K_{n,n^n}) \geq n + 1 \) for all \( n \).
Bounded palette

What if adversary’s ground set of colours pre-determined?
We call this the *palette* and denote it \([s] = \{1, \ldots, s\}\).
How much easier is it be to list colour?
Bounded palette list colouring

\(G = (V, E)\) is a simple undirected graph.

\([s] = \{1, \ldots, s\}\) is the palette.

Any \(L : V \to [s]^{\binom{s}{k}}\) is a \((k, s)\)-list-assignment of \(G\).

\(\forall L, c : V \to [s]\) is an \(L\)-colouring if \(c(v) \in L(v) \ \forall v \in V\).

\(G\) is \((k, s)\)-choosable if for any such \(L\) there is a proper \(L\)-colouring.
Bounded palette list colouring

$G = (V, E)$ is a simple undirected graph.

$[s] = \{1, \ldots, s\}$ is the palette.

Any $L : V \rightarrow \binom{[s]}{k}$ is a $(k, s)$-list-assignment of $G$.

$\forall L, c : V \rightarrow [s]$ is an $L$-colouring if $c(v) \in L(v) \ \forall v \in V$.

$G$ is $(k, s)$-choosable if for any such $L$ there is a proper $L$-colouring.

- $G$ is $k$-choosable iff it is $(k, s)$-choosable for every $s \geq k$.
- $\text{ch}(G)$ is the least such $k$.
- $G$ is $k$-colourable iff it is $(k, k)$-choosable.
Bounded palette list colouring

$K_{3,3}$ is not $(2, 3)$-choosable.

$K_{2,4}$ is not $(2, 4)$-choosable.
TCS ‘overflow’ question

How many distinct colors are needed to lower-bound the choosability of a graph?

A graph is \( k \)-choosable (also known as \( k \)-list-colorable) if, for every function \( f \) that maps vertices to sets of \( k \) colors, there is a color assignment \( c \) such that, for all vertices \( v \), \( c(v) \in f(v) \), and such that, for all edges \( vw \), \( c(v) \neq c(w) \).

Now suppose that a graph \( G \) is not \( k \)-choosable. That is, there exists a function \( f \) from vertices to \( k \)-tuples of colors that does not have a valid color assignment \( c \). What I want to know is, how few colors in total are needed? How small can \( w_{\text{vec}}(f) \) be? Is there a number \( N(k) \) (independent of \( G \)) such that we can be guaranteed to find an uncolorable \( f \) that only uses \( N(k) \) distinct colors?

The relevance to CS is that, if \( N(k) \) exists, we can test \( k \)-choosability for constant \( k \) in singly-exponential time (just try all \( \binom{n}{k} \) choices of \( f \), and for each one check that it can be colored in time \( k^c n^{O(1)} \) whereas otherwise something more quickly growing like \( n^n \) might be required.

Related

- Is it possible to have a 4-coloring for a non-planar graph?
- Complexity of edge coloring in planar graphs
- Do “outer-bounded–genus” graphs have constant treewidth?
- Coloring Planar Graphs
- Conjectures implying Four Color Theorem
- Any relation between the size of maximum independent set and the chromatic number on graph of bounded degree?
Question (Eppstein, November 2010, TCS‘overflow’)

For any $k$, is there some $s_k \geq k$ such that, if $G$ is $(k, s_k)$-choosable, then it is $k$-choosable?

Motivation: a positive answer guarantees a singly-exponential algorithm for $k$-CHOOSABILITY, a $\Pi_2^P$-complete problem.
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Question (Eppstein, November 2010, TCS‘overflow’)

For any \( k \), is there some \( s_k \geq k \) such that, if \( G \) is \((k, s_k)\)-choosable, then it is \( k \)-choosable?

In words, is there \( s_k \geq k \) (independent of the graph) such that \( k \)-choosability is guaranteed by just checking palettes of size \( s_k \)?

Motivation: a positive answer guarantees a singly-exponential algorithm for \( k \)-CHOOSABILITY, a \( \Pi_2^P \)-complete problem.
TCS ‘overflow’ answer

1 Answer

Daniel Král and Jiří Sgall answered your question to the negative. From the abstract of their paper:

A graph $G$ is said to be $(k,\ell)$-choosable if its vertices can be colored from any lists $L(v)$ with $|L(v)| \geq k$, for all $v \in V(G)$, and with $|\bigcup_{v \in V(G)} L(v)| \leq \ell$. For each $3 \leq k \leq \ell$, we construct a graph $G$ that is $(k,\ell)$-choosable but not $(k,\ell + 1)$-choosable.

So, $N(k)$ does not exist if $k \geq 3$. Král and Sgall also show that $N(2) = 4$. Of course, $N(1) = 1$.

Daniel Král, Jiří Sgall: Coloring graphs from lists with bounded size of their union. Journal of Graph Theory 49(3): 177-186 (2005)

answered Feb 11 '11 at 12:45

Serge Gaspers
3,421 16 31

Wow. This settles the question, although negatively. Thank you @Serge! And I wish I could give thanks to Daniel and Jiří tool – Hsien-Chih Chang 潇湘之 Feb 11 '11 at 13:27

I would also have preferred a positive answer to the question. – Serge Gaspers Feb 11 '11 at 14:34
An earlier counterexample

It turns out question was (asked and) answered five years earlier:

Theorem (Král’ & Sgall, 2005)

For all $s \geq k \geq 3$, there exists $G_{k,s}$ that is $(k, s)$-choosable but not $(k, s + 1)$-choosable.

$\implies$ No, mostly.
An earlier counterexample

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\[ \Rightarrow \quad \text{No, mostly.} \]

• On the other hand, \((2, 4)\)-choosability implies 2-choosability.
• \(|V(G_{k,s})| = O(s^2)\); uses precolouring (non)extension.
Follow-up questions

\((k, s)\)-choosability doesn’t guarantee \(k\)-choosability in general.

Question (1)

*Does it imply \(C\)-choosability for some large \(C = C(k, s)\)?*
(k, s)-choosability doesn’t guarantee k-choosability in general.

Question (1)

*Does it imply C-choosability for some large C = C(k, s)?*

Question (2)

*Does it imply (k + 1)-choosability if s = s_{k+1}(k) is large?*
The first follow-up

Does \((k, s)\)-choosability imply \(C\)-choosability for some \(C\)?

No, if \(s < 2k - 1\).

All bipartite graphs \((k, 2k - 2)\)-choosable. (Halve palette.)
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Yes, if \(s \geq 2k - 1\).

Theorem (Král’ & Sgall, 2005, cf. K., 2013)

For \(k \geq 2, s \geq 2k - 1\), there exists \(C = C(k, s)\) such that, if \(G\) is \((k, s)\)-choosable, then it is \(C\)-choosable.
The first follow-up

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Theorem (Král’ & Sgall, 2005, cf. K., 2013)

For \(k \geq 2, s \geq 2k − 1\), there exists \(C = C(k, s)\) such that, if \(G\) is \((k, s)\)-choosable, then it is \(C\)-choosable.

- The choice of \(C\) satisfies \(4^{(1+o(1))k} \leq C \leq 16^{(1+o(1))k} \).
How is this $C$ so large?

For $k \geq 2, s \geq 2k - 1$, there exists $C = C(k,s)$ such that, if $G$ is $(k,s)$-choosable, then it is $C$-choosable.

- The choice of $C$ satisfies $4^{(1+o(1))k} \leq C \leq 16^{(1+o(1))k}$.

Probabilistic proof built upon connection between choosability and degeneracy established by Alon (1993/2000). It relies on the choice

$$C = 12M^2 \ln M \ln k,$$

where $M = M(k,s)$ is an extremal parameter for ‘Property B’.
Property B

Bernstein, 1908: A family $\mathcal{F}$ of sets has *Property B* if

$\exists$ set $B$ that meets every set of $\mathcal{F}$ but contains no set of $\mathcal{F}$.

(Or $\exists$ bipartition of $\bigcup \mathcal{F}$ where no set is contained in one part.
Property B is equivalent to weak 2-colourability of hypergraphs.)
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Property $B$ is equivalent to weak 2-colourability of hypergraphs.)

$M(k, s)$ is size of smallest $\mathcal{F} \subseteq \binom{[s]}{k}$ without Property $B$.

(Or $M(k, s)$ is the least number of hyperedges in a $k$-uniform hypergraph
on $s$ vertices that is not weakly 2-colourable.)
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- e.g. $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ does not have Property B, but any proper subfamily of $\binom{[3]}{2}$ does. So $M(2, 3) = 3$.
- For $k \geq 2$, $M(k, 2k - 1) = \binom{2^{k-1}}{k}$, while $M(k, 2k - 2) = \infty$. 
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$\text{ch}(K_{n,n}) \sim \log_2 n$ (Erdős, Rubin & Taylor, 1980).
Bounds on $M$ (and $C$)

Clearly, $M(k, s)$ is non-increasing in $s$.

\[
\Rightarrow M(k, s) \leq M(k, 2k - 1) = \binom{2k-1}{k} < 2^{2k-1}
\]

\[
\Rightarrow C(k, s) = 12M(k, s)^2 \ln M(k, s) \ln k \leq 16^{(1+o(1))k}
\]
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$$\implies M(k, s) \leq M(k, 2k - 1) = \binom{2k-1}{k} < 2^{2k-1}$$

$$\implies C(k, s) = 12M(k, s)^2 \ln M(k, s) \ln k \leq 16^{(1+o(1))k}$$

Theorem (Erdős, 1969, “On a combinatorial problem III”)

*There is some algebraic decreasing function $f : [2, \infty) \to \mathbb{R}$ satisfying $\lim_{c \downarrow 2} f(c) = 4$ and $\lim_{c \to \infty} f(c) = 2$ such that, if $s \geq 2k - 1$ and $s \sim ck$ as $k \to \infty$, then $M(k, s) = f(c)^{(1+o(1))k}$.*

$$\implies C(k, s) \geq 4^{(1+o(1))k}$$
Bounds on $M$ (and $C$)

Clearly, $M(k, s)$ is non-increasing in $s$.

\[ M(k, s) \leq M(k, 2k - 1) = \binom{2k-1}{k} < 2^{2k-1} \]

\[ \Rightarrow \quad C(k, s) = 12M(k, s)^2 \log M(k, s) \log k \leq 16^{(1+o(1))k} \]

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\[ \Rightarrow \quad C(k, s) \geq 4^{(1+o(1))k} \]

$M(k) = \inf_{s \geq 2k-1} M(k, s)$. Radakrishnan & Srinivasan (2000).
Does $C$ have to be so large?

For $k \geq 2, s \geq 2k - 1$, there exists $C = C(k, s)$ such that, if $G$ is $(k, s)$-choosable, then it is $C$-choosable.

Question (Král’ & Sgall, 2005)

Must $C(k, 2k - 1)$ be exponentially large in $k$?
Does $C$ have to be so large?

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Yes.

Theorem (Bonamy & K.)

For $k \geq 2$, $s \geq 2k - 1$, there exists $R = R(k, s) \geq \exp((k - 1)^2 / s)$ s.t. $K_{R-1, (R-1)^{R-1}}$ is $(k, s)$-choosable but not $(R - 1)$-choosable.
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$\implies$ alternative to Král’ & Sgall construction when $R(k, s) > k$. 
Property K

A family \( \mathcal{F} \subseteq \binom{[s]}{k} \) of sets has *Property* \( K(k, s) \) if 
\[ \exists \text{ set } K \in \binom{[s]}{k-1} \text{ that meets every set of } \mathcal{F}. \]
Property K

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$\exists$ set $K \in \binom{[s]}{k-1}$ that meets every set of $\mathcal{F}$.

$R(k, s)$ is size of smallest $\mathcal{F} \subseteq \binom{[s]}{k}$ not having Property $K(k, s)$.
(Or $R(k, s)$ is the least number of hyperedges in a $k$-uniform hypergraph on $s$ vertices that has no dominating set of size $k - 1$.)
A family $\mathcal{F} \subseteq {[s]\choose k}$ of sets has Property $K(k, s)$ if
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- e.g. $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ doesn’t have Property $R(2, 3)$, but any proper subfamily of ${[3]\choose 2}$ does. So $R(2, 3) = 3$.
- For $k \geq 2$, $R(k, 2k - 1) = \binom{2k-1}{k}$, while $R(k, 2k - 2) = \infty$. 

A family $\mathcal{F} \subseteq \binom{[s]}{k}$ of sets has Property $K(k, s)$ if there exists a set $K \in \binom{[s]}{k-1}$ that meets every set of $\mathcal{F}$.

$R(k, s)$ is the size of the smallest $\mathcal{F} \subseteq \binom{[s]}{k}$ not having Property $K(k, s)$.

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- For $k \geq 2$, $R(k, 2k - 1) = \binom{2k-1}{k}$, while $R(k, 2k - 2) = \infty$.
- $R(k, k^2) \leq k$ by taking an arbitrary $k$-partition of $[k^2]$ as $\mathcal{F}$. 

Theorem (Bonamy & K.)

For $2 \leq k \leq s$, if $G$ admits a bipartition $V = V_1 \cup V_2$ with $|V_1| < R(k, s)$, then $G$ is $(k, s)$-choosable.

$\implies K_{R-1,(R-1)^{R-1}}$ is $(k, s)$-choosable but not $(R - 1)$-choosable.
Property K and bipartite graphs

(Property K(k, s): ∃ set $K \in \binom{[s]}{k-1}$ that meets every set of $\mathcal{F}$
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For $2 \leq k \leq s$, if $G$ admits a bipartition $V = V_1 \cup V_2$ with
$|V_1| < R(k, s)$, then $G$ is $(k, s)$-choosable.

Proof.

∀ $(k, s)$-list-assignment $L$, $\{L(v) : v \in V_1\}$ has Property K(k, s).

$\exists K \in \binom{[s]}{k-1}$ such that $L(u) \cap K \neq \emptyset$ for all $v_1 \in V_1$.

Since $|K| = k - 1$, $L(v) \setminus K \neq \emptyset$ for all $v_2 \in V_2$.  \qed
Bounds on $R$

Theorem (Bonamy & K.)

For $k \geq 2$, $s \geq 2k - 1$,

$$\frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \leq R(k, s) \leq \frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \ln \left( \frac{s}{k - 1} \right).$$
Bounds on $R$

Theorem (Bonamy & K.)

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$$\frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \leq R(k, s)$$

$$< \frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \ln \left( \frac{s}{k - 1} \right).$$

Proof by probabilistic method.

Lower bound: fix $\mathcal{F} \subseteq \binom{[s]}{k}$ then choose $K \in \binom{s}{k-1}$ u.a.r. . . .

Upper bound: fix $K \in \binom{s}{k-1}$ then choose $\mathcal{F} \subseteq \binom{[s]}{k}$ u.a.r. . . .
Theorem (Bonamy & K.)

For $k \geq 2, s \geq 2k - 1$,

$$\frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \leq R(k, s) \leq \frac{s!(s - 2k + 1)!}{(s - k)!(s - k + 1)!} \ln \left( \binom{s}{k - 1} \right).$$

Proof by probabilistic method.

Lower bound: fix $\mathcal{F} \subseteq \binom{[s]}{k}$ then choose $K \in \binom{[s]}{k - 1}$ u.a.r.

Upper bound: fix $K \in \binom{[s]}{k - 1}$ then choose $\mathcal{F} \subseteq \binom{[s]}{k}$ u.a.r.

Stirling’s $\Rightarrow R(k, s) \geq \exp((k - 1)^2/s)$
The second follow-up

Let $f$ be an increasing positive integer function $f$. Does $(k, s_f(k))$-choosability imply $f(k)$-choosability for some large enough $s_f(k) = s_f(k)(k)$?
Let $f$ be an increasing positive integer function $f$. 
Does $(k, s_{f(k)})$-choosability imply $f(k)$-choosability for some large enough $s_{f(k)} = s_{f(k)}(k)$?

$s_k$ does not exist except $s_2(2) = 4$.

If $G$ is $(2, 3)$-choosable, then it is $3$-choosable: $s_3(2) = 3$.

For any polynomial $f(k)$, $s_{f(k)} = \Omega \left( \frac{k^2}{\ln k} \right)$ if it exists.

$s_{4.01^k}$ exists and $s_{16.01^k} = 2k - 1$ for $k$ large enough.
Further aspects

Every planar graph is \((5, s)\)-choosable (Thomassen, 1994), but there is a non-\((4, 5)\)-choosable one (Mirzakhani, 1996).

We view bounded palette as way to refine list colouring problems (which are often quite hard).
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Every planar graph is \((5, s)\)-choosable (Thomassen, 1994), but there is a non-\((4, 5)\)-choosable one (Mirzakhani, 1996).

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Theorem (Bonamy & K.)

Any planar graph with max degree 7 is \((8, 9)\)-edge-choosable.

Weak List Colouring Conjecture, restricted to planar graphs:

Any planar graph with max degree \(\Delta\) is \((\Delta + 1)\)-edge-choosable if \(\Delta \leq 4\) (Vizing (1976), Juwan, Mohar & Škrekovski (1999)) and if \(\Delta \geq 8\) (Bonamy (2013+), Borodin (1991)).