For most graphs H, most H-free graphs have a linear homogeneous set

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Abstract

Erdős and Hajnal conjectured that for every graph H there is a constant $\varepsilon = \varepsilon(H) > 0$ such that every graph G that does not have H as an induced subgraph contains a clique or a stable set of order $\Omega(|V(G)|^{\varepsilon})$.

The conjecture would be false if we set $\varepsilon = 1$; however, in an asymptotic setting, we obtain this strengthened form of Erdős and Hajnal's conjecture for almost every graph H, and in particular for a large class of graphs H defined by variants of the colouring number.

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1 Introduction

A *homogeneous set* in a graph G is a clique or stable set. We study the size of a largest homogeneous set in G, denoted by $h(G)$. We are interested in determining how forbidding a fixed graph H as an induced subgraph affects this parameter. In contrast to most earlier work on this topic, we focus on typical rather than extreme behaviour.

Determining general lower bounds on $h(G)$ is a central focus of Ramsey theory. Let $h(n) := \min\{h(G) : |V(G)| = n\}$. The *(diagonal) Ramsey number* $R(k)$ may be defined as the least n such that $h(n) \geq k$. Thus the upper bound $R(k) \leq 2^{2k-2}$ due to Erdős and Szekeres [14] implies that $h(G) \geq \frac{1}{2}\log|V(G)|$ for all G and so $h(n) \geq \frac{1}{2}\log n$ for any n. (All logarithms are to base 2, unless specified otherwise.) Also, the classical probabilistic argument of Erdős [12] giving a lower bound on $R(k)$ shows that $h(G) \leq 2\log|V(G)|$ for almost every[1] graph G; and it follows that $h(n) \leq 2\log n$ for large n. See Conlon [10] for recent work in this area.

Erdős and Hajnal showed that if any fixed graph H is not an induced subgraph of G, then $h(G)$ is significantly larger than for a typical graph with $|V(G)|$ vertices. In fact, they showed a super-logarithmic lower bound: for every graph H there exists $\varepsilon' = \varepsilon'(H) > 0$ such that if G does not contain H as an induced subgraph, then $h(G) \geq \exp(\varepsilon'\sqrt{\log|V(G)|})$. The celebrated Erdős-Hajnal conjecture asserts that in fact a much stronger bound holds.

Conjecture 1.1 (Erdős and Hajnal [13]). *For every graph H, there exists $\varepsilon = \varepsilon(H) > 0$ such that, if G does not contain H as an induced subgraph then $h(G) \geq |V(G)|^{\varepsilon}$.*

Conjecture 1.1 motivates the following definition: a graph H is said to have the *Erdős-Hajnal property* if there exists a constant $\varepsilon = \varepsilon(H) > 0$ such that $h(G) \geq |V(G)|^{\varepsilon}$ for each graph G that does not contain H as an induced subgraph. Rephrased, the Erdős-Hajnal conjecture asserts that every graph has the Erdős-Hajnal property.

Conjecture 1.1 remains open, even when H is a cycle or path on five vertices. Resolving it may require a mixture of probabilistic and structural arguments. Most efforts so far have been on structural decompositions, but in this paper we focus on *random* graphs that do not contain H as an induced

[1] that is, if p_n is the proportion of graphs on vertex set $[n] := \{1, \ldots, n\}$ for which this holds, then $p_n \to 1$ as $n \to \infty$

subgraph. Our goal is to show that, for almost all H, almost every such graph G contains a homogeneous set of size $\Omega(|V(G)|)$. Thus, having forbidden a *typical* H as an induced subgraph, a *typical* G has $h(G)$ within a constant multiplicative factor of the trivial upper bound. We will make this more precise below.

Given a graph H, let $\mathrm{Forb}(H)$ denote the class of all graphs that do not contain H as an induced subgraph. Also, given a class $\mathcal{P}$ of graphs, let $\mathcal{P}^n$ denote the set of graphs in $\mathcal{P}$ on the vertex set $[n]$. The asymptotic behaviour of $|\mathrm{Forb}(H)^n|$ is governed by the *colouring number* $\tau(H)$ of H, which also plays a major part in this paper. It is defined as the least integer t such that for any non-negative integers a and b with $a+b=t$ the vertices of H can be partitioned into a cliques and b stable sets. Prömel and Steger [28] showed that for each graph H (with at least one edge)

$$|\mathrm{Forb}(H)^n| = 2^{\left(1-\frac{1}{\tau(H)-1}+o(1)\right)\binom{n}{2}}. \tag{1}$$

(An extension of this result to all hereditary graph classes was obtained independently by Alexeev [1] and by Bollobás and Thomason [7]. For recent work in this area see [4, 2, 5].)

Recently, an asymptotic version of Conjecture 1.1 was established by Loebl, Reed, Scott, Thomason and Thomassé [22], involving the quantity $|\mathrm{Forb}(H)^n|$ estimated in (1). We say that a graph H has the *asymptotic Erdős-Hajnal property* if there exists a constant $\varepsilon = \varepsilon(H) > 0$ such that

$$\frac{|\{G \in \mathrm{Forb}(H)^n : h(G) \geq n^{\varepsilon}\}|}{|\mathrm{Forb}(H)^n|} \to 1 \text{ as } n \to \infty.$$

Using Szemerédi's Regularity Lemma [30] and a result of Chudnovsky and Safra [8] (which we discuss below), they proved the following theorem.

Theorem 1.2 (Loebl *et al.* [22])**.** *Every graph has the asymptotic Erdős-Hajnal property.*

For the special cases of the cycles C_4 and C_5 (on four and five vertices respectively), a stronger asymptotic property holds. We say that a graph H has the *asymptotic linear Erdős-Hajnal property* if there exists a constant $b = b(H) > 0$ such that

$$\frac{|\{G \in \mathrm{Forb}(H)^n : h(G) \geq bn\}|}{|\mathrm{Forb}(H)^n|} \to 1 \text{ as } n \to \infty.$$

It follows easily from known results that C_4 and C_5 have the asymptotic linear Erdős-Hajnal property, while the path P_3 (on three vertices) does not — see Subsection 1.2 below. It was thus natural for Loebl *et al.* [22] to propose the problem of characterising those H having the asymptotic linear Erdős-Hajnal property. They indicated that P_3 and P_4 might be the only exceptional cases. We are able to give a partial solution to this problem by establishing the asymptotic linear Erdős-Hajnal property in the case when the forbidden induced subgraph is a typical (i.e. random) graph.

Theorem 1.3. *Almost every graph H has the asymptotic linear Erdős-Hajnal property.*

In other words, $p_h \to 1$ as $h \to \infty$, where p_h is the proportion of graphs H on $[h]$ with the asymptotic linear Erdős-Hajnal property. This is our main theorem. In the subsections below, we first describe the plan of the proof and then give further background material.

1.1 Plan of the proof

The statement of Theorem 1.3 implicitly involves 'almost every' twice, but the proof separates them. We define a class $\mathcal{H}$ of graphs, and show that (a) almost every graph is in $\mathcal{H}$, and (b) every graph in $\mathcal{H}$ has the asymptotic linear Erdős-Hajnal property.

In more detail, in Section 2 we introduce a variant $\tau_1(H)$ of the colouring number $\tau(H)$ of a graph H, which satisfies $\tau_1(H) \leq \tau(H)$. We define $\mathcal{H}$ to be the class of graphs H such that $\tau_1(H) < \tau(H)$. We shall observe at the end of that section that $\mathcal{H}$ contains C_4 and C_5 but neither P_3 nor P_4. Theorem 1.3 follows immediately from the next two lemmas. From Section 3 onwards, the rest of the paper falls into two quite separate parts where we prove these two key lemmas.

The first lemma tells us that we can restrict our attention to $\mathcal{H}$.

Lemma 1.4. *Almost every graph is in $\mathcal{H}$.*

The second lemma says that, for each H such that $\tau_1(H) < \tau(H)$, almost all graphs in $\mathrm{Forb}(H)$ have a linear-sized homogeneous set.

Lemma 1.5. *Every graph in $\mathcal{H}$ has the asymptotic linear Erdős-Hajnal property.*

In Section 2, as well as defining the colouring numbers τ and τ_1 discussed above, we also define another colouring number τ_2, which is simpler than τ_1. Lemma 1.4 is proved in Section 3, which starts with a lemma that allows us to work with τ_2 rather than τ_1. Then we adapt methods used recently to obtain precise upper bounds on the chromatic number $\chi(G_p^n)$ of the random graph G_p^n with vertex set $[n]$ and edge probability p to obtain an upper bound on $\tau_2(G_{1/2}^n)$. This, combined with recent precise lower bounds on $\chi(G_p^n)$, and thus on $\tau(G_{1/2}^n)$, yields Lemma 1.4.

The proof of Lemma 1.5 is given in Section 4. It is a modification of the proof of Theorem 1.2 by Loebl *et al.* That proof depends on a decomposition result (Lemma 3 in [22]) of which we give an analogue for our purposes, namely Lemma 4.6 below. We note that the proof of our decomposition result, in contrast to that of the analogous result in [22], does not rely on the bull result of Chudnovsky and Safra (discussed below).

1.2 Further background and related work

Erdős and Hajnal observed in [13] that Conjecture 1.1 holds for H the path P_4 on four vertices. They also proved that the class of graphs H for which the conjecture holds is closed under disjoint union and complementation. Alon, Pach and Solymosi [3] demonstrated that the Erdős-Hajnal property is moreover closed under substitution. More precisely, if H_1 and H_2 have the Erdős-Hajnal property, then any graph obtained by replacing an arbitrary vertex v of H_1 with a copy of H_2 (preserving adjacencies, by including all edges between the copy of H_2 and every neighbour of v in H_1) also has the property. Chudnovsky and Safra [8] showed that the bull (the five-vertex graph which consists of a triangle with pendant vertices added to two vertices) has the Erdős-Hajnal property, with $\varepsilon = 1/4$.

Now let us sketch the easy lower bound part of the inequality (1) of Prömel and Steger [28] — we shall return to this inequality later. Let a and b be non-negative integers such that $a + b = \tau(H) - 1$ and H cannot be partitioned into a cliques and b stable sets. Then in any graph G that can be partitioned into a cliques and b stable sets there cannot be an induced subgraph that is isomorphic to H. By considering a fixed partition of $[n]$ into $\tau(H) - 1$ parts, each of which has size within 1 of $n/(\tau(H) - 1)$, and counting all graphs where the edges between these parts are freely chosen (while a parts induce cliques and b parts induce stable sets), we see that the number of graphs in $\mathrm{Forb}(H)^n$ is at least as given in (1).

We next describe the strong asymptotic structural results of Prömel and Steger [26, 27] for $\mathrm{Forb}(C_4)$ and $\mathrm{Forb}(C_5)$. They showed that the class of split graphs forms almost all of $\mathrm{Forb}(C_4)$ (G is a *split graph* if the vertices of G can be partitioned into a clique and a stable set). This immediately implies that C_4 has the asymptotic linear Erdős-Hajnal property. They also showed that the class of generalised split graphs forms almost all of $\mathrm{Forb}(C_5)$ (G is a *generalised split graph* if the vertices of G or its complement $\overline{G}$ can be partitioned into two parts U and W, one of which is a single clique and the other of which is the disjoint union of cliques). Since almost all of the generalised split graphs admit a partition into U and W such that each part has about half of the vertices (cf. [27]), it follows that C_5 has the asymptotic linear Erdős-Hajnal property.

On the other hand, we note that $\mathrm{Forb}(P_3)$ is the class of all graphs which are the disjoint union of cliques. Note that graphs of order n formed as a disjoint union of cliques are equivalent to set partitions of $[n]$. Aleksandrovskii (cf. Yakubovich [31]) showed that all of the blocks of a uniformly chosen set partition of $[n]$ have length at most $(1+o(1))\ln n$ with probability tending to 1 as $n \to \infty$. Thus there is a sub-class $\mathcal{Q} \subseteq \mathrm{Forb}(P_3)$ such that $|\mathcal{Q}^n|/|\mathrm{Forb}(P_3)^n| \to 1$ as $n \to \infty$ and $h(G) = \Theta(n/\log n)$ for all $G \in \mathcal{Q}^n$. In other words, P_3 does *not* have the asymptotic linear Erdős-Hajnal property.

Extensions to Erdős and Hajnal's super-logarithmic lower bound on h for $\mathrm{Forb}(H)$ have been obtained by Prömel and Rödl [25] and Fox and Sudakov [16, 17]. Very recently, Conlon, Fox and Sudakov [11] considered a form of the Erdős-Hajnal conjecture for k-uniform hypergraphs. Chudnovsky and Zwols [9] showed that every graph of order n which does not contain the path P_5 or the complement $\overline{P_6}$ of the path P_6 as an induced subgraph has a homogeneous set of size $n^{1/6}$.

To close this section, let us note that the truth of Conjecture 1.1 would imply that graphs G with $h(G)$ near $h(|V(G)|)$ must be reasonably random. For if the conjecture is true, then for any $\delta > 0$ every graph G which contains an induced subgraph F with $|V(G)|^{\delta}$ vertices and no induced copy of H has a homogeneous set of size $|V(G)|^{\varepsilon\delta}$, which is much larger than $h(|V(G)|)$ (for large enough n). Thus, if $h(G)$ is near $h(|V(G)|)$, then no such F can exist. Considering a random choice for the vertex set of F allows us to deduce that the number of copies of H in G must be close to that in a typical graph on $|V(G)|$ vertices and the copies of H must be spread throughout G in a fairly typical fashion.

2 Colouring numbers

In this section, we define the colouring numbers τ, τ_1 and τ_2 mentioned earlier, and make some simple observations concerning them.

Let $\mathcal{F}$ be a family of $t \geq 1$ graph classes $\mathcal{A}_i$, $i \in \{1, \ldots, t\}$, where it is assumed that each $\mathcal{A}_i$ contains K_1. Given a graph H, the *$\mathcal{F}$-colouring number* $\tau_{\mathcal{F}}(H)$ of H is the least integer k such that the following holds: for each t-tuple $(n_1, \ldots, n_t)$ of non-negative integers with $\sum_i n_i = k$, there is a partition of $V(H)$ into k sets V_i^j for $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, n_i\}$ such that each induced subgraph $H[V_i^j]$ is isomorphic to a graph in $\mathcal{A}_i$. Observe that $\tau_{\mathcal{F}}(H) \leq |V(H)|$ since each $\mathcal{A}_i$ contains K_1.

A comment on notation: We often permit ourselves an abuse of terminology by saying that a vertex subset $V' \subseteq V(H)$ is in $\mathcal{A}_i$ and write $V' \in \mathcal{A}_i$ when we mean that $H[V']$ is isomorphic to a graph in $\mathcal{A}_i$.

We define eight classes of graphs. We assume that each class contains the null graph and K_1. In the following, we use the standard notation $\overline{G}$ for the complement of the graph G. Also, the notation $G \cup H$ indicates the disjoint union of graphs G and H.

$\mathcal{A}_0$: the class of edgeless graphs.

$\mathcal{A}_1$: the class of complements of the graphs in $\mathcal{A}_0$, i.e. complete graphs.

$\mathcal{B}_1$: the class of graphs $K_a \cup K_b$ for any $a, b \geq 0$.

$\mathcal{B}_2$: the class of complements of the graphs in $\mathcal{B}_1$, i.e. complete bipartite graphs.

$\mathcal{B}_3$: the class of graphs $K_a \cup \overline{K_b}$ for any $a, b \geq 0$.

$\mathcal{B}_4$: the class of complements of the graphs in $\mathcal{B}_3$.

$\mathcal{B}_5$: the class of graphs such that by deleting at most one vertex we obtain a complete graph.

$\mathcal{B}_6$: the class of complements of the graphs in $\mathcal{B}_5$, i.e. graphs such that by deleting at most one vertex we obtain an edgeless graph.

Clearly, the $(\mathcal{A}_0)$-colouring number is the usual chromatic number and the $(\mathcal{A}_1)$-colouring number is the clique cover number. The classes $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_1$, $\ldots$, $\mathcal{B}_6$ are illustrated in Figure 1.

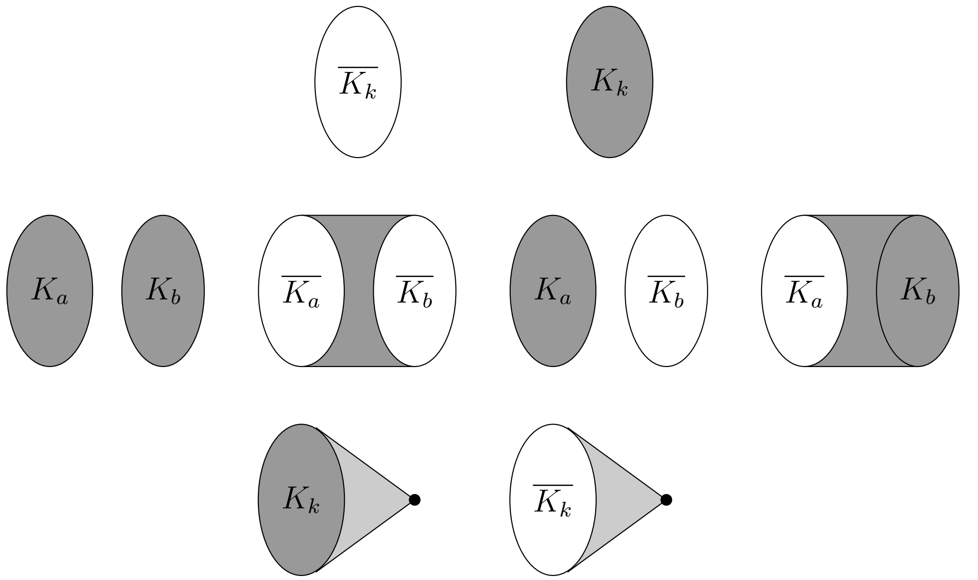

Figure 1: A depiction of the eight classes for colouring numbers.

For our purposes, we are interested in three families composed from the classes above:

- $\mathcal{F}_0 = (\mathcal{A}_0, \mathcal{A}_1)$,
- $\mathcal{F}_1 = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6)$, and
- $\mathcal{F}_2 = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$.

We consider the three associated $\mathcal{F}$-colouring numbers. For convenience, we denote $\tau_{\mathcal{F}_i}(H)$ by $\tau_i(H)$. Note that $\tau_0(H)$ is just the colouring number $\tau(H)$ of H as defined by Prömel and Steger [28, 29].

Let us make two simple observations about $\tau_i(H)$ for $i \in \{0, 1, 2\}$. First, each $\mathcal{F}_i$ is closed under taking complements, so for each graph H, $\tau_i(H) = \tau_i(\overline{H})$. Second, since each class $\mathcal{A}_i$ and $\mathcal{B}_j$ is closed under forming induced subgraphs, if H' is an induced subgraph of H, then $\tau_i(H') \leq \tau_i(H)$.

Clearly, $\tau_2(H) \leq \tau_1(H)$ for any H. Since $\mathcal{A}_0$ is a subclass of each of $\mathcal{B}_2$, $\mathcal{B}_4$, $\mathcal{B}_6$, while $\mathcal{A}_1$ is a subclass of each of $\mathcal{B}_1$, $\mathcal{B}_3$, $\mathcal{B}_5$, we also have $\tau_1(H) \leq \tau(H)$ for any H.

Now let us discuss the class $\mathcal{H}$. Recall that we defined $\mathcal{H}$ to be all graphs H such that $\tau_1(H) < \tau(H)$. Clearly, $H \in \mathcal{H}$ if and only if $\overline{H} \in \mathcal{H}$. Also,

$K_1 \notin \mathcal{H}$. Consider a graph H with $k \geq 2$ vertices. It is easily checked that if H is an induced subgraph of the path P_4, then $\tau(H) = 2$; otherwise, $\tau(H) \geq 3$. If $k = 2$ then $\tau_1(H) = 1$, while if $k \in \{3, 4\}$ then $\tau_1(H) = 2$. We see that K_2 and $\overline{K_2}$ are in $\mathcal{H}$, but of course Lemma 1.5 is trivial for these two graphs. Other graphs in $\mathcal{H}$ have $\tau(H) \geq 3$. We will assume that $\tau(H) \geq 3$ when we prove Lemma 1.5 in Section 4.

We mentioned earlier that C_4 and C_5 were known to have the asymptotic linear Erdős-Hajnal property. From the above we see that C_4 satisfies $\tau_1(H) = 2 < 3 = \tau(H)$ so $C_4 \in \mathcal{H}$; and it is not hard to check that the same result holds for C_5.

3 Colouring numbers of random graphs

In this section we consider colouring numbers of the random graph $G^n_{1/2}$, and prove Lemma 1.4. (We now use n rather than h for the number of vertices.) Recall that $G^n_{1/2}$ is sampled uniformly from the graphs on $[n]$; and that a property holds *asymptotically almost surely (a.a.s.)* if it holds with probability tending to 1 as $n \to \infty$. We may rephrase Lemma 1.4 as saying that $\tau_1(G^n_{1/2}) < \tau(G^n_{1/2})$ a.a.s. We first give a lemma which will allow us to work with τ_2 rather than with τ_1.

3.1 Working with τ_2 rather than with τ_1

Lemma 3.1. *Let H be a graph such that $\tau_2(H) + \alpha(H) + \omega(H) \leq \tau(H) + 1$. Then $\tau_1(H) < \tau(H)$.*

Proof. Let $t = \tau(H) - 1$. Let $n_1, \ldots, n_6$ be non-negative integers with $\sum_i n_i = t$. We must show that H has a colouring consisting of t colour sets, with n_i colour sets in $\mathcal{B}_i$ for each $i \in \{1, \ldots, 6\}$.

Let $m_0 = n_2 + n_4 + n_6$ and $m_1 = n_1 + n_3 + n_5$. Suppose first that $n_5 \geq \omega(H)$. Now H has a colouring with m_0 stable sets and $m_1 + 1$ cliques (since $m_0 + m_1 + 1 = \tau(H)$). Choose one of these cliques, with say j vertices, and redistribute these vertices amongst the m_1 other cliques, at most one to any clique (which we can do since $j \leq \omega(H) \leq m_1$). This yields a colouring consisting of t colour sets, m_0 of which are stable sets, $m_1 - j$ of which are cliques, and j of which are colour sets in $\mathcal{B}_5$. Since cliques are in each of $\mathcal{B}_1$, $\mathcal{B}_3$, $\mathcal{B}_5$ and stable sets are in each of $\mathcal{B}_2$, $\mathcal{B}_4$, $\mathcal{B}_6$, these colour sets may be re-designated to show that this colouring is as desired.

Similarly, if $n_6 \geq \alpha(H)$, then H has a colouring with $m_0 + 1$ stable sets and m_1 cliques: choose one of these stable sets and redistribute the vertices amongst the other stable sets, at most one to any stable set; this yields a colouring as required.

It remains to consider the case when $n_5 < \omega(H)$ and $n_6 < \alpha(H)$. Then H has a colouring with n_i colour sets in $\mathcal{B}_i$ for each $i \in \{1, \dots, 4\}$ and no other colour sets, since $n_1 + \cdots + n_4 \geq t - (\alpha(H) - 1) - (\omega(H) - 1) \geq \tau_2(H)$. Adding $n_5 + n_6$ empty sets yields a colouring as required. $\square$

It will now suffice to prove that

$$\tau_2(G^n_{1/2}) + \alpha(G^n_{1/2}) + \omega(G^n_{1/2}) \leq \tau(G^n_{1/2}) \text{ a.a.s.}, \tag{2}$$

for then the above lemma immediately gives Lemma 1.4. It is well known that both $\alpha(G^n_{1/2})$ and $\omega(G^n_{1/2})$ are less than $2 \log n$ a.a.s. (and indeed we noted this when discussing Ramsey numbers in Section 1). Let

$$\beta(n) = 2 \log n - 2 \log \log n. \tag{3}$$

Results of [15, 24], extending work in [6, 23], show that

$$\frac{n}{\beta(n) - 2 + o(1)} \leq \chi(G^n_{1/2}) \leq \frac{n}{\beta(n) - 3 + o(1)} \text{ a.a.s.} \tag{4}$$

We shall use the lower bound here, together with the fact that $\tau(H) \geq \chi(H)$ for each graph H. Our main task is to show that $\tau_2(G^n_{1/2})$ is a.a.s. much smaller than $\chi(G^n_{1/2})$. We shall follow the approach used in [15] to prove the upper bound in (4), in order to show that

$$\tau_2(G^n_{1/2}) \leq \frac{n}{\beta(n) - 1 + o(1)} \text{ a.a.s.} \tag{5}$$

Since

$$\frac{n}{\beta(n) - 2 + o(1)} - \frac{n}{\beta(n) - 1 + o(1)} = \left(\frac{1}{4} + o(1)\right) \frac{n}{(\log n)^2},$$

we obtain (2) from the inequalities (4) and (5), together with the upper bounds on $\alpha(G^n_{1/2})$ and $\omega(G^n_{1/2})$.

We actually prove a stronger version of (5), where we insist that each colour set is 'balanced'. Let $\tilde{\mathcal{B}}_1$ be the class of graphs $K_a \cup K_b$ for any $a, b \geq 0$ with $|a - b| \leq 1$; and we similarly define $\tilde{\mathcal{B}}_2$, $\tilde{\mathcal{B}}_3$ and $\tilde{\mathcal{B}}_4$. Let $\tilde{\tau}_2(H)$ denote the corresponding colouring number. Of course, always $\tau_2(H) \leq \tilde{\tau}_2(H)$ for each graph H.

Lemma 3.2.

$$\tilde{\tau}_2(G^n_{1/2}) \le \frac{n}{\beta(n) - 1 + o(1)}.$$

Our remaining task in this section is to prove this lemma. The method involves estimating first and second moments, so that we may employ Janson's inequality to show that there are induced subgraphs as required with very small failure probability: this will allow the use of a natural greedy algorithm to find colourings as required.

3.2 An expectation calculation

For $i \in \{1, \dots, 4\}$, let $\beta_i(G)$ be the maximum size of a set $W \subseteq V(G)$ such that W is in $\mathcal{B}_i$ (that is, the induced subgraph $G[W] \in \mathcal{B}_i$); and, given a positive integer k, let $\beta_i^{(k)}(G)$ be the number of k-sets $W \subseteq V(G)$ such that W is in $\mathcal{B}_i$. We define $\tilde{\beta}_i(G)$ and $\tilde{\beta}_i^{(k)}(G)$ similarly, referring to $\tilde{\mathcal{B}}_i$. Recall that $\beta(n)$ was defined in (3). Let

$$\alpha(n) = \beta(n) + 2\log e - 1 \tag{6}$$

and note that $\alpha(n)$ is the same as $\alpha_{0,1/2}(n)$ in the notation in [15]. Let $0 < \delta < 1$ be fixed and let

$$\gamma(n) = \gamma_\delta(n) = \lfloor \alpha(n) + 1 - \delta \rfloor. \tag{7}$$

Let $\mathcal{B}_i^n$ denote the set of graphs in $\mathcal{B}_i$ on $[n]$. Let $\tilde{\mathcal{B}}_i^n$ denote the set of graphs in $\tilde{\mathcal{B}}_i$ on $[n]$. Note that

$$|\tilde{\mathcal{B}}_i^n| = \begin{cases} \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor} & \text{if } i \in \{1,2\} \text{ and } n \ge 2 \text{ even;} \\ \binom{n}{\lfloor n/2 \rfloor} & \text{if } i \in \{1,2\} \text{ and } n \ge 3 \text{ odd;} \\ \binom{n}{\lfloor n/2 \rfloor} & \text{if } i \in \{3,4\} \text{ and } n \ge 4 \text{ even;} \\ 2\binom{n}{\lfloor n/2 \rfloor} & \text{if } i \in \{3,4\} \text{ and } n \ge 3 \text{ odd.} \end{cases} \tag{8}$$

In any case, $|\tilde{\mathcal{B}}_i^n| = 2^{(1+o(1))n}$ for each $i \in \{1, \dots, 4\}$.

Lemma 3.3. *Fix $i \in \{1, \dots, 4\}$. For $k = \gamma(n)$,*

$$\mathbb{E}[\tilde{\beta}_i^{(k)}(G^n_{1/2})] \ge n^{1+\delta+o(1)}.$$

Proof. For $k = k(n) = \Theta(\log n)$, we have

$$\begin{aligned}
\mathbb{E}[\tilde{\beta}_i^{(k)}(G^n_{1/2})] &= \binom{n}{k} |\tilde{\mathcal{B}}_i^k| 2^{-\binom{k}{2}} \\
&\sim \left(\frac{ne}{k}\right)^k (2\pi k)^{-1/2} |\tilde{\mathcal{B}}_i^k| (2^{-(k-1)/2})^k \\
&= \left(\frac{ne}{k} \ |\tilde{\mathcal{B}}_i^k|^{1/k} \ 2^{-(k-1)/2}\right)^k \ n^{o(1)}.
\end{aligned}$$

Let (x_n) be a bounded sequence of real numbers such that $k = k(n) = \alpha(n) + x_n$ is integer-valued. Then

$$\log k = 1 + \log\log n + \log\left(1 - \frac{\log\log n + O(1)}{\log n}\right) = 1 + \log\log n + o(1)$$

and so, using $|\tilde{\mathcal{B}}_i^k| = 2^{(1+o(1))k}$ as $n \to \infty$,

$$\begin{aligned}
&\log\left(\frac{ne}{k} \ |\tilde{\mathcal{B}}_i^k|^{1/k} \ 2^{-(k-1)/2}\right) \\
&= \log n + \log e - 1 - \log\log n + \frac{1}{k}\log|\tilde{\mathcal{B}}_i^k| - \frac{k-1}{2} + o(1) \\
&= 1 - \frac{x_n}{2} + o(1) \geq \frac{1+\delta}{2} + o(1)
\end{aligned}$$

if $x_n \leq 1 - \delta$; and then, since $k = (2 + o(1))\log n$,

$$\mathbb{E}[\tilde{\beta}_i^{(k)}(G^n_{1/2})] \geq n^{1+\delta+o(1)}.$$

The lemma now follows on taking k as $\gamma(n)$. $\square$

3.3 Using Janson's inequality

Let (x_n) be a bounded sequence of real numbers such that for

$$k = \beta(n) + x_n = 2\log n - 2\log\log n + x_n \in \mathbb{N}$$

we have $\mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) \to \infty$ as $n \to \infty$. Consider an $i \in \{1, \ldots, 4\}$. In this section, we prove that with extremely small failure probability there is a k-subset of $[n]$ which is in $\tilde{\mathcal{B}}_i$. For this, we use Janson's Inequality (see [19, 20] or Theorems 2.14, 2.18 in [21]):

$$\mathbb{P}(\tilde{\beta}_i^{(k)}(G^n_{1/2}) = 0) \leq \exp\left(-\frac{\mathbb{E}^2(\tilde{\beta}_i^{(k)}(G^n_{1/2}))}{\mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) + \Delta}\right), \tag{9}$$

where

$$\Delta = \sum_{A,B} \mathbb{P}(A, B \in \tilde{\mathcal{B}}_i)$$

and the sum is over k-subsets A and B of $[n]$ with $|A \cap B| \in \{2, \ldots, k-1\}$. Let $p(k, \ell)$ be the probability that two k-subsets of $[n]$ that overlap on exactly ℓ vertices are both in $\tilde{\mathcal{B}}_i$. Then

$$\Delta = \binom{n}{k} \sum_{\ell=2}^{k-1} \binom{k}{\ell} \binom{n-k}{k-\ell} p(k, \ell).$$

Now let A and B be two k-subsets of $[n]$ that overlap on exactly ℓ vertices, i.e. $|A \cap B| = \ell$. Then, letting X denote the graph induced on $A \cap B$ and letting G run over the graphs on $A \cap B$,

$$\begin{aligned} p(k, \ell) &= \mathbb{P}(A, B \in \tilde{\mathcal{B}}_i) \\ &= \sum_G \mathbb{P}(A \in \tilde{\mathcal{B}}_i | X = G)\, \mathbb{P}(X = G)\, \mathbb{P}(B \in \tilde{\mathcal{B}}_i | X = G) \\ &\leq \max_G \mathbb{P}(A \in \tilde{\mathcal{B}}_i | X = G) \cdot \mathbb{P}(B \in \tilde{\mathcal{B}}_i). \end{aligned}$$

We need two upper bounds on the first factor here. First, note that for each G, $\mathbb{P}(G^{\ell}_{1/2} = G) = 2^{-\binom{\ell}{2}}$, and so

$$\max_G \mathbb{P}(A \in \tilde{\mathcal{B}}_i | X = G) \leq 2^{\binom{\ell}{2}}\, \mathbb{P}(A \in \tilde{\mathcal{B}}_i).$$

We now derive a second bound. Note first that $\mathbb{P}(A \in \tilde{\mathcal{B}}_i | X = G) = 0$ unless $G \in \mathcal{B}^{\ell}_i$. Each graph G on $A \cap B$ in $\mathcal{B}^{\ell}_1$ has a unique corresponding unordered partition into two cliques, and so we may extend G to a graph on A in $\tilde{\mathcal{B}}^k_1$ in at most $2^{k-\ell}$ ways; and $\mathcal{B}^{\ell}_2$ behaves similarly. If a graph G on $A \cap B$ in $\mathcal{B}^{\ell}_3$ has an edge, then it has a unique corresponding partition into a clique and a stable set; and if it has no edges, then it has $\ell + 1 \leq k$ such partitions. Thus each graph in G on $A \cap B$ in $\mathcal{B}^{\ell}_3$ can be extended to a graph on A in $\tilde{\mathcal{B}}^k_3$ in at most $k2^{k-\ell}$ ways; and $\mathcal{B}^{\ell}_4$ behaves similarly. Thus

$$\max_G \mathbb{P}(A \in \tilde{\mathcal{B}}_i | X = G) \leq k 2^{k-\ell} 2^{-\binom{k}{2} + \binom{\ell}{2}}.$$

We break the sum for Δ into two parts. Let ℓ_0 be an integer with $\ell_0 \sim k - \sqrt{\log n}$. Then $\Delta \leq \Delta_1 + \Delta_2$, where

$$\Delta_1 = \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) \cdot \sum_{\ell=2}^{\ell_0} \binom{k}{\ell}\binom{n-k}{k-\ell} 2^{\binom{\ell}{2}} \, \mathbb{P}(A \in \tilde{\mathcal{B}}_i) \quad \text{and}$$

$$\Delta_2 = \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) \cdot \sum_{\ell=\ell_0+1}^{k-1} \binom{k}{\ell}\binom{n-k}{k-\ell} k 2^{k-\ell} 2^{-\binom{k}{2}+\binom{\ell}{2}}.$$

First consider Δ_1. For every $\ell \leq k$,

$$\binom{k}{\ell}\binom{n-k}{k-\ell} \leq k^\ell \frac{k^\ell}{(n-k)^\ell}\binom{n}{k}.$$

Hence

$$\Delta_1 \leq \mathbb{E}^2(\tilde{\beta}_i^{(k)}(G^n_{1/2})) \sum_{\ell=2}^{\ell_0} \left(\frac{k^2}{n-k}\right)^\ell 2^{\binom{\ell}{2}}. \tag{10}$$

If we set $s_\ell = (k^2/(n-k))^\ell 2^{\binom{\ell}{2}}$, then $s_{\ell+1}/s_\ell = 2^\ell k^2/(n-k)$. So the sequence (s_ℓ) is decreasing and then inreasing. Therefore,

$$\max_{\ell \in \{2,\dots,\ell_0\}} \{s_\ell\} = \max\{s_2, s_{\ell_0}\}.$$

Now $s_2 = 2k^4/(n-k)^2$ and

$$s_{\ell_0} \leq \left(\frac{k^2}{n-k} 2^{\ell_0/2}\right)^{\ell_0} \leq \left(2^{-(1/2+o(1))\sqrt{\log n}}\right)^{\ell_0} = o(s_2).$$

Thus the inequality (10) now becomes for n large enough

$$\Delta_1 \leq \frac{2k^5}{(n-k)^2} \mathbb{E}^2(\tilde{\beta}_i^{(k)}(G^n_{1/2})) = O\left(\frac{(\log n)^5}{n^2}\right) \mathbb{E}^2(\tilde{\beta}_i^{(k)}(G^n_{1/2})).$$

Now consider Δ_2. We have

$$\binom{k}{\ell}\binom{n-k}{k-\ell} \leq \frac{k^{k-\ell}}{(k-\ell)!}\frac{n^{k-\ell}}{(k-\ell)!} \leq 4\left(\frac{kn}{2}\right)^{k-\ell}$$

since $(k-\ell)! \geq 2^{k-\ell-1}$. Thus

$$\Delta_2 \leq 4k\, \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) \sum_{\ell=\ell_0}^{k-1} (kn)^{k-\ell} 2^{-\binom{k}{2}+\binom{\ell}{2}}.$$

Now let $t_\ell = (kn)^{k-\ell} 2^{-\binom{k}{2}+\binom{\ell}{2}}$ for $\ell \in \{\ell_0, \ldots, k-1\}$. Note that $\log(t_\ell) = (k-\ell)\log(kn) - \binom{k}{2} + \binom{\ell}{2}$, which is convex in ℓ. Thus $\log(t_\ell)$ has its maximum value at $\ell = \ell_0$ or $\ell = k-1$; and hence the same must hold for t_ℓ. But $\log(t_{\ell_0}) = -(1+o(1))(\log n)^{3/2}$ so $t_{\ell_0} = o(1/n)$. Also $\binom{k}{2} - \binom{k-1}{2} = k-1$, and so $t_{k-1} = kn2^{-k+1} = O((\log n)^3/n)$. Hence

$$\Delta_2 = o((\log n)^5/n) \cdot \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) = o(1) \cdot \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})).$$

It follows that

$$\mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) + \Delta = \mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})) + \Delta_1 + \Delta_2 \leq \max\{2\,\mathbb{E}(\tilde{\beta}_i^{(k)}(G^n_{1/2})), 3\Delta_1\}$$

for n sufficiently large. Now, by (9) and the upper bound on Δ_1, we obtain the following lemma. (Recall that $\gamma(n)$ is defined in (7).)

Lemma 3.4. *Let $i \in \{1, \ldots, 4\}$ be fixed. For $k = \gamma(n)$,*

$$\mathbb{P}(\tilde{\beta}_i^{(k)}(G^n_{1/2}) = 0) \leq \max\left\{\exp\left(-\frac{1}{2}\mathbb{E}\left(\tilde{\beta}_i^{(k)}(G^n_{1/2})\right)\right), \exp\left(-\Omega\left(\frac{n^2}{(\log n)^5}\right)\right)\right\}.$$

3.4 A greedy colouring algorithm

Lemma 3.5. *It holds a.a.s. that for each $i \in \{1,2,3,4\}$ and for all $V' \subseteq [n]$ with $|V'| \geq n/(\log n)^3$, we have $\tilde{\beta}_i(G^n_{1/2}[V']) \geq \gamma(|V'|)$.*

Proof. Note that Lemma 3.3 implies that for any $V' \subseteq [n]$ with $|V'| \geq n/(\log n)^3$, we have

$$\mathbb{E}\left(\tilde{\beta}_i^{(\gamma(|V'|))}(G^n_{1/2}[V'])\right) \geq |V'|^{1+\delta+o(1)}.$$

So, applying Lemma 3.4, we deduce that

$$\begin{aligned}\mathbb{P}\left(\tilde{\beta}_i(G^n_{1/2}[V']) < \gamma(|V'|)\right) &\leq \exp\left(-|V'|^{1+\delta+o(1)}\right)\\ &\leq \exp\left(-\left(\frac{n}{(\log n)^3}\right)^{1+\delta+o(1)}\right).\end{aligned}$$

Since there are at most 2^n choices for V', the probability that there exists a set $V' \subseteq [n]$ with $|V'| \geq n/(\log n)^3$ and $\tilde{\beta}_i(G^n_{1/2}[V']) < \gamma(|V'|)$ is at most $2^n \exp\left(-(n/(\log n)^3)^{1+\delta+o(1)}\right) = o(1)$. $\square$

We consider the following algorithm for colouring $G^n_{1/2}$. Let $V' = [n]$. While $|V'| \geq n/(\log n)^3$, we choose and remove a colour set S from $G^n_{1/2}[V']$ of size $\gamma(|V'|)$, with S in $\tilde{\mathcal{B}}_i$ for any desired $i \in \{1, 2, 3, 4\}$. At the end, we obtain a collection of colour sets in $\tilde{\mathcal{B}}_i$, $i \in \{1, 2, 3, 4\}$, each of them corresponding to a colour in the colouring of $G^n_{1/2}$. Lemma 3.5 implies that a.a.s. we will be able to perform this algorithm and be left with at most $n/(\log n)^3$ uncoloured vertices. We may assign a new different colour to each of these remaining vertices. Thus, if the above algorithm stops after $f(n)$ steps, then $\tilde{\tau}_2(G^n_{1/2}) \leq f(n) + n/(\log n)^3$.

Recall that $\gamma(n) = \lfloor \beta(n) + 2\log e - \delta \rfloor$. Clearly, $\gamma(s)$ is non-decreasing and

$$\gamma\left(\left\lceil \frac{n}{(\log n)^3} \right\rceil\right) = \gamma(n)\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right).$$

Thus, for all integers $s \in \{\lceil n/(\log n)^3 \rceil, \ldots, n\}$,

$$\gamma(s) = \gamma(n)\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right), \tag{11}$$

and furthermore

$$f(n) = \frac{n}{\gamma(n)}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right). \tag{12}$$

Assume that there are n_t vertices available when we have removed t colour sets from $[n]$. Thus the set picked during the $(t+1)$st iteration will have size $\gamma(n_t)$. Since the colouring algorithm stops as soon as there are fewer than $n/(\log n)^3$ vertices available, the following inequality holds:

$$\sum_{t=0}^{f(n)-2} \gamma(n_t) \leq n\left(1 - \frac{1}{(\log n)^3}\right) \leq n. \tag{13}$$

Note that for all $t \geq 0$, $n_t \in \{\lceil n/(\log n)^3 \rceil, \ldots, n\}$ and $n_t = n - \sum_{j=0}^{t-1} \gamma(n_j)$. Therefore,

$$\log n_t = \log\left(n - \sum_{j=0}^{t-1} \gamma(n_j)\right) = \log n + \log\left(1 - \frac{\sum_{j=0}^{t-1} \gamma(n_j)}{n}\right).$$

We have[2]

$$
\begin{aligned}
\sum_{t=0}^{f(n)-2} \log\left(1-\frac{\sum_{j=0}^{t-1}\gamma(n_j)}{n}\right) &= \frac{1}{\ln 2}\sum_{t=0}^{f(n)-2}\frac{n}{\gamma(n_t)}\ln\left(1-\frac{\sum_{j=0}^{t-1}\gamma(n_j)}{n}\right)\frac{\gamma(n_t)}{n}\\
&\stackrel{(11)}{=} \frac{n(1+o(1))}{\gamma(n)\ln 2}\sum_{t=0}^{f(n)-2}\ln\left(1-\frac{\sum_{j=0}^{t-1}\gamma(n_j)}{n}\right)\frac{\gamma(n_t)}{n}\\
&= \frac{n(1+o(1))}{\gamma(n)\ln 2}\int_0^1 \ln(1-x)dx = -\frac{n(1+o(1))}{\gamma(n)\ln 2}\\
&\stackrel{(12)}{=} -(f(n)-1)(\log e+o(1)).
\end{aligned}
$$

Also, $\log\log n_t \le \log\log n$ and

$$
\log\log n_t \ge \log\log\left(\frac{n}{(\log n)^3}\right) = \log\log n - O\left(\frac{\log\log n}{\log n}\right).
$$

So

$$
\begin{aligned}
\sum_{t=0}^{f(n)-2}\gamma(n_t) &\ge \sum_{t=0}^{f(n)-2}(2\log n_t - 2\log\log n_t + 2\log e - \delta - 1)\\
&= (f(n)-1)(\beta(n)-\delta-1+o(1))
\end{aligned}
$$

and with (13) we obtain

$$
f(n)-1 \le \frac{n}{\beta(n)-\delta-1+o(1)}.
$$

This now completes the proof of Lemma 3.2 and hence of Lemma 1.4.

4 Every graph in $\mathcal{H}$ has the asymptotic linear Erdős-Hajnal property

In this section, we prove Lemma 1.5. This will complete the proof of Theorem 1.3. The proof follows the method used by Loebl *et al.* [22] for their Theorem 1.2. The only major difference is around their Lemma 5, but for completeness we include the full proof.

[2]Note that $\int \ln(1-x)dx = -(1-x)\ln(1-x)+1-x$.

Fix $H \in \mathcal{H}$ with k vertices. As discussed at the end of Section 2, we may assume that $\tau(H) \geq 3$. Let $t = \tau(H) - 1$, so that $t \geq 2$ and $\tau_1(H) \leq t$ (by the definition of $\mathcal{H}$). We choose $b = b(H) > 0$ small enough to satisfy certain inequalities given below. Let $\mathcal{Q}^n = \{G \in \mathrm{Forb}(H)^n : h(G) \geq bn\}$ and $\mathcal{R}^n = \mathrm{Forb}(H)^n \setminus \mathcal{Q}^n$. We need to show that $|\mathcal{R}^n|/|\mathrm{Forb}(H)^n| \to 0$ as $n \to \infty$. The first step is an easy lower bound on $\mathcal{Q}^n$. We may argue as we did in Subsection 1.2 to prove the lower bound in (1), to show that, as long as $b \leq 1/t$, we have

$$|\mathcal{Q}^n| \geq 2^{(1-1/t+o(1))\binom{n}{2}}. \tag{14}$$

To show that $|\mathcal{R}^n| = o(|\mathcal{Q}^n|)$, we need to combine this lower bound with an upper bound on the number of graphs in $\mathcal{R}^n$. This upper bound is obtained by a combination of a standard application of Szemerédi's Regularity Lemma and some extremal graph theory. We begin by introducing the necessary terminology.

Given a graph G, a pair of non-empty subsets U and W of $V(G)$ is said to be *η-uniform* if $|d(U,W) - d(U',W')| < \eta$ whenever $U' \subseteq U$, $|U'| > \eta|U|$, and $W' \subseteq W$, $|W'| > \eta|W|$ (where $d(U,W)$ stands for the density of the bipartite graph between U and W). A *coloured partition* π is a colouring of the edges of the complete graph K_m with colours red, blue, green and grey, where m is the *order* of π, denoted by $|\pi|$. Given a graph G and constants $0 < \lambda, \eta < 1$, we say that a partition of the vertex set $V(G)$ of G into $|\pi|$ classes $V_1, \ldots, V_{|\pi|}$ *satisfies* π with respect to λ and η if $|V_1| \leq |V_2| \leq \cdots \leq |V_{|\pi|}| \leq |V_1| + 1$ and the pair (V_i, V_j) is not η-regular only if ij is grey, and otherwise $0 \leq d(V_i,V_j) \leq \lambda$, $\lambda < d(V_i,V_j) < 1-\lambda$ or $1-\lambda \leq d(V_i,V_j) \leq 1$ according to whether ij is red, green or blue. We say that G *satisfies* π with respect to λ and η if there is a partition of $V(G)$ satisfying π with respect to λ and η. Here we shall be concerned only with which edges of π are green.

Szemerédi's Regularity Lemma asserts that, given λ, η and some integer ℓ, there exists an integer $L = L(\ell, \eta)$ such that any graph G with at least ℓ vertices satisfies some coloured partition π with respect to λ and η, where $\ell \leq |\pi| < L$ and where π has at most $\eta\binom{|\pi|}{2}$ grey edges.

Our next results will lead to a proof of the upper bound on $|\mathrm{Forb}(H)^n|$ in equation (1), and will form the basis for our proof of Lemma 1.5. Turán's theorem tells us that, if the proportion of the edges of a given coloured partition π which are green exceeds $(1-1/t)$, then π contains a green clique of order $t+1$, and indeed π contains a green complete $(t+1)$-partite graph with each part of a given size. In this case, an application of an embedding lemma (as is standard in many similar applications of the Szemerédi Regularity Lemma) tells us that, for some a, b with $a + b = t + 1$, each graph G which satisfies π contains every small graph that can be partitioned into a cliques and b stable sets. Combining these two results, we obtain the following.

Proposition 4.1 ([7, Theorem 3.1]). *Let $t, k \in \mathbb{N}$, let $\nu > 0$ and let $0 < \lambda < 1$ be given. Then there exist positive constants ℓ_0 and η_0 with the following property. Let π be a coloured partition with $|\pi| \geq \ell_0$, having at most $\eta_0\binom{|\pi|}{2}$ grey edges and at least $(1-1/t+\nu)\binom{|\pi|}{2}$ green edges. Then there are integers a and b with $a+b = t+1$, such that, if G is a graph with at least $|\pi|$ vertices that satisfies π with respect to λ and η_0, then G contains as an induced subgraph every graph with at most k vertices that can be partitioned into a cliques and b stable sets.*

Observe that any $\ell \geq \ell_0$ and $0 < \eta \leq \eta_0$ could serve instead of ℓ_0 and η_0.

This proposition allows us to obtain an upper bound on the size of $\mathcal{R}^n$ using the following lemma. For any coloured partition π, let $n_g = n_g(\pi)$ be the proportion of the edges which are green. Note that we can choose $c = c(\lambda) > 0$ which goes to zero with λ such that if n_1 and n_2 are large enough, then there are at most $2^{cn_1n_2}$ ways of choosing the edges of a bipartite graph with one side of size n_1, the other of size n_2 and either (i) fewer than $\lambda n_1 n_2$ edges or (ii) more than $(1-\lambda)n_1n_2$ edges.

Lemma 4.2. *For each coloured partition π with at most $\eta\binom{|\pi|}{2}$ grey edges, for every sufficiently large n, and for every partition $\mathcal{P}$ of $[n]$ (with $|\pi|$ parts), the total number of graphs on $[n]$ for which $\mathcal{P}$ satisfies π with respect to λ and η is at most*

$$2^{(n_g+\eta+c(\lambda)+1/|\pi|)\binom{n}{2}}.$$

Proof. Denote the given partition $\mathcal{P}$ of $[n]$ by classes $V_1, \ldots, V_{|\pi|}$ with $|V_1| \leq |V_2| \leq \cdots \leq |V_{|\pi|}| \leq |V_1| + 1$. If the pair V_i, V_j corresponds to a green edge or a grey edge, then there are at most $2^{|V_i||V_j|}$ ways to join V_i to V_j. But for red and blue edges there are at most $2^{c|V_i||V_j|}$ ways if n is large enough. Furthermore, there are at most $\binom{n}{2}/|\pi|$ edges within the partition classes. So we see that the total number of graphs such that the partition $\mathcal{P}$ satisfies π is bounded as above. □

We may now easily obtain the upper bound on $|\operatorname{Forb}(H)^n|$ in equation (1), which matches the earlier lower bound (14) on $|\mathcal{Q}^n|$ (though this is not yet strong enough to be useful to us).

Corollary 4.3.
$$|\operatorname{Forb}(H)^n| \leq 2^{(1-1/t+o(1))\binom{n}{2}}.$$

Proof. Let $G \in \operatorname{Forb}(H)^n$, where n is suitably large. By Szemerédi's Regularity Lemma, G satisfies some coloured partition π with respect to λ and η, where $\ell \leq |\pi| < L$. By Proposition 4.1 we may assume that $n_g < 1 - 1/t + \nu$ (since otherwise $G \notin \operatorname{Forb}(H)$). But the number of coloured partitions does not depend on n, and for each one we may use Lemma 4.2 to see that

$$|\operatorname{Forb}(H)^n| \leq 2^{(1-1/t+\nu+\eta+c+1/\ell+o(1))\binom{n}{2}},$$

and the result follows (on choosing η, ν and λ suitably small and ℓ suitably large). □

In order to continue with our project to prove Lemma 1.5, we pick λ and ν to certify certain inequalities given below. We choose η_0 and ℓ_0 satisfying Proposition 4.1 with respect to k, t, λ, and ν. We choose $\ell \geq \ell_0$ and $\eta \leq \eta_0$ which satisfy some inequalities given below. We choose L satisfying the Szemerédi Regularity Lemma for this choice of λ, η and ℓ. Thus, if $G \in \operatorname{Forb}(H)^n$ for some $n \geq \ell$ then G satisfies some π with respect to λ and η where $\ell \leq |\pi| < L$ and π has at most $\eta\binom{|\pi|}{2}$ grey edges. We let $b = 1/(5L \cdot R(k))$ (recalling that $R(k)$ is a Ramsey number). This then determines $\mathcal{Q}^n$ and $\mathcal{R}^n$, which were defined in terms of b.

The key to our proof is to strengthen the upper bound of the above corollary, by exploiting the fact that we are counting only graphs in $\mathcal{R}^n$. An easy computation using our earlier lower bound (14) on $|\mathcal{Q}^n|$ and Lemma 4.2 yields that the number of graphs satisfying partitions with $n_g < 1 - 1/t - \eta - 2c - 1/\ell$ is $o(|\mathcal{Q}^n|)$. Thus, in proving our strengthening, we need only consider graphs for which n_g exceeds $1 - 1/t - \eta - 2c - 1/\ell$. Key to doing so are the following two lemmas, the proofs of which are postponed to the end of the section.

Lemma 4.4. *Suppose that π contains s edge-disjoint cliques of order t all of whose edges are green. Then, for $\beta = 2/(9(2k+1)^2|\pi|^2)$ and large enough n, the number of graphs in $\mathcal{R}^n$ for which a given partition satisfies π with respect to λ and η is at most*

$$2^{(n_g+\eta+c+1/|\pi|-s\beta)\binom{n}{2}}.$$

Lemma 4.5. *If $n_g \geq 1 - 1/(t-1) + 1/(2t^2)$, then π contains at least $|\pi|^2/t^4$ edge-disjoint cliques of order t all of whose edges are green.*

With these two lemmas, it is straightforward to prove Lemma 1.5. We choose λ so that c is less than $1/(1000t^4k^2)$. We choose $\nu = 1/(1000t^4k^2)$. We choose $\ell \geq \max\{\ell_0, 1000t^4k^2\}$ and $\eta \leq \min\{\eta_0, 1/(1000t^4k^2)\}$.

Now, we have that $\eta + c + 1/|\pi| \leq 3/(1000t^4k^2)$. So, if π is a partition for which $n_g < 1 - 1/(t-1) + 1/(2t^2)$, then there are at most $2^{(1-1/(t-1)+2/3t^2)\binom{n}{2}}$ graphs $G \in \mathcal{R}^n$ for which a given partition satisfies π with respect to λ and η. On the other hand, for any partition π with $n_g \geq 1 - 1/(t-1) + 1/(2t^2)$, we know by Proposition 4.1 that $n_g \leq 1 - 1/t + \nu$. Hence, combining Lemmas 4.4 and 4.5, we see that at most

$$2^{(1-1/t-1/(9(2k+1)^2t^4)+3/(1000t^4k^2))\binom{n}{2}}$$

graphs in $\mathcal{R}^n$ satisfy π with respect to λ and η. In either case, there are at most $2^{(1-1/t-1/(500t^4k^2))\binom{n}{2}}$ graphs in $\mathcal{R}^n$ for which a particular partition satisfies π. But the number of choices for π is independent of n, and the number of partitions of the vertex set is at most n^L which is $o(2^{\binom{n}{2}})$. So, for large n, the number of elements of $\mathcal{R}^n$ is at most $2^{(1-1/t-1/(600t^4k^2))\binom{n}{2}}$, and hence $|\mathcal{R}^n| = o(|\mathcal{Q}^n|)$. This completes the proof of Lemma 1.5.

We turn now to the proofs of Lemmas 4.4 and 4.5. The proof of Lemma 4.5 is straightforward and is also found in [22].

Proof of Lemma 4.5. We repeatedly rip out the edges of a green clique of size t in π until no such cliques remain. Turán's theorem tells us that when we stop at most $(1 - 1/(t-1))\binom{|\pi|}{2}$ green edges can remain. But by the assumption, π has at least $(1 - 1/(t-1) + 1/(2t^2))\binom{|\pi|}{2}$ green edges. So we must have ripped out at least $\binom{|\pi|}{2}/(2t^2)$ edges and hence at least $\binom{|\pi|}{2}/t^4$ cliques. $\square$

The result in [22] (Lemma 5) which is analogous to Lemma 4.4, rather than counting the graphs in $\mathcal{R}^n$ satisfying the partition, counts the members of $\mathrm{Forb}(H)^n$ which contain no homogeneous set of size n^ε (for some fixed $\varepsilon = \varepsilon(H) > 0$) satisfying the partition. In order to strengthen this result to obtain Lemma 4.4, we need the following new technical lemma.

For $k \geq 1$, we define the *split $2k$-clique* to be K_{2k} together with an additional vertex of degree k and the *split $2k$-stable set* to be $\overline{K_{2k}}$ together

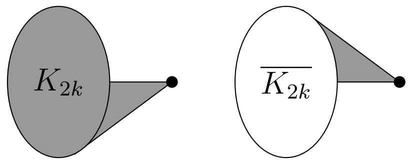

Figure 2: A split $2k$-clique and a split $2k$-stable set.

with an additional vertex of degree k, see Figure 2. (A split $2k$-clique is the complement of a split $2k$-stable set.) Let $\mathcal{B}(k)$ be the collection of graphs

$$K_k \cup K_k,\ \overline{K_k \cup K_k},\ \overline{K_k} \cup K_k,\ \overline{\overline{K_k} \cup K_k}$$

together with the split $2k$-clique and the split $2k$-stable set.

Lemma 4.6. *Fix a positive integer k. Let $c = 1/(2R(k))$ and let G be a graph of order n, where $n \geq \max\{R(k^2+k), 2(R(k)+k^2+k)\}$. Then G contains either a homogeneous set of size at least cn or an induced subgraph isomorphic to a member of $\mathcal{B}(k)$.*

Proof. Since $n \geq R(k^2+k)$, G contains a homogeneous set C of order k^2+k. We assume C is a clique; the complementary case is symmetric. We may assume that G does not contain a split $2k$-clique, as otherwise we are done.

Let $V_S = \{v \notin C : \deg_C(v) < k\}$ and $V_L = \{v \notin C : \deg_C(v) > |C| - k\}$, where $\deg_C(v)$ is the number of edges between v and C. (Here V_S is for small degrees and V_L is for large degrees.) Since G has no split $2k$-clique, $V(C) \cup V_S \cup V_L$ is a partition of $V(G)$.

Suppose that $|V_S| \geq R(k)$. Then V_S contains a homogeneous set A of order k. There are at most $k(k-1)$ edges between A and C, so there is a set B of k vertices in C with no edges between A and B. But now $A \cup B$ induces either $K_k \cup K_k$ or $K_k \cup \overline{K_k}$, and we are done. So we may assume that $|V_S| < R(k)$.

Now, $|V_L| > n - |C| - R(k) \geq n/2$. Suppose that each vertex in V_L has at most $R(k)-1$ non-edges to V_L. Then we may greedily pick a clique in V_L of size at least $|V_L|/R(k) \geq cn$, and we are done. So we may assume that this is not the case.

Now some vertex $v \in V_L$ has no edges to a subset D_0 (not containing v) of V_L of size $R(k)$. Within D_0 there is a homogeneous set D_1 of size k. Since each vertex in D_1 is adjacent to all but at most $k-1$ vertices of C, there is a subset D_2 of $V(C)$ with $|D_2| \geq |C| - k(k-1) = 2k$ such that all edges between D_1 and D_2 are in G. Now D_2 is clique. If D_1 is a stable set, then we have the complement of $K_k \cup \overline{K_k}$, and we are done. So we may assume that D_1 is a clique. But now $D_3 = D_1 \cup D_2$ forms a clique, and v has at least $|D_1| = k$ non-edges to D_3 and at least $|D_2| - (k-1) \geq k$ edges to D_3. Thus we have a split $2k$-clique, which contradicts our initial assumption, and we are done. □

We end this section with the promised proof of Lemma 4.4.

Proof of Lemma 4.4. Denote the edge-disjoint green cliques by $C(1), \dots, C(s)$. For each $C(z)$, let $m_{C(z)}$ be the sum of $|V_i||V_j|$ over the unordered pairs of partition classes V_i and V_j in $C(z)$. We claim that for each of the s cliques, having fixed the edges within the partition classes, there are at most $2^{m_{C(z)} - \beta\binom{n}{2}}$ ways to pick the edges for G within the green edges of the clique. Assuming the claim is true, the total number of choices corresponding to green edges of π within the cliques is at most

$$\prod_{z=1}^{s} 2^{m_{C(z)} - \beta\binom{n}{2}} = 2^{\sum_{z=1}^{s} m_{C(z)} - s\beta\binom{n}{2}}$$

which, combined with the argument for Lemma 4.2, gives the required result.

It remains to prove the claim. To have $G \in \mathcal{R}^n$, we must also have that $h(G[V_i]) < bn$ for each i. With a view to applying Lemma 4.6 and recalling $c = 1/(2R(k))$, observe that $bn \leq (c/2)\lfloor n/|\pi| \rfloor \leq (c/2)|V_i|$ and so $h(G[V_i]) < (c/2)|V_i|$. But now in $G[V_i]$ we can find at least $|V_i|/(2(2k+1))$ vertex-disjoint copies of graphs in $\mathcal{B}(k)$. For, if G' denotes the graph remaining after deleting $j < |V_i|/(2(2k+1))$ copies of graphs in $\mathcal{B}(k)$ from $G[V_i]$, then $|V(G')| \geq |V_i| - j(2k+1) \geq |V_i|/2$ and so $h(G') < c|V(G')|$; hence G' contains a graph in $\mathcal{B}(k)$ by Lemma 4.6, assuming n is sufficiently large.

Let p be a prime in

$$\left\{ \left\lceil \frac{n}{3(2k+1)|\pi|} \right\rceil, \dots, \frac{1}{2(2k+1)} \left\lfloor \frac{n}{|\pi|} \right\rfloor \right\}$$

Such a p is guaranteed to exist if n is sufficiently large, by the prime number theorem, see for example Section 22.19 in [18]. As we have just seen, for any graph G in $\mathcal{R}^n$, we can find within each $G[V_i]$ p vertex-disjoint copies $B_{i,j}$, $j \in \{1, \dots, p\}$, of graphs in $\mathcal{B}(k)$.

Consider the green clique $C(z)$ and suppose without loss of generality for simplicity that its vertices correspond to the classes $V_1, \ldots, V_t$. For $r, s \in \{1, \ldots, p\}$, consider the t-tuple

$$(B_{1,r}, B_{2,r+s}, B_{3,r+2s}, \ldots, B_{t,r+(t-1)s})$$

where the second subscripts are taken modulo p. For each t-tuple, there is at least one way to join the classes to obtain a copy of H, since $\tau_1(H) \leq t$. Also, as p is prime, no pair of t-tuples coincide in more than one coordinate, and so no edge between classes is spanned by more than one t-tuple. As there are p^2 t-tuples, it follows that the number of ways of choosing the edges between pairs from$V_1, \ldots, V_t$ is at most $2^{m_{C(z)}-p^2} \leq 2^{m_{C(z)}-\beta\binom{n}{2}}$. This establishes the claim, and thus completes the proof of the lemma. $\square$

5 Concluding remarks

We have seen that almost all graphs have the asymptotic linear Erdős-Hajnal property. We noted that the three-vertex path P_3 does not have this property, and indeed Loebl *et al.* [22] suggested that P_3 and P_4 might be the only graphs which do not have the property. Let us spell this out.

Question 1. Does P_4 have the asymptotic linear Erdős-Hajnal property?

Question 2. Does every connected graph (with at least two vertices) other than P_3 and P_4 have the asymptotic linear Erdős-Hajnal property?

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