Circular choosability of planar graphs

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21 July 2006
Horizons of Combinatorics
Balatonalmádi
circular chromatic number

Introduced by Vince (1988).

Optimises over all \((p, q)\)-colourings:
circular chromatic number

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\[
V(G) \rightarrow \tau(k)
\]
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Optimises over all \((p, q)\)-colourings:

\[
\chi_c(G) := \inf \left\{ \frac{p}{q} : G \text{ admits a } (p, q)\text{-colouring} \right\}
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Optimises over all \( r \)-circular colourings:
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Optimises over all \( r \)-circular colourings:

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\chi_c(G) = \inf \{ r : G \text{ admits an } r \text{-circular colouring} \}
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circular chromatic number

Close connections to homomorphisms, fractional chromatic number, traffic lights, . . .
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\[ \chi - 1 < \chi_c \leq \chi \]
circular chromatic number

Close connections to homomorphisms, fractional chromatic number, traffic lights, ... 

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always rational
circular chromatic number

Close connections to homomorphisms, fractional chromatic number, traffic lights, . . .

\[ \chi - 1 < \chi_c \leq \chi \]
\[ \uparrow \]
always rational

Many interesting questions; consult survey by Zhu (2001).
A natural list variant for $\chi_c$ was introduced by Mohar (2003) and Zhu (2005).
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Fix $t$. 
circular choosability

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$$cch(G) := \inf\{ t \geq 1 : G \text{ is circularly } t\text{-choosable} \}$$
circular chromatic number

Optimises over all \((p, q)\)-colourings:

\[
\chi_c(G) := \inf \left\{ \frac{p}{q} : G \text{ admits a (p, q)-colouring} \right\}
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circular choosability: Zhu (2005)

\[ \text{cch} \geq \chi_c \text{ and } \text{cch} \geq \text{ch} - 1 \]
circular choosability: Zhu (2005)

\[ c_{\text{ch}} \geq \chi_c \text{ and } c_{\text{ch}} \geq \chi - 1 \]

\textbf{but} \; c_{\text{ch}} \not\leq \chi
<table>
<thead>
<tr>
<th>$\chi_c$</th>
<th>cch</th>
<th>$6 \leq \tau \leq 8$</th>
<th>$\tau(k)$</th>
<th>$\tau_0(k)$</th>
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</table>

### Circular Choosability: Zhu (2005)

\[
cch \geq \chi_c \text{ and } cch \geq ch - 1
\]

**but** \( cch \nsubseteq ch \)

In particular, \( cch(K_{k,m^k}) \geq (2 - 2k/m)k \)
circular choosability: Zhu (2005)

\[ \text{cch} \geq \chi_c \text{ and } \text{cch} \geq \text{ch} - 1 \]

\[ \text{but } \text{cch} \not\leq \text{ch} \]

in particular, \( \text{cch}(K_{k,m^k}) \geq (2 - 2k/m)k \)

\[ \text{cch} \leq 2 \cdot \delta^* \]
| $\chi_c$ | cch $| 6 \leq \tau \leq 8$ | $\tau(k)$ | $\tau_o(k)$ |
|---------|----------------|----------|------------|

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$cch \leq 2 \chi$ ???
circular choosability: Zhu (2005)

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in particular, \[ cch(K_{k,m^k}) \geq (2 - 2k/m)k \]

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\[ cch \leq 2 \text{ch} \quad ??? \quad \text{cch attained} ??? \]
upper bound for planar graphs

Define

\[ \tau := \sup\{ \text{cch}(G) : G \text{ is planar} \} \]
upper bound for planar graphs

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Mohar asked the following: is \( 4 \leq \tau \leq 5 \)?
Recall the (now) classical theorem of Thomassen (1994).

**Theorem**

_Every planar graph is 5-choosable._
upper bound for planar graphs

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**Theorem**
Every planar graph is 5-choosable.

**Proposition**
Let $G$ be a near triangulation with outer face $C$. Let $L$ be a list-assignment such that

$$|L(v)| \geq \begin{cases} 
3 & \text{if } v \in C \\
5 & \text{otherwise} 
\end{cases}.$$ 

Then any precolouring of two adjacent vertices of $C$ can be extended to a colouring of $G$. 
Recall the (now) classical theorem of Thomassen (1994).

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Every planar graph is $5$-choosable.

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Proposition

Let $G$ be a near triangulation with outer face $C$. Let $L$ be a $(p, q)$-list-assignment such that

$$|L(v)| \geq \begin{cases} 4q - 1 & \text{if } v \in C \\ 8q - 3 & \text{otherwise} \end{cases}.$$

Then any $L$-$(p, q)$-precolouring of two adjacent vertices of $C$ can be extended to a $L$-$(p, q)$-colouring of $G$. 

upper bound for planar graphs
upper bound for planar graphs

Theorem
Every planar graph is circularly 8-choosable, i.e. $\tau \leq 8$.

Proposition
Let $G$ be a near triangulation with outer face $C$. Let $L$ be a $(p, q)$-list-assignment such that

$$|L(v)| \geq \begin{cases} 4q - 1 & \text{if } v \in C \\ 8q - 3 & \text{otherwise} \end{cases}.$$

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Voigt (1993) described a non-4-choosable planar graph. We show that there exist *circularly* non-$(6 - \varepsilon)$-choosable graphs.
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**Theorem**

*For any $n \geq 2$, there exists planar $G_n$ with $cch(G_n) \geq 6 - \frac{1}{n}$.***
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**Theorem**

*For any* \( n \geq 2 \), *there exists planar* \( G_n \) *with* \( \text{cch}(G_n) \geq 6 - \frac{1}{n} \).

Our examples are relatively simple.

In the next frames, we denote \( t = 6 - \frac{1}{n} \).
\begin{tabular}{|c|c|c|c|}
\hline
$\chi_c$ & cch & $6 \leq \tau \leq 8$ & $\tau(k)$ & $\tau_0(k)$ \\
\hline
\end{tabular}

planar graphs with high cch

\[ [r, r + tq - 1] \]

\[ [s, s + tq - 1] \]
planar graphs with high cch

\[ \chi_c \quad \text{cch} \quad 6 \leq \tau \leq 8 \quad \tau(k) \quad \tau_0(k) \]
planar graphs with high cch
planar graphs of prescribed girth

\( \tau(k) := \sup\{\text{cch}(G) : G \text{ is planar and has girth } \geq k\} \).
planar graphs of prescribed girth

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<table>
<thead>
<tr>
<th>girth</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>( k \geq 11 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>upper</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>( 2 + \frac{4}{2 \lceil (k-2)/4 \rceil} )</td>
</tr>
<tr>
<td>lower</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
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**Table:** Bounds for \( \tau(k) \).
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planar graphs of prescribed girth with high cch
planar graphs of prescribed girth with high cch
Theorem

\[ \tau_0(k) \geq 2 + \frac{4}{k-2} \text{ for all integers } k \geq 3. \]
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planar graphs of prescribed girth with high cch

**Theorem**

$\tau_o(k) \geq 2 + \frac{4}{k-2}$ for all integers $k \geq 3$.

Recall

$$\text{Mad}(G) := \max \left\{ \frac{2|E(H)|}{|V(G)|} : H \subset G \right\}$$
planar graphs of prescribed girth with high cch

**Theorem**
\[ \tau_o(k) \geq 2 + \frac{4}{k-2} \quad \text{for all integers } k \geq 3. \]

Recall
\[
\text{Mad}(G) := \max \left\{ \frac{2|E(H)|}{|V(G)|} : H \subset G \right\}
\]

Euler’s formula and girth \( k \) \( \implies \) \( \text{Mad} < 2 + \frac{4}{k-2} \)
\( \chi_c \)  |  cch  |  6 \( \leq \tau \leq 8 \)  |  \( \tau(k) \)  |  \( \tau_0(k) \)
--- | --- | --- | --- | ---
planar graphs of prescribed girth

\[ \tau(k) := \sup \{ \text{cch}(G) : G \text{ is planar and has girth} \geq k \}. \]

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<td>2 ( \frac{4}{5} )</td>
<td>( 2 \frac{2}{3} )</td>
<td>( 2 \frac{4}{7} )</td>
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**Table:** Bounds for \( \tau(k) \).
Define

$$\tau_o(k) := \sup\{\text{cch}(G) : G \text{ is outerplanar and has girth } \geq k\}.$$  

**Theorem**

$$\tau_o(k) = 2 + \frac{2}{k-2} \text{ for all integers } k \geq 3.$$
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outerplanar graphs with prescribed girth
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