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INDUCED MATCHINGS IN SUBCUBIC PLANAR GRAPHS*

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Abstract. We present a linear-time algorithm that, given a planar graph with m edges and maximum degree 3, finds an induced matching of size at least $m/9$. This is best possible.

Key words. induced matchings, subcubic planar graphs, strong chromatic index

AMS subject classifications. 05C70, 05C85, 68R10

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1. Introduction.

For a graph $G = (V, E)$, an *induced matching* is a set $M \subseteq E$ of edges such that the graph induced by the endpoints of M is a disjoint union of edges. In other words, a shortest path in G between any two edges in M has length at least 2. In this article, we prove that every planar graph with maximum degree 3 has an induced matching of size at least $|E(G)|/9$ (which is best possible), and we give a linear-time algorithm that finds such an induced matching.

The problem of computing the size of a largest induced matching was introduced in 1982 by Stockmeyer and Vazirani [15] as a variant of the maximum matching problem. They proposed it as the "risk-free" marriage problem: find the maximum number of married couples such that no married person is compatible with a married person other than her/his spouse. Recently, the induced matching problem has been used to model the capacity of packet transmission in wireless ad hoc networks, under interference constraints [2].

In contrast to the maximum matching problem, as shown by Stockmeyer and Vazirani, the maximum induced matching problem is NP-hard even for quite a restricted class of graphs: bipartite graphs of maximum degree 4. Other classes in which this problem is NP-hard include planar bipartite graphs and line graphs. Despite these discouraging negative findings, there is a large body of work showing that the maximum induced matching number can be computed in polynomial time in other classes of graphs, e.g., chordal graphs, cocomparability graphs, asteroidal-triple free graphs, and graphs of bounded cliquewidth. Consult the survey article of Duckworth, Manlove, and Zito [4] for references to these results.

Since our main focus in this paper is the class of planar graphs of maximum degree 3, we point out that Lozin [10] showed the maximum induced matching problem to

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be NP-hard for this class; on the other hand, the problem admits a polynomial-time approximation scheme for this class [4].

There have been recent efforts to determine the parameterized complexity of the maximum induced matching problem. In general, the problem of deciding whether there is an induced matching of size k is W[1]-hard with respect to k [13]. It is even W[1]-hard for the class of bipartite graphs, as shown by Moser and Sikdar [12]. Therefore, the maximum induced matching problem is unlikely to be in the class of fixed-parameter tractable problems (FPT). Consult the monograph of Niedermeier [14] for a recent detailed account of fixed-parameter algorithms. On the positive side, Moser and Sikdar showed that the problem is in FPT (and even has a linear kernel) for the class of planar graphs as well as for the class of bounded degree graphs. Notably, by examining a greedy algorithm, they showed that for subcubic graphs (that is, graphs of maximum degree at most 3), the maximum induced matching problem has a problem kernel of size at most $26k$ [12]. Furthermore, Kanj et al. [9], using combinatorial methods to bound the size of a largest induced matching in *twinless* planar graphs, contributed an explicit bound of $40k$ on kernel size for the *general* planar maximum induced matching problem; this was subsequently improved to $28k$ by Erman et al. [5]. (A graph is *twinless* if it contains no pair of vertices both having the same neighborhood.)

We provide a result similar in spirit to the last-mentioned results. In particular, we promote the use of a structural approach to derive explicit kernel size bounds for planar graph classes. Our main result relies on graph properties proved using a discharging procedure. The discharging method was developed to establish the famous Four Color Theorem.

THEOREM 1. *There is a linear-time algorithm that, given as input a planar graph of maximum degree* 3 *with* m *edges, outputs an induced matching of size at least* $m/9$*.*

Let us note two direct corollaries before justifying a corollary concerning explicit kernel size bounds.

COROLLARY 2. *Every planar graph of maximum degree* 3 *with* m *edges has an induced matching of size at least* $m/9$*.*

COROLLARY 3. *Every* 3*-regular planar graph with* n *vertices has an induced matching of size at least* $n/6$*.*

COROLLARY 4. *The problem of determining whether a subcubic planar graph has an induced matching of size at least* k *has a problem kernel of size at most* $9k$*. The problem of determining whether a* 3*-regular planar graph has an induced matching of size at least* k *has a problem kernel of size at most* $6k$*.*

Proof. Here is the kernelization: take as input $G = (V, E)$; if $k \leq |E|/9$, then answer "yes" and produce an appropriate matching by way of the algorithm guaranteed by Theorem 1; otherwise, $|E| < 9k$, and we have obtained a problem kernel with fewer than $9k$ edges. A similar argument demonstrates a problem kernel of size at most $6k$ for 3-regular planar graphs. ◻

In Corollaries 2 and 3, our result gives lower bounds on the maximum induced matching number for subcubic or 3-regular planar graphs that are best possible: consider the disjoint union of multiple copies of the triangular prism. See Figure 1.

The condition on maximum degree in Corollary 2 cannot be weakened: the disjoint union of multiple copies of the octahedron is a 4-regular planar graph with m edges that has no induced matching with more than $m/12$ edges. See Figure 2. Also, the condition on planarity cannot be dropped: the disjoint union of multiple copies of the graph in Figure 3 is a subcubic graph with m edges that has no induced matching

FIG. 1. *A 3-regular planar graph with n vertices, $m = 3n/2$ edges, and no induced matching of size more than $n/6 = m/9$.*

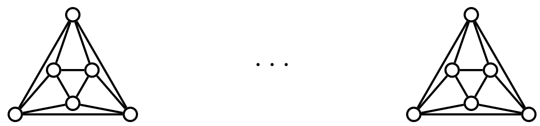

FIG. 2. *A 4-regular planar graph with m edges (and $n = m/2$ vertices) and no induced matching of size more than $m/12$ ($= n/6$).*

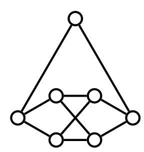

FIG. 3. *A subcubic graph with no induced matching of size 2.*

with more than $m/10$ edges.

There has been considerable interest in the induced matching problem due to its connection with the strong chromatic index. A *strong edge k-coloring* of G is a proper k-coloring of the edges such that no edge is adjacent to two edges of the same color, i.e., a partition of the edge set into k induced matchings. If G has m edges and admits a strong edge k-coloring, then a largest induced matching in G has size at least m/k. Thus, Theorem 1 is related to problems surrounding the long-standing Erdős–Nešetřil conjecture, which concerns the extremal behavior of the strong chromatic index for bounded degree graphs (cf. Faudree et al. [6, 7], Chung et al. [3]).

In particular, our work lends support to a conjecture of Faudree et al. [7] that every planar graph of maximum degree 3 is strongly edge 9-colorable. This conjecture has an earlier origin: it is implied by one case of a thirty-year-old conjecture of Wegner [16], asserting that the square of a planar graph with maximum degree 4 can be 9-colored. (Observe that the line graph of a planar graph with maximum degree 3 is a planar graph with maximum degree 4.) Independently, Andersen [1] and Horák, He, and Trotter [8] demonstrated that every subcubic graph has a strong edge 10-coloring, which implies that every subcubic graph with m edges has an induced matching of size at least $m/10$.

For graphs with larger maximum degree, Faudree et al. [7, Theorem 10] observed using the Four Color Theorem that every planar graph of maximum degree Δ with m edges admits a strong edge $(4\Delta + 4)$-coloring and thus contains an induced matching of size at least $m/(4\Delta + 4)$. They also observed that the disjoint union of multiple copies of the graph in Figure 4 is a planar graph of maximum degree Δ with m edges that has no induced matching with more than $m/(4\Delta - 4)$ edges. Narrowing the gap between these bounds for induced matchings in graphs of maximum degree $\Delta \geq 4$ is left for future work.

The remainder of this paper is organized as follows. We describe the linear-

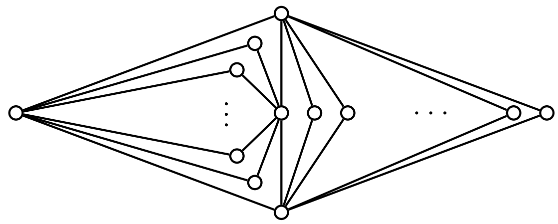

FIG. 4. *A planar graph of maximum degree* Δ *with no induced matching of size* 2.

time algorithm in section 3. The main structural result on which this algorithm relies is provided in section 4: the details of the discharging procedure are given in subsection 4.1, and we analyze the structures guaranteed by this procedure in subsection 4.2. Before continuing, we introduce some necessary terminology.

2. Notation and preliminaries. We remind the reader that a *plane graph* is a planar graph for which an embedding in the plane is fixed. The algorithm that we shall present in section 3 does not need any information about the embedding of the input graph. However, later on, Lemmas 9 and 10 do make use of any particular embedding of the graph under consideration. Throughout this paper, G will be a subcubic planar graph with vertex set V and edge set E, with $|V| = n$ and $|E| = m$. In cases when we have also fixed the embedding, we will denote the set of faces by F.

We assume the standard convention that a vertex and face (respectively, cycle) are called *incident* if the vertex lies on the face (respectively, cycle).

A vertex of degree d is called a *d-vertex*. A vertex is an *$(\leqslant d)$-vertex* if its degree is at most d and an *$(\geqslant d)$-vertex* if its degree is at least d. The notions of *d-face*, *$(\leqslant d)$-face*, *$(\geqslant d)$-face*, *d-cycle*, *$(\leqslant d)$-cycle*, and *$(\geqslant d)$-cycle* are defined analogously as for the vertices, where the degree of a face or cycle is the number of edges along it, with the exception that a cut-edge on a face is counted twice. Let $\deg(v)$, respectively, $\deg(f)$, denote the degree of vertex v, respectively, face f.

Given $u, v \in V$, the *distance* $\mathrm{dist}(u, v)$ between u and v in G is the length (in edges) of a shortest path from u to v. Given two subgraphs $G_1, G_2 \subseteq G$, the *distance* $\mathrm{dist}(G_1, G_2)$ between G_1 and G_2 is defined as the minimum distance $\mathrm{dist}(v_1, v_2)$ over all vertex pairs $(v_1, v_2) \in V(G_1) \times V(G_2)$.

Note that another way to say that $M \subseteq E$ is an induced matching is that

$$\mathrm{dist}(e, f) \geq 2 \text{ for all distinct } e, f \in M.$$

For a set $E' \subseteq E$ of edges we will set

$$\Psi(E') := \{e \in E : \mathrm{dist}(e, E') < 2\}. \tag{1}$$

Given $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and for $k \in \mathbb{N}$ let $N^k(v)$ denote the set of vertices at distance at most k from v. For a subgraph $H \subseteq G$, we will set $N^k(H) := \bigcup_{v \in V(H)} N^k(v)$.

For a subgraph $H \subseteq G$, we will also use the notation Ψ_H, N_H, N^k_H to refer to the analogous sets restricted to H.

Two distinct cycles or faces are *adjacent* if they share at least one edge. Let $C_1, \ldots, C_k$ be a collection of cycles or faces. We say that C_1 and C_2 are *in sequence (through e_1)* if there exists a path $e_A e_1 e_B$ (e_i are edges) along C_1 such that only e_1 is also part of C_2. We say that $C_1, \ldots, C_k$ are *in sequence* if there are vertices

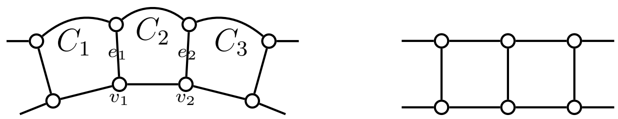

FIG. 5. *Three faces in sequences and a double 4-face.*

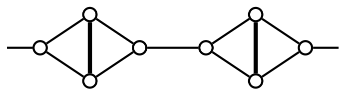

FIG. 6. *An edge subset (in bold) that is good, but not minimally good.*

$v_0, \ldots, v_k$ and edges $e_1, \ldots, e_{k-1}$ such that $v_0 \cdots v_k$ is a path, v_i is an endpoint of e_i for $i \in \{1, \ldots, k-1\}$, and C_i and C_{i+1} are in sequence through e_i for $i \in \{1, \ldots, k-1\}$. A *double* 4-*face* refers to two 4-faces in sequence. See Figure 5.

3. The algorithm. Our result will rely on building up the desired induced matching by augmenting it iteratively each time by up to five edges. We say that a set of edges $E' \subseteq E$ is *good* if E' is an induced matching, $1 \le |E'| \le 5$, and $|\Psi(E')| \le 9|E'|$, with Ψ as defined by (1). We will want E' to be *minimally good*, i.e., so that it is good and no proper subset of E' is good. See Figure 6.

THEOREM 5. *Every subcubic planar graph with at least one edge contains a good set of edges.*

Theorem 5 follows immediately from Lemmas 9 and 10 in section 4 below, which are proved using structural arguments. It will be illustrative to give the main approach for the algorithm in a direct proof of Corollary 2, using Theorem 5, before justifying the linear running time claimed in Theorem 1.

Proof of Corollary 2. Theorem 5 allows us to adopt a greedy approach for building up the induced matching. We start from $M = \emptyset$ and $H = G$. At each iteration, we find a minimally good E' in H and then augment M by E' and delete $\Psi_H(E')$ from H. Removing $\Psi_H(E')$ from H ensures that any edge moved from H to M at a later iteration is compatible with the edges of E', i.e., that M is maintained as an induced matching. Since we delete only edges, H is subcubic and planar throughout the process. The theorem then guarantees that we may iterate until H is the edgeless graph. By the definition of a good set of edges—in particular, that Ψ is at most nine times the number of edges in the set—the matching M at the end of the process must have size at least $|E|/9$. $\square$

The algorithm uses exactly the above approach; however, we need to be more careful to ensure that the running time is linear. For this, we require the following brief observation.

LEMMA 6. *If $E' \subseteq E$ is minimally good, then $2 \le \mathrm{dist}(e, f) \le 15$ for all distinct $e, f \in E'$.*

Proof. That $\mathrm{dist}(e, f) \ge 2$ for all distinct $e, f \in E'$ follows from the fact that E' is an induced matching. Let us now note that no $E'' \subseteq E'$ can exist with $\mathrm{dist}(e, f) \ge 4$ for all $e \in E''$, $f \in E' \setminus E''$. This is because otherwise $\Psi(E'') \cap \Psi(E' \setminus E'') = \emptyset$, which implies that $|\Psi(E')| = |\Psi(E'')| + |\Psi(E' \setminus E'')|$ and at least one of E'' and $E' \setminus E''$ must be good, contradicting that E' is minimally good. We can then write $E' = \{e_1, e_2, e_3, e_4, e_5\}$ with $\mathrm{dist}(e_i, \{e_1, \ldots, e_{i-1}\}) \le 3$ for all i. This shows that for any $e, f \in E'$ there is a path of length at most 15 between an endpoint of e and an endpoint of f. (Note that the distance is not necessarily at most 12, because we may

Initialize as follows.

$Q := V$ (in some arbitrary order), $M := \emptyset$, and $H := G$.

While Q is nonempty, iterate the following.

Letting v_0 denote the beginning element of Q,

1. if v_0 is isolated, then remove it from Q;
2. else, check for a minimally good set of edges E' such that v_0 is the endpoint of some edge of E', and
   - 2a. if such an E' does not exist, then move v_0 to the end of Q,
   - 2b. else, remove the vertices of $N_H^{20}(E')$ from Q and replace them at the beginning of Q in some arbitrary order, then set $M := M \cup E'$ and $H := H \setminus \Psi_H(E')$.

FIG. 7. *An algorithm that, given a subcubic planar graph $G = (V, E)$, generates an induced matching M in G of size at least $|E|/9$.*

need to use up to three of the edges e_i in a shortest path between e and f.) ∎

We now present our algorithm formally and then argue that its time complexity is linear. For convenience, we adopt the random access machine (RAM) model of computation. (See, for instance, section 2.2 of [11] for a detailed description of the RAM model.) We may assume that the algorithm takes as input an adjacency list for $G = (V, E)$, i.e., an array with an entry for each vertex v, each of which contains a list of pointers to the (up to three) neighbors of v. If G is instead given as a list of edges or in a $|V| \times |V|$ adjacency matrix (which is a rather inefficient means of storing a bounded degree graph), then we can first perform a routine conversion to an adjacency list in time that is linear in the size of the input, and thus the overall time complexity to output the desired induced matching remains linear in the input size.

The algorithm examines the vertices of H (i.e., of G) one at a time according to a queue Q. We store Q by means of a doubly linked list each element of which is doubly linked to its corresponding vertex in H. Each element of Q stores a pointer to the corresponding vertex in H as well as pointers to the next and previous elements in Q, and we maintain two special pointers to the beginning and end elements of Q; the graph H is stored in an adjacency list, except that each entry stores an extra pointer to its corresponding element in Q. This ensures that the operations of deleting arbitrary elements of H or Q and inserting such elements at the beginning or the end of Q all take constant time. Our algorithm is described in Figure 7 and uses a more refined version of the greedy approach taken in the proof of Corollary 2. Theorem 1 follows immediately from the following.

THEOREM 7. *Given as input a subcubic planar graph $G = (V, E)$, the algorithm described in Figure 7 outputs in M an induced matching of G of size at least $|E|/9$ in linear time.*

Proof. As shown in the proof of Corollary 2, Theorem 5 implies that the greedy approach produces a matching of the promised size. And so the algorithm must terminate. It remains only to show that the running time of the algorithm is linear.

Let us first show that each of steps 1, 2, 2a, and 2b of the algorithm can be performed in constant time. (Steps 1 and 2a are obvious.)

The check for a suitable E' at the beginning of step 2 can be done in constant time: by Lemma 6, we need only consider sets E' of up to five edges such that each edge $e \in E'$ has at least one endpoint at distance at most 16 from v_0. Hence, all vertices incident to edges of $\Psi_H(E')$ will be within distance 18 from v_0. Thus, to find a minimally good E' with at least one edge incident to v_0, we need only examine the subgraph $H[N_H^{18}(v_0)]$. Now, this subgraph has fewer than $3 \cdot 2^{18} = O(1)$ vertices, and it can be determined in constant time from the adjacency list data structure for H. (We can read in constant time which are the neighbors of v_0, then in constant time which are the neighbors of the neighbors, and so on, until depth 18.) Since $H[N_H^{18}(v_0)]$ has constant size, we can clearly search for a set E' of the required form in constant time.

For step 2b, since the set $N_H^{20}(E')$ has constant size, we can in constant time determine the vertices of $N_H^{20}(E')$ and move them to the beginning of Q. Similarly, we can remove an edge uv from the adjacency list for H in constant time, since we need only update the entries for u and v (and each entry contains a list of constant size). Thus, removing $\Psi_H(E')$ from H can also be done in constant time.

For the rest of the proof, it will be convenient to index the different iterations of the while loop by a "time parameter" $t \in \{1, 2, \dots\}$. Let $H(t)$, respectively, $Q(t)$, denote the state of H, respectively, Q, at the start of iteration t.

It suffices to show that there are only $O(|V|)$ iterations. We do this by showing that each vertex u may be the beginning element of Q a bounded number of times. To this end, fix an arbitrary vertex $u \in V(G)$ and let $t_1 < t_2 < \cdots < t_N$ be those iterations in which u is at the beginning of the queue $Q(t)$. (Observe that the algorithm deletes u from H in iteration t_N.) We assert that the following holds.

Claim 8. *For each $i \in \{1, \dots, N-2\}$, there is an iteration $t \in \{t_i, \dots, t_{i+1}-1\}$ in which at least one edge of $N_{H(t)}^{21}(u)$ is deleted from $H(t)$.*

Proof of Claim 8. If in iteration t_i a minimally good E' is found (one of whose edges is incident to u), then step 2b is taken and the statement is clearly satisfied. So we may assume that in iteration t_i no such minimally good E' is found. Thus, in iteration t_i we move u to the end of Q by step 2a.

Next observe that, if in some iteration $t \in \{t_i+1, \dots, t_{i+1}-1\}$ a minimally good E' is found with $u \in N_{H(t)}^{20}(E')$, then at least one edge of $N_{H(t)}^{21}(u)$ is removed. So we may assume that this is not the case so that, in particular, u is not replaced at the beginning of Q by step 2b during any iteration $t \in \{t_i, \dots, t_{i+1}-1\}$.

Let us write $Q(t_{i+1}) = (v_0 = u, v_1, \dots, v_k)$, and let us consider an arbitrary index $\ell \in \{1, \dots, k\}$. Since u was not replaced at the beginning of the queue during any iteration in $\{t_i, \dots, t_{i+1}-1\}$ and v_ℓ is behind u in iteration t_{i+1}, there must be at least one iteration $t \in \{t_i, \dots, t_{i+1}-1\}$ in which v_ℓ was first in the queue and was moved to the end by step 2a. Let $s \in \{t_i, \dots, t_{i+1}-1\}$ be the last iteration before t_{i+1} in which v_ℓ was moved to the end of the queue. Since v_ℓ remained behind u in the queue during each iteration in $\{s+1, \dots, t_{i+1}-1\}$, in none of these iterations was a minimally good E' found with $v_\ell \in N_{H(t)}^{20}(E')$. So, in particular, no edges of $N_{H(s)}^{18}(v_\ell)$ were deleted in the iterations $t \in \{s, \dots, t_{i+1}-1\}$. This implies that

$$N_{H(t_{i+1})}^{18}(v_\ell) = N_{H(s)}^{18}(v_\ell).$$

As noted previously, we can determine from $H[N_H^{18}(v_\ell)]$ whether there is a minimally good E' with one edge incident to v_ℓ. It thus follows that there is no such minimally good E' incident to v_ℓ in $H(t_{i+1})$; otherwise, step 2a would not have been taken in iteration s. Since $\ell \in \{1, \dots, k\}$ was arbitrary, there is in fact no minimally good E' in $H(t_{i+1})$ at all. But this contradicts Theorem 5! (Since $i+1 \leq N-1$, there is at least one iteration after t_{i+1} in which u occurs at the beginning of the queue. Hence u is not isolated at time t_{i+1}—otherwise it would get deleted—and in particular $H(t_{i+1})$ has at least one edge.)

It follows that either step 2b was taken in iteration t_i, or there was an iteration $t \in \{t_i+1, \ldots, t_{i+1}-1\}$ in which step 2b was taken and some edge of $N^{21}_{H(t)}(u)$ was deleted from $H(t)$. This concludes the proof of Claim 8. $\square$

By Claim 8, the vertex u occurs as the first element of the queue Q in at most

$$N \leq |E(N^{21}_G(u))| + 2 \leq 3 \cdot 2^{21} + 2 = O(1)$$

iterations of the while loop. Since u was arbitrary, the while loop is iterated at most $|V| \cdot (3 \cdot 2^{21} + 2) = O(|V|)$ times, which concludes the proof of Theorem 7. $\square$

We comment here that our distance estimate in Lemma 6 is sufficient for arguing that the time complexity is linear without knowing the full details of Theorem 5. As we have just seen, the upper bound 15 in Lemma 6 leads to a bound on the number of iterations of the while loop of at most $(3 \cdot 2^{21} + 2) \cdot |V|$. However, a closer examination of the proof of Theorem 5 (and the claims used to prove Lemma 9 in particular) demonstrates that we are in fact guaranteed a good set such that no two of its edges are at distance greater than 6. This implies that indeed the number of iterations is at most $(3 \cdot 2^{12} + 2) \cdot |V|$, an improvement of a factor approximately 2^9. Furthermore, if we consider only neighborhoods at smaller distances, we will obtain a similar improvement on the number of computations within each iteration. This suggests that our algorithm could reasonably be implemented.

4. The proof of Theorem 5. Theorem 5 is a direct consequence of the following two lemmas. Recall that a plane graph is a planar graph with a fixed embedding in the plane. Fixing an embedding has the advantage that we can speak unambiguously of the faces of the graph. Although it was difficult to develop, we do not claim that the following collection of twelve structures is optimal. As is often the case with discharging methods, later improvements may be found. Roughly speaking, what is most important about this collection for induced matchings is that the structures are locally sparse.

LEMMA 9. *Let G be a subcubic plane graph. If G contains one of the following structures, then G contains a good set of edges:*

(C1) *a 1-vertex;*
(C2) *a 2-vertex incident to an $(\leqslant 6)$-cycle or 7-face;*
(C3) *a 2-vertex at distance at most 2 from another 2-vertex;*
(C4) *a 2-vertex at distance at most 2 from an $(\leqslant 5)$-cycle;*
(C5) *a 3-cycle adjacent to an $(\leqslant 7)$-cycle;*
(C6) *a 4- or 5-cycle in sequence with a 5- or 6-cycle;*
(C7) *a 3-cycle at distance 1 from an $(\leqslant 5)$-cycle;*
(C8) *a double 4-face adjacent to an $(\leqslant 7)$-cycle;*
(C9) *a 4-cycle, $(\leqslant 8)$-cycle, and 4-cycle in sequence;*
(C10) *a 4-cycle, 7-cycle, and 5-cycle in sequence;*
(C11) *a 3-cycle or double 4-face at distance at most 2 from a 3-cycle or double 4-face; and*
(C12) *a double 4-face at distance 1 from a 5-cycle.*

LEMMA 10. *Every subcubic plane graph with at least one edge contains one of the structures* (C1)–(C12) *listed in Lemma* 9.

The proof of Lemma 9 is a rather lengthy case analysis, which we postpone to subsection 4.2. We now prove Lemma 10 by means of a discharging procedure.

4.1. The proof of Lemma 10. Suppose that G is a subcubic plane graph with at least one edge, and that G does not contain any of the structures (C1)–(C12). Without loss of generality, we may assume that G has no isolated vertices. (Note also that the removal of isolated vertices does not affect whether a graph has a good set or not.)

We will obtain a contradiction by using the discharging method, which is commonly used in graph coloring. The rough idea of this method is as follows: each vertex and face of G is assigned an initial "charge." Here the charges are chosen such that their total sum is negative. We then apply certain redistribution rules (the discharging procedure) for exchanging charge between the vertices and faces. These redistribution rules are chosen such that the total sum of charges is invariant. However, we will prove by a case analysis that if G contains none of (C1)–(C12), then each vertex and each face will have nonnegative charge after the discharging procedure has finished. This contradicts that the total sum of the charges is negative. Hence, G must have at least one of (C1)–(C12).

Initial charge. For every vertex $v \in V$, we define the initial charge $\mathrm{ch}(v)$ to be $2\deg(v) - 6$, while for every face $f \in F$, we define the initial charge $\mathrm{ch}(f)$ to be $\deg(f) - 6$. We claim that this way the total sum of initial charges is negative. To see this, note that by Euler's formula $6|E| - 6|V| - 6|F| = -12$. It follows from $\sum_{v\in V} \deg(v) = 2|E| = \sum_{f\in F} \deg(f)$ that

$$-12 = (4|E| - 6|V|) + (2|E| - 6|F|) = \sum_{v\in V}(2\deg(v) - 6) + \sum_{f\in F}(\deg(f) - 6).$$

Discharging procedure. To describe a discharging procedure, it suffices to fix how much each vertex or face sends to each of the other vertices and faces. In our case, vertices and ($\leqslant$6)-faces do not send any charge. The ($\geqslant$7)-faces redistribute their charge as follows:

Each ($\geqslant$7)-face f sends
- (R1) 1 to each incident 2-vertex,
- (R2) 1 to each adjacent 3-face,
- (R3) 1 to each adjacent 4-face in a double 4-face if f and the two 4-faces are in sequence,
- (R4) 1/2 to each other adjacent 4-face, and
- (R5) 1/5 to each adjacent 5-face.

When we say that an ($\geqslant$7)-face sends a charge to an adjacent face or incident vertex, we mean that the charge is sent as many times as these elements are adjacent or incident to each other. For $v \in V$ and $f \in F$, we denote their final charges—that is, the charges after the redistribution—by $\mathrm{ch}^*(v)$ and $\mathrm{ch}^*(f)$, respectively.

In the following analysis of the final charges of vertices and faces, we will often say that something holds "by (Cx)" for some $x \in \{1, \ldots, 12\}$. By this we of course mean "by the absence of (Cx)."

Final charge of 2-vertices. The initial charge of a 2-vertex is -2. By (C2) it is adjacent to two ($\geqslant$8)-faces. Hence, it receives 2 by (R1), so that its final charge is 0.

Final charge of ($\geqslant$3)-vertices. An ($\geqslant$3)-vertex has nonnegative initial charge. Since it sends no charge, its final charge is nonnegative.

Final charge of 3-faces. A 3-face has initial charge -3. By (C5) it is adjacent only to ($\geqslant$8)-faces. Hence, it receives a charge of 3 by (R2), and its final charge is 0.

Final charge of 4-faces. Let f be a 4-face; then its initial charge is $\mathrm{ch}(f) = -2$. If f is not in a double 4-face, then by (C5) and (C6) f is adjacent only to ($\geqslant$7)-faces and receives a charge of $1/2$ from each by (R4); thus, $\mathrm{ch}^*(f) = 0$. Otherwise, if f is in a double 4-face, then f is adjacent to exactly one 4-face and three ($\geqslant$8)-faces by (C8). Thus, f receives a charge of 1 from an ($\geqslant$8)-face by (R3) and charges of $1/2$ from the other two by (R4), and so $\mathrm{ch}^*(f) = 0$.

Final charge of 5-faces. Let f be a 5-face; then its initial charge is $\mathrm{ch}(f) = -1$. Since f is not adjacent to any ($\leqslant$6)-faces by (C5) and (C6), it receives a charge of $1/5$ from each adjacent face by (R5), and so $\mathrm{ch}^*(f) = 0$.

Final charge of 6-faces. The initial charge of a 6-face is 0, and it sends no charge, so its final charge is 0.

Final charge of 7-faces. Let f be a 7-face; then its initial charge is $\mathrm{ch}(f) = 1$. By (C5), (C8), (C9), and (C10), f is adjacent to no 3-faces, no double 4-faces, and at most two 4- or 5-faces. Thus, f sends a charge of at most $2 \cdot 1/2$ by (R4) or (R5), and so $\mathrm{ch}^*(f) \geq 0$.

Final charge of 8-faces. Let f be an 8-face; then its initial charge is $\mathrm{ch}(f) = 2$. We consider several cases.

First, suppose that f is incident to a 2-vertex. By (C3), f is incident to at most two 2-vertices. However, if f is incident to exactly two 2-vertices (and hence by (R1) sends a charge of 1 to each), then by (C2) and (C4) f is adjacent only to ($\geqslant$6)-faces; thus, $\mathrm{ch}^*(f) = 0$. So assume that f is incident to exactly one 2-vertex v to which f sends charge 1 by (R1). By (C4), faces that are adjacent to f but at distance at most 2 from v must be ($\geqslant$6)-faces. There remain two other faces adjacent to f, which are adjacent to each other. If one of these is a 3-face (sent charge 1 by (R2)), then the other is an ($\geqslant$8)-face by (C5) so that $\mathrm{ch}^*(f) = 0$. If both are 4-faces, then neither is in sequence with f and some other 4-face by (C8), and so both receive $1/2$ from f by (R4) so that $\mathrm{ch}^*(f) = 0$. If one is a 4-face and the other is a 5-face, then by (C8) the 4-face is not in a double 4-face, and each is sent charge at most $1/2$ by (R4) and (R5) so that again $\mathrm{ch}^*(f) \geq 0$. If one is a 4-face (sent charge at most 1 by (R3) or (R4)) and the other is an ($\geqslant$6)-face, then $\mathrm{ch}^*(f) \geq 0$. Finally, if both are ($\geqslant$5)-faces, then f sends to each a charge of at most $1/5$ by (R5), and $\mathrm{ch}^*(f) > 0$. We may hereafter assume that f is not incident to a 2-vertex.

Second, suppose that f is adjacent to a 3-face f'. By (C5) and (C7), faces that are adjacent to f but at distance at most 1 from f' must be ($\geqslant$6)-faces. There remain three other faces adjacent to f, call them f_1, f_2, f_3, in sequence. By (C7), (C8), (C9), (C11), and (C12), if one of these is sent charge 1 by (R2) or (R3), then the others are ($\geqslant$6)-faces so that $\mathrm{ch}^*(f) \geq 0$. Furthermore, if two of these are in the same double 4-face (each sent $1/2$ by (R4)), say f_1 and f_2, then f_3 is an ($\geqslant$6)-face and $\mathrm{ch}^*(f) \geq 0$. We can thus suppose that none of f_1, f_2, f_3 is a 3-face or part of a double 4-face. By (C9), at most one of f_1, f_2, f_3 is a 4-face, and by (C6) at most two are 5-faces. Hence, f sends total charge at most $1 + 1/2 + 2/5 < 2$ by (R4) and (R5), and $\mathrm{ch}^*(f) > 0$. We may hereafter assume that f is not adjacent to a 3-face.

Third, suppose that f is adjacent to a 4-face f'. Assume that f' is part of a double 4-face and f'' is its 4-face partner. By (C6), (C8), (C9), (C11), and (C12), all faces, with the possible exception of f'', that are adjacent to f but at distance at most 1 from f' must be ($\geqslant$6)-faces. There remain at most three other faces adjacent to f, and we can proceed as in the previous case. Thus f' is not part of a double 4-face. It follows by (C9) that, of the faces that are adjacent to f but at distance at most 1 from f', none is a 4-face; furthermore, by (C6), at most two of these are 5-faces. Thus, by (R4) and (R5), in total at most $1/2 + 2/5 < 1$ charge is sent to f' and these four faces. Again, there remain at most three other faces adjacent to f, and we proceed as in the previous paragraph. We may hereafter assume that f is not adjacent to an ($\leqslant$4)-face.

Finally, by (C6), f is adjacent to at most four 5-faces, and so by (R5) f sends a total charge of at most $4/5 < 2$, and $\mathrm{ch}^*(f) > 0$.

Final charge of ($\geqslant$9)-faces. Let f be an ($\geqslant$9)-face, and let $v_1e_1v_2e_2v_3e_3v_4e_4v_5$ be a path of four edges along f. Denote by f_i the face adjacent to f via the edge e_i. We first show that the combined charge sent through these four edges (counting half of the charge contributed to the end-vertices v_1, v_5) is at most $3/2$.

First, suppose that at least one of the v_i is a 2-vertex. By (C3), at most two are 2-vertices. If two are, then, without loss of generality, either v_1 and v_4 are 2-vertices, or v_1 and v_5 are 2-vertices. In both cases, $f_1, \dots, f_4$ are all ($\geqslant$6)-faces by (C4) and the total charge sent is at most $3/2$ (by (R1), except that one contribution is halved). If exactly one of the v_i is a 2-vertex, then without loss of generality, either v_1 is a 2-vertex, or one of v_2 or v_3 is. In the former case, we have by (C4) that f_1, f_2, and f_3 are ($\geqslant$6)-faces and the total charge sent is at most $3/2$ (since the (R1) contribution to v_1 is halved and f_4 is sent charge at most 1). In the latter case, we have by (C4) that all four faces are ($\geqslant$6)-faces and the total charge sent is 1 by (R1). We may hereafter assume that none of the v_i is a 2-vertex.

Second, suppose that some f_i is a 3-face. By symmetry, there are two cases to consider: $i = 1$ or $i = 2$. In the former case, we have by (C5), (C7), and (C11) that f_2 and f_3 are both ($\geqslant$6)-faces and f_4 is forbidden from being a 3-face or part of a double 4-face, in which case the total charge sent is at most $3/2$ (by (R2) and (R4) or (R5)). In the latter case, we have by (C5) and (C7) that f_1, f_3, and f_4 are ($\geqslant$6)-faces, in which case the total charge sent is 1 (to f_2 by (R2)).

Third, suppose that some f_i is part of a double 4-face. Without loss of generality, there are four subcases to consider: (a) f_1 is part of a double 4-face, but f_2 is not, (b) f_1 and f_2 are part of the same double 4-face, (c) f_2 is part of a double 4-face, but neither f_1 nor f_3 is, and (d) f_2 and f_3 are part of the same double 4-face. In case (a), f_1 is sent charge 1 by (R3). By (C6), (C8), and (C11), at most one of f_2, f_3, f_4 is a 4- or 5-face and none is part of a double 4-face, in which case, by (R4) or (R5), the total charge sent is at most $3/2$. In (b), we have by (C8) and (C11) that f_3 is an ($\geqslant$6)-face and f_4 is not part of a double 4-face; thus, by (R4) and (R5), the total charge sent is at most $3/2$. In case (c), we have by (C8) that f_1, f_3 are ($\geqslant$6)-faces and by (C11) that f_4 is not part of a double 4-face (and hence sent charge at most $1/2$ by (R4) or (R5)), so that the total charge sent is at most $3/2$. In (d), we have by (C8) that f_1, f_4 are ($\geqslant$6)-faces, so that by (R4) the total charge sent is 1. In all four subcases, the total charge sent is at most $3/2$.

We now have that none of the v_i is a 2-vertex, and none of the f_i is a 3-face or part of a double 4-face. By (C6), not every f_i is a 4- or 5-face. Thus, there is one face sent no charge while each of the others is by (R4) or (R5) sent at most $1/2$; the total charge sent is at most $3/2$, completing our proof of the claim.

It remains to complete the analysis of the final charge for f using this claim. Let us denote the facial cycle by $v_1e_1v_2e_2v_3 \cdots v_ke_kv_1$ and denote by f_i the face adjacent to f via the edge e_i. By "rotating" the labelling, we may assume without loss of generality that $\deg(v_1) = \deg(v_2) = 3$ and f_1 is an ($\geqslant$6)-face, so f_1 is sent no charge. By the claim, halving the contribution to v_{10}, the total charge sent to $f_2, \ldots, f_9$ is at most 3. Every face f_i, $i > 9$, receives a charge of at most 1 from f (including half the charge sent to v_i and v_{i+1}). Hence, f sends total charge at most $3 + \deg(f) - 9 = \deg(f) - 6 = \mathrm{ch}(f)$, and so $\mathrm{ch}^*(f) \geq 0$.

We have seen that every vertex and every face of G have nonnegative final charge, which gives the required contradiction and completes the proof.

4.2. The proof of Lemma 9. In this section, we prove Lemma 9 by analyzing the structures in order, including some intermediate structures. The order of our analysis is significant. The proofs for the presence of later structures rely in part on the absence of earlier structures.

We give a figure for each structure. We employ a visual code: a square represents an ($\leqslant$2)-vertex, a circle represents an ($\leqslant$3)-vertex, a thin solid line represents a present edge, and a bold solid line indicates membership in a good set. In Table 1, we provide for convenience a key for matching the claims and figures with the structures.

Throughout this section, we assume G to be a subcubic plane graph.

CLAIM 11. *If G has a 1-vertex, then it contains a good set of edges.*

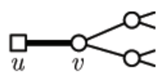

FIG. 8. *A 1-vertex.*

Proof. If u is a 1-vertex with neighbor v, then $E' = \{uv\}$ is a good set as $|\Psi(E')| \leq 7$. See Figure 8. □

CLAIM 12. *If G has two adjacent 2-vertices, then it contains a good set of edges.*

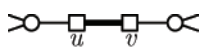

FIG. 9. *Adjacent 2-vertices.*

Proof. If u, v are adjacent 2-vertices, then $E' = \{uv\}$ is a good set as $|\Psi(E')| \leq 7$. See Figure 9. □

CLAIM 13. *If G has a 2-vertex on a 3-cycle, then it contains a good set of edges.*

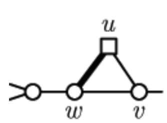

FIG. 10. *A 2-vertex on a 3-cycle.*

Proof. If u is a 2-vertex on 3-cycle uvw, then $E' = \{uw\}$ is a good set as $|\Psi(E')| \leq 7$. See Figure 10. □

CLAIM 14. *If G has a 2-vertex on a 4-cycle, then it contains a good set of edges.*

Proof. If u is a 2-vertex on 4-cycle $uvwx$, then $E' = \{ux\}$ is a good set as $|\Psi(E')| \leq 9$. See Figure 11. □

CLAIM 15. *If G has two 2-vertices at distance 2, then it contains a good set of edges.*

TABLE 1
A key to cross-referencing the claims and figures with the structures.

Claim 11	Figure 8	A 1-vertex.	(C1)
Claim 12	Figure 9	Adjacent 2-vertices.	(C3)
Claim 13	Figure 10	A 2-vertex on a 3-cycle.	(C2)
Claim 14	Figure 11	A 2-vertex on a 4-cycle.	(C2)
Claim 15	Figure 12	Two 2-vertices at distance 2.	(C3)
Claim 16	Figure 13	A 2-vertex on a 5-cycle.	(C2)
Claim 17	Figure 14	A 2-vertex on a 6-cycle.	(C2)
Claim 18	Figure 15	A 2-vertex at distance 1 from a 3-cycle.	(C4)
Claim 19	Figure 16	A 2-vertex at distance 1 from a 4-cycle.	(C4)
Claim 20	Figure 17	A 2-vertex at distance 1 from a 5-cycle.	(C4)
Claim 21	Figure 18	A 2-vertex on a 7-face.	(C2)
Claim 22	Figure 19	Adjacent 3-cycles.	(C5), (C11)
Claim 23	Figure 20	A 3-cycle adjacent to a 4-cycle.	(C5), (C8), (C11)
Claim 24	Figure 21	A 3-cycle adjacent to a 5-cycle.	(C5)
Claim 25	Figure 22	A 3-cycle adjacent to a 6-cycle.	(C5)
Claim 26	Figure 23	A 3-cycle adjacent to a 7-cycle.	(C5)
Claim 27	Figure 24	A pair of 4-cycles adjacent along two incident edges.	
Claim 28	Figure 25	A 5-cycle adjacent to a 4-cycle along two incident edges.	
Claim 29	Figure 26	Three 4-cycles in sequence.	(C8)
Claim 30	Figure 27	Three 4-cycles that are pairwise in sequence.	(C8)
Claim 31	Figure 28	Two 4-cycles and a 5-cycle that are pairwise in sequence.	
Claim 32	Figure 29	Two 4-cycles and a 6-cycle that are pairwise in sequence.	(C8)
Claim 33	Figure 30	Two 4-cycles and a 7-cycle that are pairwise in sequence.	(C8)
Claim 34	Figure 31	Two 5-cycles and a 4-cycle that are pairwise in sequence.	
Claim 35	Figure 32	A 4-cycle adjacent to a 5-cycle.	(C6), (C8)
Claim 36	Figure 33	Two adjacent 5-cycles.	(C6)
Claim 37	Figure 34	A 4-cycle adjacent to a 6-cycle along two incident edges.	
Claim 38	Figure 35	A 4-cycle in sequence with a 6-cycle.	(C6), (C8)
Claim 39	Figure 36	A 5-cycle in sequence with a 6-cycle.	(C6)
Claim 40	Figure 37	A 2-vertex at distance 2 from a 3-cycle.	(C4)
Claim 41	Figure 38	A 2-vertex at distance 2 from a 4-cycle.	(C4)
Claim 42	Figure 39	A 2-vertex at distance 2 from a 5-cycle.	(C4)
Claim 43	Figure 40	Two 3-cycles at distance 1.	(C7), (C11)
Claim 44	Figure 41	A 3-cycle at distance 1 from a 4-cycle.	(C7), (C11)
Claim 45	Figure 42	A 3-cycle at distance 1 from a 5-cycle.	(C7)
Claim 46	Figure 43	A 5-cycle at distance 1 from a double 4-face.	(C12)
Claim 47	Figure 44	A pair of double 4-faces at distance 1.	(C11)
Claim 48	Figure 45	A 4-cycle, ($\leqslant$8)-cycle, and 4-cycle in sequence.	(C9)
Claim 49	Figure 46	A sequence of 4-cycles such that one of the 4-cycles is adjacent to a 7-cycle.	(C8)
Claim 50	Figure 47	A 4-cycle, 7-cycle, and 5-cycle in sequence.	(C10)
Claim 51	Figure 48	Two 3-cycles at distance 2.	(C11)
Claim 52	Figure 49	A 3-cycle at distance 2 from a double 4-face.	(C11)
Claim 53	Figure 50	A pair of double 4-faces at distance 2.	(C11)

Proof. Let u, w be 2-vertices at distance 2. Let $N(u) \cap N(w) = \{v\}$, and let $N(u) \setminus \{v\} = \{u'\}$ and $N(w) \setminus \{v\} = \{w'\}$. Notice that by Claims 13 and 14, vertices u, u', w, w' are distinct. Since v is a 3-vertex by Claim 12, it has a neighbor $v' \notin \{u, w\}$. Because G has no 2-vertex on an ($\leqslant$4)-cycle by Claims 13 and 14, v' is distinct from u' and w', and neither $u'v'$ nor $w'v'$ is an edge. If $u'w'$ is an edge, then $E' = \{u'w', vv'\}$

FIG. 11. *A* 2*-vertex on a* 4*-cycle.*

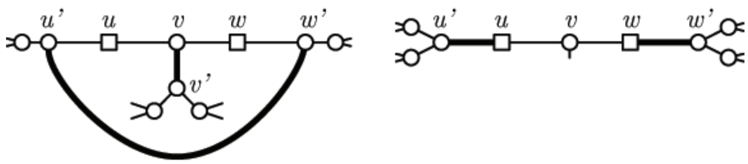

FIG. 12. *Two* 2*-vertices at distance* 2*.*

is a good set since $|\Psi(E')| \leq 18$. Otherwise, $E' = \{uu', ww'\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 12. ∎

CLAIM 16. *If G has a* 2*-vertex on a* 5*-cycle, then it contains a good set of edges.*

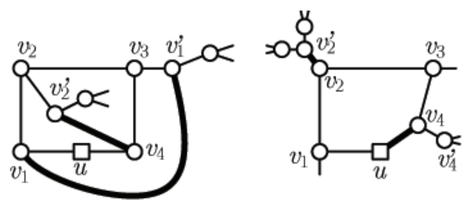

FIG. 13. *A* 2*-vertex on a* 5*-cycle.*

Proof. Let u be a 2-vertex on a 5-cycle $C = uv_1v_2v_3v_4$. Observe that v_1, v_2, v_3, v_4 are 3-vertices, by Claims 12 and 15. By Claims 13 and 14, C has no chords. For $i = 1, 2, 3$, let $N(v_i) \setminus \{u, v_1, \ldots, v_4\} = \{v_i{}'\}$. By Claim 14, $v_1{}' \neq v_4{}'$.

(1) If $v_2{}' = v_4{}'$ and $v_1{}' = v_3{}'$, then $E' = \{v_1v_1{}', v_4v_4{}'\}$ is a good set since $|\Psi(E')| \leq 15$.

(2) Suppose that $v_2{}' \neq v_4{}'$. Then $E' = \{uv_4, v_2v_2{}'\}$ is a good set since $|\Psi(E')| \leq 17$. The case that $v_1{}' \neq v_3{}'$ is handled symmetrically.

See Figure 13. ∎

CLAIM 17. *If G has a* 2*-vertex on a* 6*-cycle, then it contains a good set of edges.*

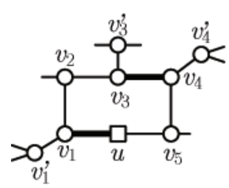

FIG. 14. *A* 2*-vertex on a* 6*-cycle.*

Proof. Let u be a 2-vertex on a 6-cycle $uv_1v_2v_3v_4v_5$. By Claims 14 and 16, neither v_1v_3 nor v_1v_4 can be an edge. Then $E' = \{uv_1, v_3v_4\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 14. ∎

CLAIM 18. *If G has a 2-vertex at distance 1 from a 3-cycle, then it contains a good set of edges.*

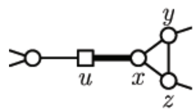

FIG. 15. *A 2-vertex at distance 1 from a 3-cycle.*

Proof. Let u be a 2-vertex at distance 1 from a 3-cycle xyz, where ux is an edge. Then $E' = \{ux\}$ is a good set since $|\Psi(E')| \leq 9$. See Figure 15. ∎

CLAIM 19. *If G has a 2-vertex at distance 1 from a 4-cycle, then it contains a good set of edges.*

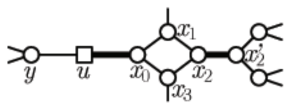

FIG. 16. *A 2-vertex at distance 1 from a 4-cycle.*

Proof. Let u be a 2-vertex at distance 1 from a 4-cycle $x_0x_1x_2x_3$, where ux_0 is an edge. By Claim 14, ux_2 is not an edge and x_2 is a 3-vertex. Let $N(x_2) \setminus \{x_1, x_3\} = \{x_2'\}$. By Claim 16, ux_2' is not an edge. So $E' = \{ux_0, x_2x_2'\}$ is a good set as $|\Psi(E')| \leq 17$. See Figure 16. ∎

CLAIM 20. *If G has a 2-vertex at distance 1 from a 5-cycle, then it contains a good set of edges.*

FIG. 17. *A 2-vertex at distance 1 from a 5-cycle.*

Proof. Let u be a 2-vertex at distance 1 from a 5-cycle $x_0x_1x_2x_3x_4$, where ux_0 is an edge. By Claim 14, neither ux_2 nor ux_3 is an edge. Then $E' = \{ux_0, x_2x_3\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 17. ∎

CLAIM 21. *If G has a 2-vertex on a 7-face, then it contains a good set of edges.*

Proof. Let u be a 2-vertex on a 7-face $C = uv_1v_2v_3v_4v_5v_6$. By Claims 12 and 15, v_1, v_2, v_5, v_6 are 3-vertices. By Claims 13, 14, 16, and 17, C has no chords. For $i = 1, 2, 5, 6$, let $N(v_i) \setminus \{u, v_1, \ldots, v_6\} = \{v_i'\}$.

(1) If v_3 or v_4 is a 2-vertex, say v_4, by symmetry, then $E' = \{uv_1, v_4v_5\}$ is a good set since $|\Psi(E')| \leq 16$.

For $i = 3, 4$, let $N(v_i) \setminus \{v_2, \ldots, v_5\} = \{v_i'\}$. By Claims 16 and 19, $v_1' \neq v_5'$ and $v_1' \neq v_3'$.

(2) If $v_2'v_4'$ is an edge or $v_2' = v_4'$, then since C is a 7-face, it follows from the Jordan Curve Theorem that $v_3' \neq v_5'$ and $v_3'v_5'$ is not an edge. Then $E' = \{uv_1, v_3v_3', v_5v_5'\}$ is a good set since $|\Psi(E')| \leq 27$. The case in which $v_3'v_5'$ is an edge or $v_3' = v_5'$ is handled similarly.

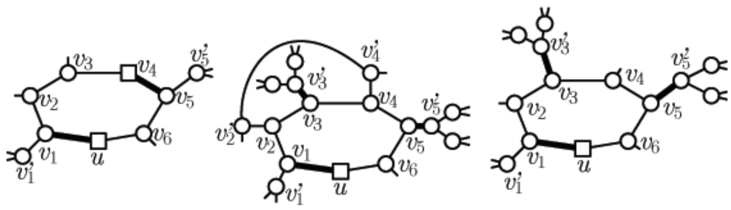

FIG. 18. *A* 2*-vertex on a* 7*-face.*

Otherwise, $E' = \{uv_1, v_3v_3', v_5v_5'\}$ is a good set since $|\Psi(E')| \le 27$. See Figure 18. ∎

CLAIM 22. *If G has a pair of adjacent* 3*-cycles, then it contains a good set of edges.*

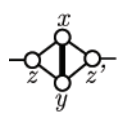

FIG. 19. *Adjacent* 3*-cycles.*

Proof. If xyz and xyz' are adjacent 3-cycles, then $E' = \{xy\}$ is a good set as $|\Psi(E')| \le 7$. See Figure 19. ∎

CLAIM 23. *If G has a* 3*-cycle adjacent to a* 4*-cycle, then it contains a good set of edges.*

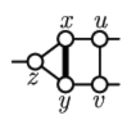

FIG. 20. *A* 3*-cycle adjacent to a* 4*-cycle.*

Proof. Let xyz be a 3-cycle sharing the edge xy with 4-cycle $xuvy$. Observe that the two cycles have no other edges in common by Claim 22. Then $E' = \{xy\}$ is a good set as $|\Psi(E')| \le 9$. See Figure 20. ∎

CLAIM 24. *If G has a* 3*-cycle adjacent to a* 5*-cycle, then it contains a good set of edges.*

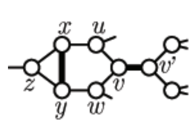

FIG. 21. *A* 3*-cycle adjacent to a* 5*-cycle.*

Proof. By Claim 23, the 3-cycle and 5-cycle share at most one edge. So let xyz be the 3-cycle, and let $xuvwy$ be the 5-cycle. Observe that v is a 3-vertex by Claim 16. Let $N(v) \setminus \{u, w\} = \{v'\}$. Then $E' = \{xy, vv'\}$ is a good set since $|\Psi(E')| \le 17$. See Figure 21. ∎

CLAIM 25. *If G has a 3-cycle adjacent to a 6-cycle, then it contains a good set of edges.*

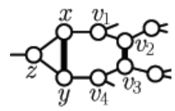

FIG. 22. *A 3-cycle adjacent to a 6-cycle.*

Proof. By Claim 24, the 3-cycle and 6-cycle share at most one edge. So let xyz be the 3-cycle, and let $xv_1v_2v_3v_4y$ be the 6-cycle. Then $E' = \{xy, v_2v_3\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 22. ◻

CLAIM 26. *If G has a 3-cycle adjacent to a 7-cycle, then it contains a good set of edges.*

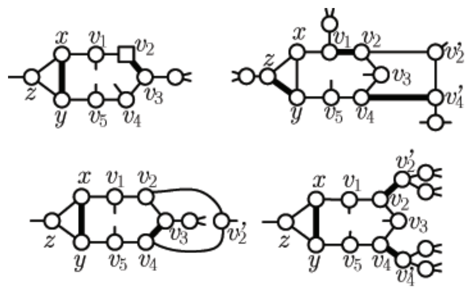

FIG. 23. *A 3-cycle adjacent to a 7-cycle.*

Proof. By Claim 25, the 3-cycle and 7-cycle share at most one edge. So let xyz be the 3-cycle, and let $C = xv_1v_2v_3v_4v_5y$ be the 7-cycle. By Claims 23 and 24, C has no chords.

(1) If v_2 or v_4 is a 2-vertex, say v_2, by symmetry, then $E' = \{xy, v_2v_3\}$ is a good set as $|\Psi(E')| \leq 16$.

For $i = 2, 4$, let $N(v_i) \setminus \{v_{i-1}, v_{i+1}\} = \{v_i{}'\}$.

(2) Suppose $v_2{}'v_4{}'$ is an edge. By Claim 25, $v_1v_4{}'$ is not an edge. Then $E' = \{yz, v_1v_2, v_4v_4{}'\}$ is a good set since $|\Psi(E')| \leq 24$.

(3) Suppose $v_2{}' = v_4{}'$. Then $E' = \{xy, v_3v_4\}$ is a good set since $|\Psi(E')| \leq 18$.

Otherwise, $E' = \{xy, v_2v_2{}', v_4v_4{}'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 23. ◻

CLAIM 27. *If G has a pair of 4-cycles adjacent along two incident edges, then it contains a good set of edges.*

Proof. Let $x_0x_1x_2x_3$ and $x_0x_1x_2x_4$ be the two 4-cycles. Furthermore, suppose that in the embedding of the graph, the vertex x_1 is in the interior of the curve formed by the cycle $x_0x_3x_2x_4$. Observe that x_1, x_3, x_4 are 3-vertices by Claim 14. For $i = 1, 3, 4$, let $N(x_i) \setminus \{x_0, x_2\} = \{x_i{}'\}$.

(1) If $x_3{}' = x_4$, then $E' = \{x_1x_1{}', x_3x_4\}$ is a good set since $|\Psi(E')| \leq 14$.

(2) Suppose $x_3{}' = x_4{}'$. Since G is a plane graph, the embedding of $x_3x_3{}'$ is necessarily exterior to the curve formed by the cycle $x_0x_4x_2x_4$. By the Jordan Curve Theorem, $x_1{}'x_3{}'$ is not an edge. Then $E' = \{x_1x_1{}', x_3x_3{}'\}$ is a good set since $|\Psi(E')| \leq 18$.

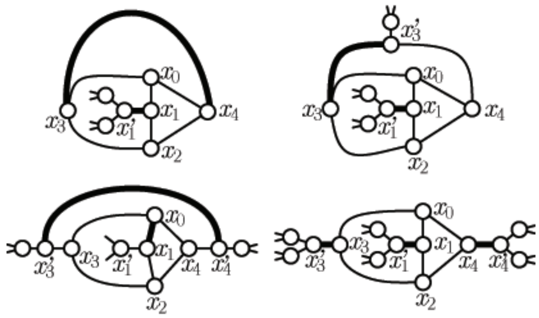

FIG. 24. *A pair of 4-cycles adjacent along two incident edges.*

(3) Suppose $x_3'x_4'$ is an edge. Since G is a plane graph, the embedding of $x_3'x_4'$ is necessarily exterior to the curve formed by the cycle $x_0x_4x_2x_4$. By the Jordan Curve Theorem, $x_1' \neq x_3'$ and $x_1' \neq x_4'$. Then $E' = \{x_0x_1, x_3'x_4'\}$ is a good set since $|\Psi(E')| \leq 18$.

All of the above cases may be repeated with the roles of x_3 and x_4 played instead by, respectively, x_1 and x_3, or, respectively, x_1 and x_4. Then $E' = \{x_1x_1', x_3x_3', x_4x_4'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 24. ∎

CLAIM 28. *If G has a 5-cycle adjacent to a 4-cycle along two incident edges, then it contains a good set of edges.*

Proof. Let $v_1v_2v_3v_4v_5$ be the 5-cycle, and let $uv_4v_5v_1$ be the 4-cycle. Furthermore, suppose that in the embedding of the graph, the vertex v_5 is in the interior of the curve formed by the cycle $uv_1v_2v_3v_4$. By Claims 14 and 16, u, v_2, v_3, v_5 are 3-vertices. Let $N(u) \setminus \{v_1, v_4\} = \{u'\}$ and, for $i = 2, 3, 5$, let $N(v_i) \setminus \{v_1, \ldots, v_5\} = \{v_i'\}$. By Claims 23 and 24, uv_2, uv_3, uv_5, v_2v_5, v_3v_5 are not edges. By Claim 27, $u' \neq v_5'$.

(1) Suppose $u'v_5'$ is an edge. By Claim 16, u' is a 3-vertex, so let $N(u') \setminus \{u, v_5'\} = \{u''\}$. Note that v_5', u', u'' must be in the interior of the curve formed by the cycle $uv_4v_5v_1$, whereas v_3 and v_3' are exterior to this curve. By the Jordan Curve Theorem, v_3u', v_3u'', $v_3'u'$, $v_3'u''$ are not edges. Then $E' = \{v_1v_5, v_3v_3', u'u''\}$ is a good set since $|\Psi(E')| \leq 26$.

(2) Suppose $v_3' = v_5'$.
  (a) If v_2' and v_3' have a common neighbor p, then $E' = \{v_3v_3', uv_1\}$ is a good set since $|\Psi(E')| \leq 16$.
  (b) If $u' = v_2'$, then $E' = \{v_3v_3', uv_1\}$ is a good set since $|\Psi(E')| \leq 15$.
  (c) Suppose we are in neither of the last two subcases. By Claim 14, v_3' is a 3-vertex. Let $N(v_3') \setminus \{v_3, v_5\} = \{v_3''\}$. Note that v_3' and hence v_3'' must be in the interior of the curve formed by the embedding of the cycle $v_1v_2v_3v_4v_5$, whereas u is exterior to this curve; thus, by the Jordan Curve Theorem, uv_3'' is not an edge. Then $E' = \{v_2v_2', v_3'v_3'', uv_4\}$ is a good set since $|\Psi(E')| \leq 26$.

The case for which $v_2' = v_5'$ is handled similarly.

(3) Suppose $u' = v_3'$. Then it must be that u' is exterior to the curve formed by the cycle $uv_1v_2v_3v_4$, and in particular $u'v_5$ is not an edge by the Jordan Curve Theorem. Then $E' = \{u'v_3, v_5v_5'\}$ is a good set since $|\Psi(E')| \leq 16$. The case for which $u' = v_2'$ is handled similarly.

Otherwise, $E' = \{uu', v_2v_3, v_5v_5'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 25. ∎

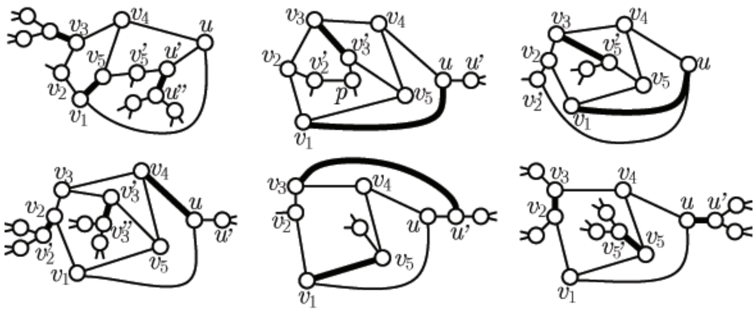

FIG. 25. *A 5-cycle adjacent to a 4-cycle along two incident edges.*

CLAIM 29. *If G has three 4-cycles in sequence, then it contains a good set of edges.*

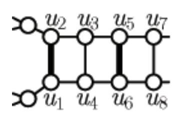

FIG. 26. *Three 4-cycles in sequence.*

Proof. Let $u_1u_2u_3u_4$, $u_3u_5u_6u_4$, $u_5u_7u_8u_6$ be three 4-cycles that are in sequence. Then $E' = \{u_1u_2, u_5u_6\}$ is a good set since $|\Psi(E')| \leq 18$. See Figure 26. ◻

CLAIM 30. *If G has three 4-cycles that are pairwise in sequence, then it contains a good set of edges.*

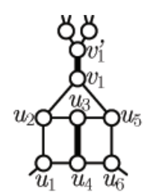

FIG. 27. *Three 4-cycles that are pairwise in sequence.*

Proof. Let $u_1u_2u_3u_4$, $u_3u_5u_6u_4$, and $v_1u_5u_3u_2$ be the 4-cycles. By Claim 14, v_1 is a 3-vertex, so let $N(v_1) \setminus \{u_2, u_5\} = v_1{}'$. By Claim 22, $v_1{}' \neq u_1$ and $v_1{}' \neq u_6$. Then $E' = \{u_3u_4, v_1v_1{}'\}$ is a good set since $|\Psi(E')| \leq 18$. See Figure 27. ◻

CLAIM 31. *If G has two 4-cycles and a 5-cycle that are pairwise in sequence, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4$, $u_3u_5u_6u_4$ be the 4-cycles, and let $v_1v_2u_5u_3u_2$ be the 5-cycle. Then $E' = \{u_3u_4, v_1v_2\}$ is a good set since $|\Psi(E')| \leq 18$. See Figure 28. ◻

CLAIM 32. *If G has two 4-cycles and a 6-cycle that are pairwise in sequence, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4$, $u_3u_5u_6u_4$ be the 4-cycles, and let $v_1v_2v_3u_5u_3u_2$ be the 6-cycle. By Claim 23, u_1v_1 is not an edge. By Claim 24, v_1v_3 is not an edge. By Claim 28, u_1v_3 is not an edge. By Claim 16, v_1 is a 3-vertex, so let $N(v_1) \setminus \{u_2, v_2\} = \{v_1{}'\}$. By Claim 29, $u_1v_1{}'$ is not an edge. If $v_1{}'v_3$ is an edge, then $E' = \{v_1u_2, v_3u_5\}$

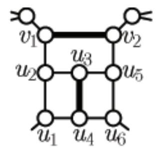

FIG. 28. *Two* 4-*cycles and a* 5-*cycle that are pairwise in sequence.*

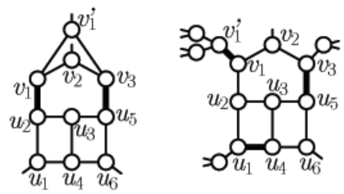

FIG. 29. *Two* 4-*cycles and a* 6-*cycle that are pairwise in sequence.*

is a good set since $|\Psi(E')| \leq 17$. Otherwise, $E' = \{u_1u_4, v_1v_1{}', v_3u_5\}$ is a good set since $|\Psi(E')| \leq 26$. See Figure 29. ∎

CLAIM 33. *If G has two* 4-*cycles and a* 7-*cycle that are pairwise in sequence, then it contains a good set of edges.*

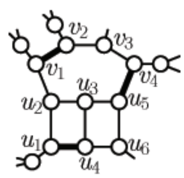

FIG. 30. *Two* 4-*cycles and a* 7-*cycle that are pairwise in sequence.*

Proof. Let $u_1u_2u_3u_4$, $u_3u_5u_6u_4$ be the 4-cycles, and let $v_1v_2v_3v_4u_5u_3u_2$ be the 7-cycle. By Claim 23, u_1v_1 is not an edge. By Claim 29, u_1v_2 is not an edge. By Claim 28, u_1v_4 is not an edge. By Claim 31, v_1v_4 is not an edge. By Claim 32, v_2v_4 is not an edge. Then $E' = \{u_1u_4, v_1v_2, v_4u_5\}$ is a good set since $|\Psi(E')| \leq 26$. See Figure 30. ∎

CLAIM 34. *If G has two* 5-*cycles and a* 4-*cycle that are pairwise in sequence, then it contains a good set of edges.*

Proof. Let $v_1v_2v_3v_4v_5, v_2v_1v_6v_7v_8$ be the 5-cycles, and let $v_3v_2v_8v_9$ be the 4-cycle. Notice that v_4, v_5, v_6, and v_7 are 3-vertices by Claim 16, and v_9 is a 3-vertex by Claim 14. By Claim 23, v_4v_9 and v_7v_9 are not edges. By Claim 24, v_5v_6 is not an edge. By Claim 28, v_4v_6, v_4v_7, v_5v_7, v_5v_9, v_6v_9 are not edges. For $i = 4, 5, 6, 7, 9$, let $N(v_i) \setminus \{v_1, \dots, v_9\} = \{v_i{}'\}$.

(1) Suppose $v_5{}' = v_6{}'$. By Claim 14, $v_5{}'$ is a 3-vertex, so let $N(v_5{}') \setminus \{v_5, v_6\} = \{v_5{}''\}$. Then set $E_1' = \{v_1v_2, v_9v_9{}', v_5{}'v_5{}''\}$ and $E_2' = \{v_1v_2, v_4v_4{}', v_7v_7{}'\}$, so that both $|\Psi(E_1')| \leq 27$ and $|\Psi(E_2')| \leq 27$. By planarity and the Jordan Curve Theorem, one of E_1' or E_2' is an induced matching and hence a good set.
(2) Suppose $v_4{}' = v_6{}'$. Observe that, by the Jordan Curve Theorem, $v_5{}' \neq v_7{}'$. Then $E' = \{v_2v_3, v_5v_5{}', v_6v_7\}$ is a good set since $|\Psi(E')| \leq 25$.

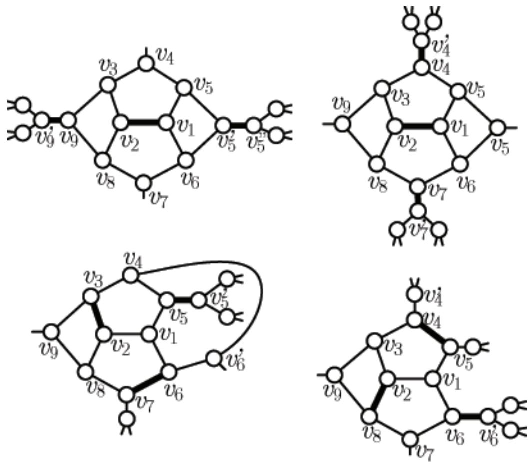

FIG. 31. *Two* 5*-cycles and a* 4*-cycle that are pairwise in sequence.*

Otherwise, $E' = \{v_2v_8, v_4v_5, v_6v_6'\}$ is a good set since $|\Psi(E')| \leq 26$. See Figure 31. □

CLAIM 35. *If G has a* 4*-cycle adjacent to a* 5*-cycle, then it contains a good set of edges.*

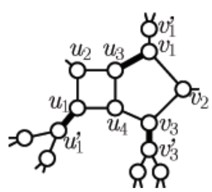

FIG. 32. *A* 4*-cycle adjacent to a* 5*-cycle.*

Proof. The 4-cycle and 5-cycle share at most one edge, due to Claims 23 and 28. So let $u_1u_2u_3u_4$ be the 4-cycle, and let $u_4u_3v_1v_2v_3$ be the 5-cycle. By Claim 14, u_1 is a 3-vertex, so let $N(u_1) \setminus \{u_2, u_4\} = \{u_1'\}$. By Claim 16, v_3 is a 3-vertex, so let $N(v_3) \setminus \{u_4, v_2\} = \{v_3'\}$. By Claim 27, u_1v_1 is not an edge. By Claim 28, $u_1'v_1$ is not an edge. By Claim 23, u_1v_3 is not an edge. By Claim 31, $u_1' \neq v_3'$. By Claim 34, $u_1'v_3'$ is not an edge. By Claim 23, v_1v_3 is not an edge. By Claim 28, v_1v_3' is not an edge. Then $E' = \{u_1u_1', u_3v_1, v_3v_3'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 32. □

CLAIM 36. *If G has a pair of adjacent* 5*-cycles, then it contains a good set of edges.*

Proof. By Claim 28, the cycles have at most two edges in common. Suppose they have two. Let $v_1v_2v_3v_4v_5$ and $v_5v_4v_6v_7v_1$ be the two 5-cycles. (The case in which the two common edges are not incident is excluded by Claim 23.) By Claim 35, v_2v_6 is not an edge. Then $E' = \{v_1v_2, v_4v_8\}$ is a good set since $|\Psi(E')| \leq 17$.

Otherwise, let $v_1v_2v_3v_4v_5$ and $v_5v_4v_6v_7v_8$ be the two 5-cycles. By Claim 16, v_8 is a 3-vertex, so let $N(v_8) \setminus \{v_5, v_7\} = \{v_8'\}$. By Claim 28, v_1v_6 is not an edge. By Claim 24, $v_1 \neq v_8'$. By Claim 34, v_1v_8' is not an edge. By Claim 28, v_2v_6 is not an edge. By Claim 28, $v_2 \neq v_8'$. If v_2v_8' is an edge, then we may identify two 5-cycles with exactly two common edges, handled in the paragraph above. By Claim 23,

$v_6 \neq v_8'$. By Claim 28, v_6v_8' is not an edge. Then $E' = \{v_1v_2, v_4v_6, v_8v_8'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 33. ∎

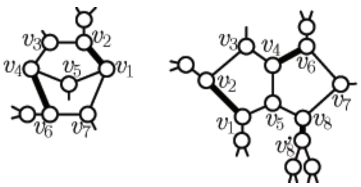

FIG. 33. *Two adjacent 5-cycles.*

CLAIM 37. *If G has a 4-cycle adjacent to a 6-cycle along two incident edges, then it contains a good set of edges.*

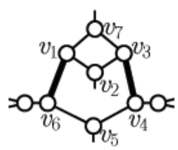

FIG. 34. *A 4-cycle adjacent to a 6-cycle along two incident edges.*

Proof. Let $v_1v_2v_3v_4v_5v_6$ be the 6-cycle, and let $v_3v_2v_1v_7$ be the 4-cycle. By Claim 28, v_4v_6 is not an edge. Then $E' = \{v_1v_6, v_3v_4\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 34. ∎

CLAIM 38. *If G has a 4-cycle in sequence with a 6-cycle, then it contains a good set of edges.*

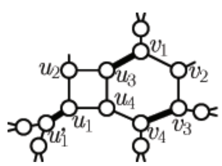

FIG. 35. *A 4-cycle in sequence with a 6-cycle.*

Proof. Let $u_1u_2u_3u_4$ be the 4-cycle, and let $u_4u_3v_1v_2v_3v_4$ be the 6-cycle. By Claim 27, u_1v_1 is not an edge. By Claim 37, u_1v_3 is not an edge. By Claim 23, u_1v_4 is not an edge. By Claim 28, $u_1'v_1$ is not an edge. By Claim 35, $u_1'v_3$ is not an edge. By Claim 32, $u_1'v_4$ is not an edge. By Claim 24, v_1v_3 is not an edge. By Claim 29, v_1v_4 is not an edge. Then $E' = \{u_1u_1', u_3v_1, v_3v_4\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 35. ∎

CLAIM 39. *If G has a 5-cycle in sequence with a 6-cycle, then it contains a good set of edges.*

Proof. Let $v_1v_2v_3v_4v_5$ be the 5-cycle, and let $v_5v_4v_6v_7v_8v_9$ be the 6-cycle. By Claim 16, v_2 is a 3-vertex; let $N(v_2) \setminus \{v_1, v_3\} = \{v_2'\}$. Note that $v_2' \neq v_6$ and $v_2' \neq v_9$ by Claim 28.

(1) If $v_2' = v_7$, then $E' = \{v_2v_7, v_4v_5\}$ is a good set as $|\Psi(E')| \leq 16$. The case $v_2' = v_8$ is handled similarly.

(2) Suppose $v_2'v_7$ is an edge. By Claim 28, v_1v_6 is not an edge. By Claim 35, v_1v_8 is not an edge. By Claim 24, v_1v_9 is not an edge. By Claim 24, v_6v_8 is

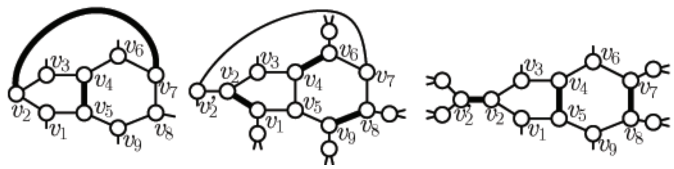

FIG. 36. *A* 5*-cycle in sequence with a* 6*-cycle.*

not an edge. By Claim 35, v_6v_9 is not an edge. Then $E' = \{v_1v_2, v_4v_6, v_8v_9\}$ is a good set since $|\Psi(E')| \leq 26$. The case that $v_2{'}v_8$ is an edge is handled similarly.

Otherwise, $E' = \{v_2v_2{'}, v_4v_5, v_7v_8\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 36. $\square$

CLAIM 40. *If G has a* 2*-vertex at distance* 2 *from a* 3*-cycle, then it contains a good set of edges.*

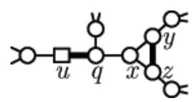

FIG. 37. *A* 2*-vertex at distance* 2 *from a* 3*-cycle.*

Proof. Let u be a 2-vertex at distance 2 from a 3-cycle xyz, where u and x have a common neighbor q. By Claim 18, qy and qz are not edges. Then $E' = \{uq, yz\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 37. $\square$

CLAIM 41. *If G has a* 2*-vertex at distance* 2 *from a* 4*-cycle, then it contains a good set of edges.*

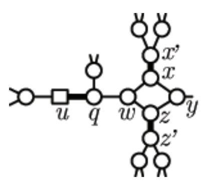

FIG. 38. *A* 2*-vertex at distance* 2 *from a* 4*-cycle.*

Proof. Let u be a 2-vertex at distance 2 from a 4-cycle $wxyz$, where u and w have a common neighbor q. By Claim 18, qx and qz are not edges. By Claim 22, xz is not an edge. By Claim 14, ux and uz are not edges, and both x and z are 3-vertices. Let $N(x) \setminus \{w, y\} = \{x'\}$ and $N(z) \setminus \{w, y\} = \{z'\}$. By Claim 16, ux' and uz' are not edges. By Claim 19, qx' and qz' are not edges. By Claim 27, $x' \neq z'$. By Claim 28, $x'z'$ is not an edge. Then $E' = \{uq, xx', zz'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 38. $\square$

CLAIM 42. *If G has a* 2*-vertex at distance* 2 *from a* 5*-cycle, then it contains a good set of edges.*

Proof. Let u be a 2-vertex at distance 2 from a 5-cycle $x_0x_1x_2x_3x_4$, where u and x_0 have a common neighbor q. By Claim 16, x_1 is a 3-vertex, so let $N(x_1) \setminus \{x_0, x_2\} = \{x_1{'}\}$. By Claim 14, ux_1 and ux_4 are not edges. By Claim 16, $ux_1{'}$ and ux_3 are not edges. By Claim 18, qx_1 and qx_4 are not edges. By Claim 19, $qx_1{'}$ and qx_3 are not edges. By Claim 23, $x_1{'} \neq x_3$ and $x_1{'} \neq x_4$. By Claim 28, $x_1{'}x_3$ and $x_1{'}x_4$

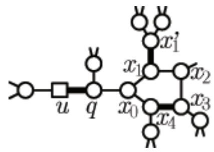

FIG. 39. *A* 2*-vertex at distance* 2 *from a* 5*-cycle.*

are not edges. Then $E' = \{uq, x_1x_1', x_3x_4\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 39. ◻

CLAIM 43. *If G has two* 3*-cycles at distance* 1*, then it contains a good set of edges.*

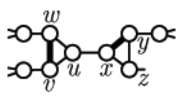

FIG. 40. *Two* 3*-cycles at distance* 1*.*

Proof. Let uvw and xyz be 3-cycles at distance 1, with ux being an edge. By Claim 23, vy and wy are not edges. Then $E' = \{vw, xy\}$ is a good set since $|\Psi(E')| \leq 17$. See Figure 40. ◻

CLAIM 44. *If G has a* 3*-cycle at distance* 1 *from a* 4*-cycle, then it contains a good set of edges.*

FIG. 41. *A* 3*-cycle at distance* 1 *from a* 4*-cycle.*

Proof. Let xyz be a 3-cycle at distance 1 to a 4-cycle $v_1v_2v_3v_4$, with xv_1 being an edge. Observe that v_2, v_4 are 3-vertices by Claim 14. Let $N(v_i) \setminus \{v_1, v_3\} = \{v_i'\}$ for $i = 2, 4$. By Claim 23, zv_2 and zv_4 are not edges. By Claim 24, zv_2' and zv_4' are not edges. By Claim 22, v_2v_4 is not an edge. By Claim 27, $v_2' \neq v_4'$. By Claim 28, $v_2'v_4'$ is not an edge. Then $E' = \{xz, v_2v_2', v_4v_4'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 41. ◻

CLAIM 45. *If G has a* 3*-cycle at distance* 1 *from a* 5*-cycle, then it contains a good set of edges.*

Proof. Let xyz be a 3-cycle at distance 1 to a 5-cycle $v_1v_2v_3v_4v_5$, with xv_1 being an edge. Observe that v_2 is a 3-vertex by Claim 16, so let $N(v_2) \setminus \{v_1, v_3\} = \{v_2'\}$. By Claim 23, zv_2 and zv_5 are not edges. By Claim 24, zv_2' and zv_4 are not edges. By Claim 23, v_2v_4 and v_2v_5 are not edges. By Claim 28, $v_2'v_4$ and $v_2'v_5$ are not edges. Then $E' = \{xz, v_2v_2', v_4v_5\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 42. ◻

CLAIM 46. *If G has a* 5*-cycle at distance* 1 *from a double* 4*-face, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4u_5$ be a 5-cycle at distance 1 from a double 4-face $v_1v_2v_3v_4$, $v_3v_2v_5v_6$, with u_1v_1 being an edge. Observe that u_5 is a 3-vertex by Claim 16, so let $N(u_5) \setminus \{u_1, u_4\} = \{u_5'\}$.

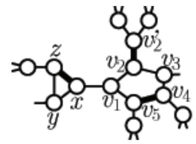

FIG. 42. *A* 3*-cycle at distance* 1 *from a* 5*-cycle.*

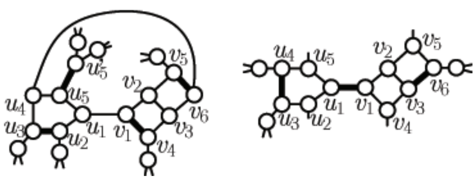

FIG. 43. *A* 5*-cycle at distance* 1 *from a double* 4*-face.*

Suppose u_4v_6 is an edge. By Claim 23, $u_2 \neq u_5{}'$ and $u_3 \neq u_5{}'$. By Claim 28, $u_2u_5{}'$ and $u_3u_5{}'$ are not edges. By Claim 27, v_4v_5 is not an edge. By Claim 29, u_2v_4 and u_3v_5 are not edges. By Claim 35, u_3v_4, u_2v_5, $u_5{}'v_4$ are not edges, and $u_5{}' \neq v_4$, $u_5{}' \neq v_5$. By Claim 38, $u_5{}'v_5$ is not an edge. Then $E' = \{u_2u_3, u_5u_5{}', v_1v_4, v_5v_6\}$ is a good set since $|\Psi(E')| \leq 33$. The case in which u_3v_6 is an edge is treated similarly.

Otherwise, $E' = \{u_3u_4, u_1v_1, v_3v_6\}$ is a good set since $|\Psi(E')| \leq 26$. See Figure 43. ∎

CLAIM 47. *If G has a pair of double* 4*-faces at distance* 1*, then it contains a good set of edges.*

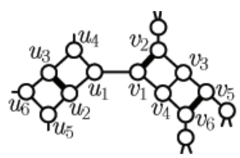

FIG. 44. *A pair of double* 4*-faces at distance* 1*.*

Proof. Let $u_1u_2u_3u_4, u_3u_2u_5u_6$ and $v_1v_2v_3v_4, v_4v_3v_5v_6$ be double 4-faces at distance 1, with u_1v_1 being an edge. By Claim 23, v_2v_5 is not an edge. By Claim 27, v_2v_6 is not an edge. Then $E' = \{u_2u_3, v_1v_2, v_5v_6\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 44. ∎

CLAIM 48. *If G has a* 4*-cycle,* ($\leqslant$8)*-cycle, and* 4*-cycle in sequence, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4, v_1v_2v_3v_4$ be 4-cycles at distance 1, with u_1v_1 being an edge, and C the adjacent ($\leqslant$8)-cycle. Suppose without loss of generality that C contains both u_4 and v_2. By Claim 14, u_2, u_3, u_4 and v_2, v_3, v_4 are 3-vertices. By Claims 22 and 29, there are no edges among u_2, u_4, v_2, v_4. For $i = 2, 3, 4$, let $N(u_i) \setminus \{u_1, u_2, u_3, u_4\} = \{u_i{}'\}$ and $N(v_i) \setminus \{v_1, v_2, v_3, v_4\} = \{v_i{}'\}$. By Claim 35, $|\{u_2{}', u_4{}', v_2{}', v_4{}'\}| = 4$. Also, by Claim 38, there are no edges among $u_2{}', u_4{}', v_2{}', v_4{}'$. Note now that C is either a 7- or an 8-cycle.

(1) Suppose $u_3{}'v_3{}'$ is an edge.
   (a) If $v_2{}'v_3{}'$ is an edge, then $E' = \{u_2u_3, v_1v_4, v_2{}'v_3{}'\}$ is a good set as $|\Psi(E')| \leq 25$. The subcases in which $v_3{}'v_4{}'$, $u_2{}'u_3{}'$, or $u_3{}'u_4{}'$ is an edge are handled similarly.

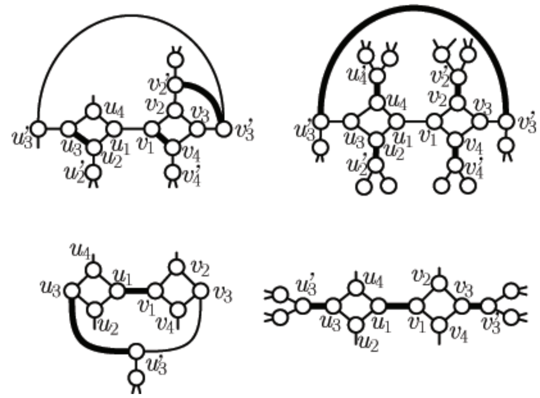

FIG. 45. *A* 4*-cycle,* (⩽8)*-cycle, and* 4*-cycle in sequence.*

(b) Otherwise, let $E' = \{u_2u_2', u_4u_4', v_2v_2', v_4v_4', u_3'v_3'\}$. By Claim 23, u_2u_3', u_4u_3', v_2v_3', v_4v_3' are not edges. By Claim 38, u_2v_3', u_4v_3', $u_3'v_2$, $u_3'v_4$ are not edges. By Claim 35, $u_2'v_3'$, $u_4'v_3'$, $u_3'v_2'$, $u_3'v_4'$ are not edges. Since C is an (⩽8)-cycle, $|\Psi(E')| \leq 45$ and hence E' is a good set.

(2) If $u_3' = v_3'$, then $E' = \{u_1v_1, u_3u_3'\}$ is a good set as $|\Psi(E')| \leq 18$.

Otherwise, $E' = \{u_1v_1, u_3u_3', v_3v_3'\}$ is a good set since $|\Psi(E')| \leq 27$. See Figure 45. ◻

CLAIM 49. *If G has a* 4*-cycle,* 4*-cycle, and* 7*-cycle in sequence, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4$, $u_4u_3u_5u_6$, and $u_6u_5v_1v_2v_3v_4v_5$ be the sequence of cycles. By Claim 14, u_1 and u_2 are 3-vertices, so let $N(u_i) \setminus \{u_1, u_2, u_3, u_4\} = \{u_i'\}$ for $i = 1, 2$. By Claims 25 and 35, note that the 7-cycle does not have any chords. By Claim 41, v_2 and v_4 are 3-vertices, so let $N(v_i) \setminus \{v_1, v_3, v_5\} = \{v_i'\}$ for $i = 2, 4$.

(1) If $v_2' = v_4'$, then $E' = \{u_3u_4, v_1v_2, v_4v_5\}$ is a good set as $|\Psi(E')| \leq 25$.

(2) If $v_2'v_4'$ is an edge, then $E' = \{u_3u_4, v_1v_2, v_4v_5\}$ is a good set as $|\Psi(E')| \leq 27$.

(3) If $u_2' = v_4'$, then $E' = \{u_3u_4, v_1v_2, v_4v_5\}$ is a good set as $|\Psi(E')| \leq 27$. The case for which $u_1' = v_2'$ is treated similarly.

Otherwise, note that, by Claim 35, $u_1 \neq v_4'$ and $u_2 \neq v_2'$. By Claim 38, $u_1' \neq v_4'$ and $u_2' \neq v_2'$, and u_1v_2, u_2v_4 are not edges. Then it follows that $E' = \{u_1u_2, u_5u_6, v_2v_2', v_4v_4'\}$ is a good set since $|\Psi(E')| \leq 36$. See Figure 46. ◻

CLAIM 50. *If G has a* 4*-cycle,* 7*-cycle, and* 5*-cycle in sequence, then it contains a good set of edges.*

Proof. Let $u_1u_2u_3u_4$ be the 4-cycle, let $v_1v_2v_3v_4v_5$ be the 5-cycle, and let $u_1u_4w_1w_2w_3v_2v_1$ be the 7-cycle. By Claim 14, u_2 and u_3 are 3-vertices, so let $N(u_i) \setminus \{u_1, u_2, u_3, u_4\} = \{u_i'\}$ for $i = 2, 3$. Also, v_3, v_4 are 3-vertices by Claim 16, so let $N(v_i) \setminus \{v_2, v_3, v_4, v_5\} = \{v_i'\}$ for $i = 3, 4$.

(1) Suppose $u_3' = v_4'$. Note that u_2w_1 and $u_2'w_1$ are not edges by the Jordan Curve Theorem. By Claim 23, u_2v_4' is not an edge. By Claim 49, $u_2'v_4'$ is not an edge. By Claim 33, $v_4'w_1$ is not an edge. Then $E' = \{u_2u_2', u_4w_1, v_1v_2, v_4v_4'\}$ is a good set since $|\Psi(E')| \leq 33$.

(2) Suppose $u_3' = v_3'$. By the Jordan Curve Theorem, w_1u_2, w_1u_2', and w_1v_5 are not edges. By Claim 23, u_2v_3' is not an edge. By Claim 35, u_2v_5 and $u_2'v_5$ are not edges. By Claim 33, $u_2'v_3'$ is not an edge. By Claim 28, v_3v_5 is not an

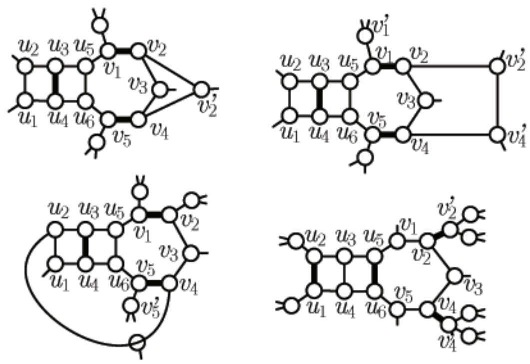

FIG. 46. *A sequence of* 4*-cycles such that one of the* 4*-cycles is adjacent to a* 7*-cycle.*

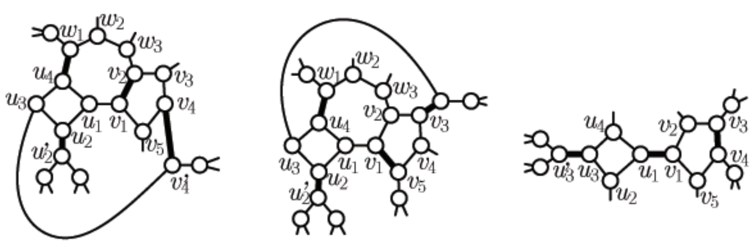

FIG. 47. *A* 4*-cycle,* 7*-cycle, and* 5*-cycle in sequence.*

edge. By Claim 33, v_3w_1 is not an edge. Then $E' = \{u_2u_2{}', u_4w_1, v_1v_5, v_3v_3{}'\}$ is a good set since $|\Psi(E')| \le 35$.

Otherwise, note that, by Claim 37, $u_3{}' \ne v_3$ and $u_3{}' \ne v_4$. It therefore follows that $E' = \{u_3u_3{}', u_1v_1, v_3v_4\}$ is a good set since $|\Psi(E')| \le 27$. See Figure 47. ∎

CLAIM 51. *If G has a pair of* 3*-cycles at distance* 2*, then it contains a good set of edges.*

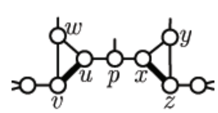

FIG. 48. *Two* 3*-cycles at distance 2.*

Proof. Let uvw and xyz be 3-cycles at distance 2, where u and x have a common neighbor p. By Claim 24, vz is not an edge. Then $E' = \{uv, xz\}$ is a good set since $|\Psi(E')| \le 17$. See Figure 48. ∎

CLAIM 52. *If G has a* 3*-cycle at distance* 2 *from a double* 4*-face, then it contains a good set of edges.*

Proof. Let xyz be a 3-cycle, and let $v_1v_2v_3v_4, v_3v_2v_5v_6$ be a double 4-face at distance 2, with x and v_1 having a common neighbor p. By Claim 22, yp and zp are not edges. By Claim 26, yv_6 and zv_6 are not edges. By Claim 28, pv_6 is not an edge. Then $E' = \{yz, pv_1, v_3v_6\}$ is a good set since $|\Psi(E')| \le 26$. See Figure 49. ∎

CLAIM 53. *If G has a pair of double* 4*-faces at distance* 2*, then it has a good set of edges.*

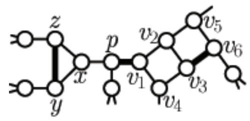

FIG. 49. *A* 3*-cycle at distance* 2 *from a double* 4*-face.*

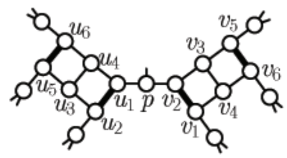

FIG. 50. *A pair of double* 4*-faces at distance* 2*.*

Proof. Let $u_1u_2u_3u_4, u_4u_3u_5u_6$ and $v_1v_2v_3v_4, v_3v_5v_6v_4$ be two double 4-faces at distance 2, such that u_1 and v_2 have a common neighbor p. Now, the following are not edges by Claim 47: u_2v_1, u_2v_5, u_2v_6, u_5v_1, u_5v_5, u_5v_6, u_6v_1, u_6v_5, and u_6v_6. Then $E' = \{u_1u_2, u_5u_6, v_1v_2, v_5v_6\}$ is a good set since $|\Psi(E')| \leq 35$. See Figure 50. □

To wrap up the proof of Lemma 9, we list the specific claims which certify the presence of a good set, given the presence of one of the structures (C1)–(C12).

(C1) Claim 11.
(C2) Claims 13, 14, 16, 17, and 21.
(C3) Claims 12 and 15.
(C4) Claims 18, 19, 20, 40, 41, and 42.
(C5) Claims 22, 23, 24, 25, and 26.
(C6) Claims 35, 38, 36, and 39.
(C7) Claims 43, 44, and 45.
(C8) Claims 23, 29, 30, 35, 38, 32, 33, and 49.
(C9) Claim 48.
(C10) Claim 50.
(C11) Claims 22, 23, 43, 44, 47, 51, 52, and 53.
(C12) Claim 46.

This concludes the proof of Lemma 9. □

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